The Interaction of Weak Gravitational Waves with a Gas

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The interaction between a weak gravitational wave (GW) and a gas is considered. The problem is solved on the background of a flat Minkowski space-time. It is shown that if prior to the passage of the GW the distribution function f_0 of the gas was isotropic, then the phase velocity of the GW is $v > c$. An anisotropy of f_0 can lead to $^p v_p < c$. In this connection we consider the interaction of a GW with a beam of ultrarelativistic particles. It is proved that in a collisionless gas there is no specific damping of the GW analogous to Landau damping of waves in a plasma. The complex index of refraction for GW in a gas is computed, taking into account the collisions. For comparison we consider the collisionless interaction of short-wave adiabatic perturbations (AP) of the metric during the early stages of expansion of the Universe with a neutrino gas. It is shown that in distinction from GW, damping of the adiabatic perturbations is possible, damping which is analogous to Landau damping. A condition is derived for which a sufficiently large anisotropy of f_0 of the neutrinos leads to replacement of the collisionless damping of these perturbations by their growth.

It is known that gravitational waves (GW) do not interact with a homogeneous Pascal fluid in the absence of dissipative processes[1-3]. In a viscous but homogeneous medium the energy of the GW is converted into heat, without provoking macroscopic motions of the medium[4,5]. In the case of nonhomogeneous viscosity[6], as well as for nonhomogeneous elasticity[7], the energy of the GW can also be converted into acoustic vibrations. In addition, a recent paper[8] has rigorously solved the problem of refraction of GW in a medium filled by quadrupoles.

In the present paper we consider the interaction of GW with a gas, when the model of a continuous medium is not applicable and the motion of the individual particles leads to a refractive index n differing from one. In particular, refraction occurs even for a collisionless gas[1)].

In order to compute n we make use of a dispersion relation derived from the kinetic equation and the linearized Einstein equations. The problem uses a small parameter α which reflects the weakness of the gravitational interaction: $\alpha = (\omega_0/\omega)^2$, where ω is the frequency of the incident wave and ω_0 is the analog of the plasma frequency: $\omega_0^2 = 4\pi GE/c^2$, where E is the energy density of the gas, and G the gravitational constant; $\omega_0^2 \approx 10^{-6}\rho$, $\rho = E/c^2\,[\text{g-cm}^{-3}]$. It is clear that a wide range of values ω and ρ corresponds to $\alpha \ll 1$. (In cosmological applications $\alpha \ll 1$ means in addition that $\lambda \ll ct$, i.e., the wavelength of the GW is much smaller than the optical horizon.)

In a collisionless ultrarelativistic gas Im n $\lesssim 0(\alpha^2)$, i.e., in the approximation used here, the collisionless damping is absent, which is a direct consequence of the small difference between v_p (the phase velocity) and c.

The computed dependence of the refractive index on the collision frequency ν in the isotropic case is such that for $\nu/\omega \gg 1$

$$|\mathrm{Re}\, n - 1| \sim \alpha\xi, \quad \mathrm{Im}\, n \sim \alpha\nu\xi/\omega; \tag{1a}$$

and for $\nu/\omega \gg 1$

$$|\mathrm{Re}\, n - 1| \sim \alpha\xi(\omega/\nu)^2, \quad \mathrm{Im}\, n \sim \alpha\xi\omega/\nu. \tag{1b}$$

Here $\xi \sim 1$ in the ultrarelativistic case, and $\xi \sim kT/mc^2 \ll 1$ in the nonrelativistic case (kT is the mean kinetic energy and m is the mass of one particle).

Here always Re n < 1 and Im n > 0. In other words, in the isotropic case $v_p > c$[2)].

In addition to the problem of GW, the same method is used to consider the following problem. During the early stages of expansion of the Universe short-wave adiabatic perturbations (AP)[1,2], which represent sound waves (their velocity is $v_{\text{sound}} = c/\sqrt{3}$), interact with neutrinos through metric perturbations that are related to the excitation of such waves. For the case of sufficiently short wavelengths the collisionless kinetic equation holds for the neutrino gas.

