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Nonlinear Theory of Dissipative Instability of a Relativistic Beam in a Plasma

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Dissipative instability of a monochromatic relativistic beam in a plasma with frequent collisions $\nu < \delta$ (δ is the beam instability increment) is investigated in the quasilinear approximation in the absence as well as in the presence of a strong external magnetic field. It is shown that the energy lost by the beam during development of such an instability is mainly expended in increasing the thermal energy of the plasma particles. The energy of the oscillations generated during the instability is lower (by a factor δ/ν).

1. In connection with experiments on the interaction of a beam of relativistic charged particles with a dense plasma, great interest attaches to the study of the dissipative instability of an electron beam in a plasma, which sets in at high collision frequencies $\nu > \delta$ (δ is the growth increment of the oscillations). At relativistic energies, this instability becomes significant also because the collisionless-instability increments decrease with the increasing relativism parameter $\gamma_0 = (1 - v_0^2/c^2)^{-1/2}$. The dissipative instability results from the fact that the system made up of the plasma and the monochromatic beam has a negative energy proportional to $\omega\delta\epsilon/\delta\omega$ ($\epsilon(\omega)$ is the dielectric constant of the system), and the presence of collisions in such a system leads to instability of these waves. Such an instability is possible only when the condition $\Delta v < \delta/k$ is satisfied (Δv is the thermal-velocity spread in the beam, and k is the wave number of the unstable harmonic), and at a large spread the beam does not influence the dispersion of the wave, so that there is no dissipative instability. The main stabilizing effect in the instability development is therefore the smearing of the velocity distribution function of the beam.

We confine ourselves to the instability of a beam of low density $n_1 \ll n_0$ (n_1 and n_0 are the beam and plasma densities, respectively), and assume also that the condition $\omega_0 \gg \nu \gg \delta$ is satisfied. In this case the maximum growth increment is possessed by the electrostatic oscillations, whose frequency is close to the natural frequency of the plasma. In the absence of an external magnetic field, the increment of the most stable harmonic with $k_z = \omega_0/v_0$ (k_z is the wave-vector component along the beam velocity v_0, and ω_0 is the electron Langmuir plasma frequency) is determined by the relation

$$\delta = \frac{1}{2^{1/2}}\frac{\omega_1}{\gamma_0^{1/2}}\left(\frac{\omega_0}{\nu}\right)^{1/2}\left[\frac{k_z^2}{k^2}\frac{1}{\gamma_0^2}+\frac{k_\perp^2}{k^2}\right]^{1/2}, \tag{1}$$

where ω_1 is the Langmuir frequency of the beam. Just as in the collisionless case (see, e.g.,[1,2]), when $\gamma_0 > 1$ the oscillations having the largest increment are those propagating at large angles to the beam, $k_\perp \gg k_z$. Taking into account the thermal motion in the beam at large values of $k_\perp/k_z$, we can determine the optimal angle at which the increment is maximal:

$$\delta = \delta_{max} = \frac{1}{2^{1/2}}\frac{\omega_1}{\gamma_0^{1/2}}\left(\frac{\omega_0}{\nu}\right)^{1/2}, \quad \left(\frac{k_\perp^2}{k_z^2}\right)^{opt} = \frac{2^{3/2}}{3}\frac{\delta_{max}c}{\omega_0\Delta v_\perp}, \tag{1'}$$

where

$$n_1(\Delta v_\perp)^2 = \int v_\perp^2 f_0\, d\mathbf{p}.$$

The parameter in the right-hand side of (1′) is large, since

$$\omega_0\Delta v_\perp/c\delta_{max} \sim k_\perp^2(\Delta v_\perp)^2/\delta^2 \ll 1$$

by virtue of the initial monochromaticity of the beam.

In the presence of a strong external magnetic field $\omega_H \gg \omega_0$ ($\omega_H = eH_0/mc$), the growth increment at $\gamma_0 > 1$ is much smaller:

$$\delta_{\mathbf{k}} = \frac{1}{2^{1/2}}\frac{\omega_1}{\gamma_0^{3/2}}\left(\frac{\omega_0}{\nu}\right)^{1/2}\left(\frac{k_z}{k}\right)^{3/2}. \tag{2}$$

The maximum of the increment corresponds to oscillations propagating along the magnetic field, $k_z \approx k$, i.e., the oscillation spectrum is close to one-dimensional. The growth increment is of the order of the maximal one determined by formulas (1) and (2) in a wide interval of k_z:

$$\Delta k_z \sim \frac{\nu}{v_0}\ln^{-1/2}\frac{E_{max}}{E(0)} \gg \frac{\delta}{v_0}$$

($E(0)$ and E_{max} are respectively the initial value of the field amplitude and its maximum value determined from (11)). In the long-wave region of the spectrum, $k_z < \omega_0/v_0$, the instability in question goes over into the weaker nonresonant collisionless instability, and in the short-wave region the increment decreases to zero. We assume that many harmonics fall into the region Δk_z of the most unstable wave numbers, and use the equations of the quasilinear approximation for the investigation of the nonlinear stage of development of the instability.