Since $v_{\text{sound}} < c$, there will always be individual neutrinos traveling in resonance with the wave. For this the neutrinos must form with the wave an angle $\theta = \cos^{-1} 3^{-1/2}$. Thus, in the isotropic case there appears a damping which is analogous to Landau damping[10].

For strong anisotropy one may expect the damping to be replaced by growth of the wave. Let us derive a condition for this to become possible.

1. In a flat metric on which perturbations are superimposed, i.e. in a metric of the form

$$ds^2 = dt^2 - (\delta_{\alpha\beta} + h_{\alpha\beta})\,dx^\alpha dx^\beta,$$
$$c = 1;\ \alpha, \beta \equiv 1, 2, 3$$

the kinetic equation has the form[11,12] (the dot means differentiation with respect to time)

$$\dot{f}_1 + \xi n^\alpha \partial f_1 / \partial x^\alpha - {}^1/_2 \hat{L} f_0 = [f]_{st}, \tag{2}$$

where f_0 is the unperturbed distribution function and f_1 is the perturbation of the distribution function. Here $[f]_{st}$ is the collision integral, $\xi = (\epsilon^2 - m^2)^{1/2}/\epsilon$, ϵ is the energy of one particle and m is its mass.

In the ultrarelativistic case $\xi \sim 1$, $n^\alpha = p^\alpha/p$, p^α is the momentum of one particle, $p^2 = p^\alpha p_\alpha$. The operator L which describes the interaction of the gas with the

[1)]A growth of GW has been discussed in Bashkov's thesis[9]. For a certain dependence of the collision frequency on the particle energy in the gas he has obtained some growth for ultrarelativistic particles. The problem of collisionless damping of gravitational waves was for the first time posed by Shvarts.

[2)]At the same time, of course, the group velocity $v_{gr} = d^\omega/dk$ is smaller than c.

perturbations of the metric in (2) has the form

$$\hat{L} = F\xi^2\varepsilon\frac{\partial}{\partial\varepsilon} + E^\alpha\frac{\partial}{\partial n^\alpha}, \tag{3}$$

$$F = \dot{h}_{\alpha\beta}n^\alpha n^\beta, \quad E^\alpha = \dot{h}^\alpha_\beta n^\beta + \xi(h^\alpha_{\beta,\gamma} - h^{,\alpha}_{\beta\gamma})n^\beta n^\gamma.$$

The linearized Einstein equations can be written in the following form

$$-{}^1/_2\ddot{h} = \gamma\delta T_0^0, \tag{4}$$

$${}^1/_2(-\dot{h}_{,\alpha} + \dot{h}^\gamma_{\alpha,\gamma}) = \gamma\delta T_\alpha^0, \tag{5}$$

$${}^1/_2(2\ddot{h} - h_{,\gamma}^{,\gamma} + h^{\alpha,\gamma}_{\gamma,\alpha}) = \gamma\delta T_\gamma^\gamma, \tag{6}$$

$${}^1/_2(-\ddot{h}^\beta_\alpha + h^{\beta,\gamma}_{\alpha,\gamma} + h^{,\beta}_{,\alpha} - h^{\gamma,\beta}_{\alpha,\gamma} - h^{\beta,\gamma}_{\gamma,\alpha}) = \gamma\delta T^\beta_\alpha \quad (\alpha\neq\beta), \tag{7}$$
$$\gamma = 8\pi G/c^4, \quad h = h_\gamma^\gamma;$$

$$\delta T_i^k = \int\int\int p_i p^k f_1 dP, \quad dP = \xi^2\varepsilon d(\xi\varepsilon)d\Omega, \tag{8}$$
$$i,\ k = 0,\ 1,\ 2,\ 3;$$

dP is the volume element in momentum space and $d\Omega$ is the element of solid angle in this space.