2. We consider first a one-dimensional case corresponding to the presence of a strong external magnetic field $\mathbf{H}_0 \parallel \mathbf{v}_0$ in the plasma. We use the following equations to describe the variation of the "background" beam distribution function f_0 and the energy of the oscillations generated during the course of the instability[3]:

$$\frac{\partial f_0}{\partial t} = e^2\sum_{k>0}\frac{2\delta_k|E_k|^2}{(kv_0-\omega_k)^2+\delta_k^2}\frac{\partial^2 f_0}{\partial p^2} - e^2\sum_{k>0}\frac{2\delta_k(kv_0-\omega_k)k|E_k|^2}{[(kv_0-\omega_k)^2+\delta_k^2]^2}\frac{\partial}{\partial p}\left[(v-v_0)\frac{\partial f_0}{\partial p}\right], \tag{3}$$

$$\frac{\partial|E_k|^2}{\partial t} = 2\delta_k|E_k|^2 = 2^{1/2}\frac{\omega_1}{\gamma^{3/2}(t)}\left(\frac{\omega_0}{\nu}\right)^{1/2}|E_k|^2; \tag{4}$$

here

$$\gamma(t) = \left[\frac{1}{n_1}\int\frac{dp\, f_0(t,p)}{(p^2/m^2c^2+1)^{3/2}}\right]^{-1/3}$$

where $f_0(t, p)$ is the beam-electron distribution function averaged over distances that are large in comparison with the wavelength of the oscillations. By determining with the aid of (3) the change of the moments of the beam-electron distribution function, we obtain ultimately the following system of equations describing the relaxation of the beam in the plasma in the case under consideration:

$$\frac{dT}{dt}=\frac{d}{dt}\frac{2}{n_1}\int[(p^2c^2+m^2c^4)^{1/2}-(\bar{p}^2c^2+m^2c^4)^{1/2}]f_0\,dp$$

$$=2^{3/2}\frac{1}{n_1}\frac{\omega_1}{\gamma^{3/2}}\left(\frac{\nu}{\omega_0}\right)^{1/2}W, \tag{5}$$

$$\frac{d\bar{p}}{dt}\equiv\frac{d}{dt}\frac{1}{n_1}\int pf_0\,dp=-2\nu\frac{W}{n_1c}, \tag{6}$$

$$\frac{dW}{dt}=2^{1/2}\frac{\omega_1}{\gamma^{3/2}}\left(\frac{\omega_0}{\nu}\right)^{1/2}W. \tag{7}$$

In these equations

$$W=\frac{1}{8\pi}\sum_k|E_k|^2,$$

where p is the average beam-electron momentum, $p = p_0$ at $t = 0$.

From (5)—(7) it follows that excitation of the oscillations brings about a thermal smearing of the beam and a deceleration of the beam particles, determined by the relations

$$\Delta\bar{p}\approx-2^{1/2}\frac{\nu}{\omega_1}\left(\frac{\nu}{\omega_0}\right)^{1/2}\gamma_0^{3/2}\frac{W}{n_1c},\qquad \Delta T\approx2\frac{\nu}{\omega_0}\frac{W}{n_1}. \tag{8}$$

Our analysis will pertain to the case of not too high energies $\gamma n_1/n_0 \ll \nu/\omega_0$, when the change of momentum at the maximum oscillation energy is small: $|\Delta p| \ll p_0$. To determine at which velocity scatter Δv in the beam the dissipative instability becomes stabilized, we use the linear-theory dispersion equation in the kinetic approximation. Assuming that the beam-particle distribution function with respect to the relative momentum $p - p_0$ is Maxwellian[1], and considering oscillations of frequency close to ω_0, we write this equation in the form

$$\frac{k^2(\Delta v)^2}{\pi^{1/2}\omega_1^2}\frac{\gamma_0^3\nu}{\omega_0}+xU(x,0)=yV(x,0). \tag{9}$$

In this equation

$$x=(\omega_k-kv_0)/2^{1/2}k\Delta v,\qquad y=\delta/2^{1/2}k\Delta v;$$

$U(x, y)$ and $V(x, y)$ are respectively the real and imaginary parts of the probability integral $w(x + iy)$. We have assumed in its derivation that $y \ll 1$.