We expand the equations (3)—(7) in a Fourier integral over space. For convenience we introduce spherical coordinates:

$$\cos\theta = \varkappa = n^\alpha e_\alpha,$$
$$n^\alpha R_\alpha = (1-\varkappa^2)^{1/2}\cos\varphi, \quad n^\alpha R_\alpha' = (1-\varkappa^2)^{1/2}\sin\varphi, \tag{9}$$

where the following notations are used: $k^\alpha = ke^\alpha$ is the wave vector, R^α and R'^α are unit vectors perpendicular to e^α and to one another.

One can further expand Eqs. (3)—(7) into a Fourier integral with respect to time. Although this procedure is not correct (as was shown by Landau[10]) it still leads to the correct dispersion law, if one chooses the integration path correctly (according to the Landau rule).

Then (2) yields (the tilde denotes the Fourier transform)

$$\tilde{f}_1 = \frac{1}{2i}(k\xi\varkappa - \omega)^{-1}(\widetilde{\hat{L}f_0}). \tag{10}$$

2. We now consider the interaction of GW with a collisionless gas. It is convenient to write $\tilde{h}^\alpha_\beta$ in the following form:

$$\tilde{h}^\alpha_\beta = h_1(R^\alpha R_\beta - R'^\alpha R_\beta') + h_2(R^\alpha R_\beta' + R'^\alpha R_\beta).$$

Contracting (7) first with $R^\alpha R_\beta - R'^\alpha R'_\beta$ and then with $R'^\alpha R_\beta + R'_\beta R^\alpha$[3], we obtain

$$(1-n^2)h_i = (\alpha/E)I_i^k h_k, \quad i,\ k = 1,\ 2, \tag{11}$$

where E is the energy density of the gas, and

$$I_i^k = -\int_0^\infty \xi^4\varepsilon^3 d(\xi\varepsilon)\int_0^{2\pi}d\varphi\int_{-1}^1 d\varkappa(1-\varkappa^2)^2 b_i \tag{12}$$
$$\times\left[\frac{\varepsilon\partial f_0/\partial\varepsilon + n\partial f_0/\partial\varkappa}{1-\xi n\varkappa}b^k + \frac{\partial f_0/\partial\varphi}{1-\varkappa^2}d^k\right],$$
$$b_i = \begin{pmatrix}\cos 2\varphi\\ \sin 2\varphi\end{pmatrix}, \quad d_i = \begin{pmatrix}\sin 2\varphi\\ \cos 2\varphi\end{pmatrix}.$$

[3]Here there is no contradiction with the fact that in (7) $\alpha\neq\beta$ since before carrying out the contraction one would have to add a term with δ^α_β, but the contraction with δ^α_β vanishes.

Finally, equating the determinant of the system (11) to zero, we obtain

$$1 - n_\pm^2 = \frac{\alpha}{2E}\{I_1^1 + I_2^2 \pm [(I_1^1 - I_2^2)^2 + 4I_1^2 I_2^1]^{1/2}\}. \tag{13}$$

In the simplest case of an ultrarelativistic gas ($\xi = 1$) with an isotropic distribution f_0, we obtain $I_1^2 = I_2^1 = 0$, $I_1^1 = I_2^2 = I$. To first order in α we obtain

$$n = 1 - {}^2/_3\alpha, \text{ i.e. } v_p = c/n > c. \tag{14}$$

For anisotropic f_0, birefringence occurs.

The absence of Landau damping to first order in α follows formally from the fact that the integral I has no singularities. We now consider a beam of ultrarelativistic particles. Then $f_0 = \Psi_0(\epsilon)\delta(\kappa - \kappa_0)\delta(\varphi - \varphi_0)$. Computing I_i^k we obtain

$$I_i^k = E(1+\varkappa_0)\left\{(5 - 3\varkappa_0 - 4\varkappa_0^2)\cdot\begin{pmatrix}\cos^2 2\varphi_0 & {}^1/_2\sin 4\varphi_0\\ {}^1/_2\sin 4\varphi_0 & \sin^2 2\varphi_0\end{pmatrix} + 2(1-\varkappa_0)\begin{pmatrix}\cos 4\varphi_0 & -\sin 4\varphi_0\\ \sin 4\varphi_0 & \cos 4\varphi_0\end{pmatrix}\right\},$$

and finally

$$n_\pm = 1 - {}^1/_4\alpha(1+\varkappa_0)\{(5 - 3\varkappa_0 - 4\varkappa_0^2) + 4(1-\varkappa_0)\cos 4\varphi_0 \pm [(5 - 3\varkappa_0 - 4\varkappa_0^2)^2 - 16(1-\varkappa_0)\sin^2 4\varphi_0]^{1/2}\}. \tag{15}$$

For example, for $\kappa_0 = 1$ (the beam goes along the wave) we have $n_+ = 1$, $n_- = 1 + 2\alpha$.