It follows from (9) that the threshold value of Δv, at which stabilization of the dissipative instability takes place, is determined by the relation

$$k^2(\Delta v)^2=\pi^{1/2}\frac{\omega_1^2}{\gamma_0^3}\frac{\omega_0}{\nu}[-xU(x,0)]_{max}\approx0{,}8\frac{\omega_1^2}{\gamma_0^3}\frac{\omega_0}{\nu}. \tag{10}$$

With the aid of (8) and (10) we find that the energy of the oscillations excited during the dissipative instability is of the order of

$$W\sim n_1mc^2\omega_1^2/\nu^2. \tag{11}$$

The losses of the translational motion of the beam particles are obtained by using the first relation of (8):

$$\Delta\mathscr{E}\approx-\mathscr{E}_0\gamma_0^{1/2}\frac{\omega_1}{(\omega_0\nu)^{1/2}}\sim-\frac{\nu}{\delta}W, \tag{12}$$

where $\mathscr{E}_0 = n_1mc^2\gamma_0$ is the initial beam energy. It follows from (12) that the energy lost by the beam in the case of dissipative instability is much larger than the energy of the excited oscillations. This energy goes mainly to increase the thermal energy of the plasma electrons, through collision heating by the high-frequency field excited during the instability,

$$n_0\Delta T_{\rm pl}\approx\frac{n_0e^2}{2m}\sum_k|E_k|^2\frac{\nu}{\delta_k(\omega_k^2+\nu^2)}\sim\frac{\nu}{\delta}W. \tag{13}$$

We note that, unlike the collisionless instability, where the plasma particles interact adiabatically with the fields, in the present case there is an irreversible increase of the kinetic energy of the plasma particles.

3. We consider now the excitation of a three-dimensional oscillation spectrum (the case when there is no external magnetic field). The relaxation of a monochromatic beam can be investigated in this case with the aid of the equations

$$\frac{\partial f_0}{\partial\tau}=\frac{1}{\xi_\perp}\frac{\partial}{\partial\xi_\perp}\left[\xi_\perp\frac{\partial f_0}{\partial\xi_\perp}-\alpha\frac{\xi_\perp^2}{(\xi_z^2+\xi_\perp^2+1)^{1/2}}\frac{\partial f_0}{\partial\xi_z}\right]$$

$$+\frac{\partial}{\partial\xi_z}\left[\beta\frac{\partial f_0}{\partial\xi_z}-\alpha\frac{\xi_\perp}{(\xi_z^2+\xi_\perp^2+1)^{1/2}}\frac{\partial f_0}{\partial\xi_\perp}\right]; \tag{14}$$

$$\frac{d|E_{\mathbf{k}}|^2}{dt}=\sqrt{2}\frac{\omega_1}{\gamma_0^{1/2}}\left(\frac{\omega_0}{\nu}\right)^{1/2}|E_{\mathbf{k}}|^2, \tag{15}$$

where $f_0(t, p)$ is the background distribution function of the beam electrons,

$$\alpha=\left(\frac{2\omega_0\nu\gamma_0}{\omega_1^2}\right)^{1/2},\qquad\beta=2\left(\frac{k_z^2}{k_\perp^2}\right)^{opt}\approx\frac{3}{2^{1/2}}\alpha\frac{\Delta v_\perp}{c}.$$

In (14) we have introduced the dimensionless variables $\xi_\perp = p_\perp/mc$, $\xi_z = p_z/mc$ and τ, the latter connected with t by the relation

$$\frac{d\tau}{dt}=\frac{1}{2^{1/2}}\omega_1\left(\frac{\nu}{\omega_0}\right)^{1/2}\gamma_0^{1/2}\frac{1}{8\pi n_1mc^2}\sum_{\mathbf{k}}|E_{\mathbf{k}}|^2. \tag{16}$$

In the derivation of (14) and (15) we have used the beam monochromaticity condition $k_\perp v_\perp/\delta_k \ll 1$ and replaced the quantities ω_k and δ_k by the values for the most unstable harmonic of the spectrum with $k_\perp \gg k_z$:

$$\omega_{\mathbf{k}}+i\delta_{\mathbf{k}}=k_zv_0+\frac{1}{2^{1/2}}(1+i)\frac{\omega_1}{\gamma_0^{1/2}}\left(\frac{\omega_0}{\nu}\right)^{1/2}.$$