However, there is still no damping to first order in α, which is a consequence of the transversality of the GW. It can be seen that in the anisotropic case both $v_p > c$ and $v_p < c$ are possible.

3. For comparison we consider the problem of interaction of AP with a neutrino gas during the early stages of expansion of the Universe (the formulation of the problem was already described above)[4].

From (4) and (6) as well as (7), projecting the latter equation onto $e_\alpha e^\beta$, we obtain

$$(\ddot{\lambda} - {}^1/_3k^2\Delta) = \gamma I(n)\Delta, \quad \ddot{\mu} + {}^2/_3k^2\Delta = 0. \tag{16}$$

Here $\Delta = \lambda + \mu \sim \delta\rho$ ($\delta\rho$ is a perturbation of the radiation density in the sound wave)

$$I(n)\Delta = i\int_0^\infty\int_0^{2\pi}\int_{-1}^1 \varepsilon^3\varkappa^2 d\varepsilon\, d\varphi\, d\varkappa\frac{F\varepsilon\,\partial f_0/\partial\varepsilon + E^\alpha\partial f_0/\partial n^\alpha}{\omega(1-\sqrt{3}n\varkappa)}, \tag{17}$$
$$F = {}^1/_3 i\omega\Delta[1 + 3\varkappa^2(2n^2-1)]; \quad E^\alpha = {}^1/_3 i\omega\Delta\{(1-2n^2)3\varkappa e^\alpha - n^\alpha + \sqrt{3}n(\varkappa n^\alpha - e^\alpha)\}.$$

Here we have introduced by analogy the refractive index $n = v_{sound}/v_p = kc/3^{1/2}\omega$. We obtain, finally,

$$n^2 - 1 = \frac{\gamma}{\omega^2}I(1), \tag{18}$$

$$I(1) = \frac{1}{3\sqrt{3}}\int_0^\infty\varepsilon^3 d\varepsilon\int_0^{2\pi}d\varphi\, P\int_{-1}^1\frac{\varkappa^2 d\varkappa}{\varkappa - 1/\sqrt{3}}\times\left\{(1+3\varkappa^2)\varepsilon\frac{\partial f_0}{\partial\varepsilon} + [4\varkappa + \sqrt{3}(1-\varkappa^2)]\frac{\partial f_0}{\partial\varkappa}\right\} - \frac{2i\pi}{9\sqrt{3}}\int_0^\infty\varepsilon^3 d\varepsilon\int_0^{2\pi}d\varphi\left(\varepsilon\frac{\partial f_0}{\partial\varepsilon} + \sqrt{3}\frac{\partial f_0}{\partial\varkappa}\right)_{\varkappa=1/\sqrt{3}}, \tag{19}$$

where P designates a principal value integral.

[4]In spite of the fact that it is important, we first neglect the expansion, and take it into account later.

Let for example, $f_0 = \Psi(\epsilon)X(\kappa)$. Then the damping δ is

$$\delta = \frac{2\pi^2\gamma}{9\sqrt{3}\omega^2}\left(\int_0^\infty \Psi(\varepsilon)\varepsilon^3 d\varepsilon\right)\left(4X - \sqrt{3}\frac{\partial X}{\partial\varkappa}\right)_{\varkappa=1/\sqrt{3}}. \quad (20)$$

For an isotropic distribution ($X \equiv 1$) we have $\delta = 4\pi\alpha/9\sqrt{3} > 0$, corresponding to damping.