From (14) we can easily obtain a system of equations for the moments of the distribution function f_0. When (15) is taken into account, the system of equations describing the beam relaxation during the dissipative instability takes the form

$$\frac{d(\Delta\theta)^2}{d\tau}=\frac{1}{n_1}\int\frac{\xi_\perp^2}{\xi_z^2+\xi_\perp^2+1}\frac{\partial f_0}{\partial\tau}dp\approx\frac{4}{\gamma_0^2}, \tag{17}$$

$$\frac{d(\Delta p_z)^2}{d\tau}=\frac{m^2c^2}{n_1}\int(\xi_z-\xi_0)^2\frac{\partial f_0}{\partial\tau}dp\approx4m^2c^2\left(\frac{k_z^2}{k_\perp^2}\right)^{opt}, \tag{18}$$

$$\frac{d\mathscr{E}}{d\tau}=mc^2\int(1+\xi_\perp^2+\xi_z^2)^{1/2}\frac{\partial f_0}{\partial\tau}dp\approx-2^{3/2}\frac{(\omega_0\nu)^{1/2}}{\omega_1}\frac{n_1mc^2}{\gamma_0^{1/2}}. \tag{19}$$

$$\frac{dW}{d\tau}=2n_1mc^2\frac{\omega_0}{\nu}\frac{1}{\gamma_0}. \tag{20}$$

Here Δp_z is the spread of the longitudinal momenta of the beam particles, $\Delta\theta$ is the angle spread of the

[1] With the aid of Eq. (3), in which we neglect the small second term, we can show that a symmetrical smearing of the distribution function with respect to the momenta $p-p_0$ takes place during the course of the beam relaxation, and that the distribution remains Maxwellian in the course of time.

momenta of the beam particles. The longitudinal and transverse velocity spreads connected with Δp_z and $\Delta\theta$ are determined by the relations[2,3]

$$\Delta v_\perp \approx c\Delta\theta, \quad \Delta v_z \approx c\Delta p_z / p\gamma^2 - c(\Delta\theta)^2.$$

The angle spread in the beam, needed to stabilize the "oblique" oscillations with maximum increment (1′), is determined by the relation

$$(\Delta\theta)^2 \sim \frac{\delta_{max}^2}{k^2c^2} \approx \frac{1}{2}\frac{\omega_1^2}{\omega_0\nu}\frac{1}{\gamma_0}\left(\frac{k_z^2}{k_\perp^2}\right)^{opt}.$$

According to (17), this corresponds to

$$\tau_{max} \approx \frac{1}{8}\frac{\omega_1^2}{\omega_0\nu}\gamma_0\left(\frac{k_z^2}{k_\perp^2}\right)^{opt}.$$

The energy of these oscillations is of the order of

$$W \sim \frac{1}{4}n_1mc^2\frac{\omega_1^2}{\nu^2}\left(\frac{k_z^2}{k_\perp^2}\right)^{opt}. \tag{21}$$

When $k_\perp\Delta v_\perp \sim \delta$ in the spectrum of the excited oscillations, according to (1′), we have $k_\perp \sim k_z$ and the oscillation energy determined by (21) is the same as in the one-dimensional case (see (11)).

The energy lost by the beam at this stage of the instability is lower by a factor γ_0 than in the one-dimensional case:

$$\Delta\mathscr{E} \sim -\frac{\nu}{\delta}W \sim -\mathscr{E}_0\frac{\omega_1}{(\omega_0\nu)^{1/2}}\frac{1}{\gamma_0^{1/2}}. \tag{22}$$

The transverse-momentum spread, which is connected with the buildup of the "oblique" oscillations, is determined from (18):

$$\Delta p_z \sim mc\omega_1\gamma_0^{1/2}/(\omega_0\nu)^{1/2}. \tag{23}$$

During this stage of the instability, at the energies under consideration (which satisfy the condition $\gamma_0 n_1/n_0 \ll \nu/\omega_0$), the beam remains monochromatic with respect to the longitudinal velocities and the buildup of almost one-dimensional oscillations with $k_\perp \approx 0$ continues. The energy of these oscillations is of the same order as in (11), i.e., the spectrum of the oscillations excited in the dissipative instability is nearly isotropic, unlike the collisionless case considered in[3].

Excitation of oscillations with $k_\perp \approx 0$ is accompanied by a considerable deceleration of the beam. The energy loss in this case is determined by formula (12). The energy lost by the beam, according to (13), goes to increase the longitudinal temperature of the plasma electrons.

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