It can be seen that the condition for growth of the sound waves is the inequality

$$\left(\frac{1}{X}\frac{\partial X}{\partial\varkappa}\right)_{\varkappa=1/\sqrt{3}} > \frac{4}{\sqrt{3}}. \quad (21)$$

This condition can be explained in the following manner.

If the distribution is isotropic there will always be damping, owing to the fact that, as a consequence of the Doppler effect, the particles having their velocity projection on the direction of propagation of the wave equal to $c/3^{1/2} + \Delta v$ transfer to the wave less energy than is taken up by the particles which have their velocity projection on the wave equal to $c/3^{1/2} - \Delta v$. But if there are more particles with velocity projection $c/3^{1/2} + \Delta v$, than particles with velocity projection $c/3^{1/2} - \Delta v$, the total contribution to the energy of the wave by all resonant particles can be positive. However, for this we see it is not sufficient that the inequality $(\partial X/\partial\kappa)_{\kappa = 3^{-1/2}} > 0$ be verified, it is also necessary that $(\partial X/\partial\kappa)_{\kappa = 3^{-1/2}} > A > 0$, which is expressed by Eq. (21).

The example involving the AP was discussed here only for the purpose of comparison with the case of GW, since Zel'dovich has noted that a collisionless interaction is of no significance in cosmology, where the presence of expansion leads to a decrease of h_α^β with time, and the term in Eq. (3) which corresponds to the collisionless interaction also decreases rapidly.

4. Let us return to the GW. We consider a collisionless nonrelativistic gas and limit ourselves to an isotropic Maxwellian distribution function

$$f_0 = A\exp(-p^2/2mkT) = A\exp(-\beta p^2),$$

where A is determined by the normalization condition:

$$A\int \exp(-\beta p^2)p^2\varepsilon\, d\Omega = E = N\varepsilon;$$

$$A = N\left(\frac{\beta}{\pi}\right)^{3/2}, \quad \varepsilon \approx mc^2.$$

Here

$$n^2 - 1 = \frac{\pi\gamma}{2\omega^2\varepsilon}\int_0^\infty p^5\, dp\frac{df_0}{dp}I(p,n),$$

$$I(p,n) = \int_{-1}^{1}\frac{(1-\varkappa^2)^2}{1-p\varkappa n/\varepsilon}d\varkappa. \quad (22)$$

In the nonrelativistic case the main contribution to the integral comes from the region $p/\epsilon \ll 1$:

$$I(p,n) \approx \int_{-1}^{1}(1-\varkappa^2)^2 d\varkappa + \frac{p}{\varepsilon}n\int_{-1}^{1}\varkappa(1-\varkappa^2)^2 d\varkappa + O\left(\frac{p^2}{\varepsilon^2}\right),$$

however the second term canishes, since the integrand is odd. We obtain finally

$$n = 1 - 2\alpha\xi. \quad (23)$$

Thus, as expected, an additional small factor $\xi = kT/mc^2 \ll 1$ appears due to the nonrelativistic limit.

5. We now take collisions into account by setting $[f]_{st} = f_1/\tau(p) = -\nu(p)f_1$; here $\nu(p)$ is the frequency of collisions, which in general depends on p (τ is the time between two collisions).

In the isotropic nonrelativistic case we have

$$I(p) = \int_{-1}^{1}\frac{(1-\varkappa^2)^2 d\varkappa}{(1-p\varkappa/\varepsilon) + i\nu(p)/\omega}. \quad (24)$$

A computation yields

$$\operatorname{Re} I(p) = \left(1+\frac{\nu^2}{\omega^2}\right)^{-1}\frac{16}{15} + O(\xi^2),$$

$$\operatorname{Im} I(p) = \left(1+\frac{\nu^2}{\omega^2}\right)^{-1}\frac{\nu}{\omega}\frac{16}{15} + O(\xi^2).$$

This transforms (22) into the following expression

$$n - 1 = \frac{4\pi\gamma}{15\omega^2\varepsilon}\int_0^\infty p^5\, dp\frac{df_0}{dp}\left(1+\frac{\nu^2(p)}{\omega^2}\right)^{-1}\left(1 - i\frac{\nu(p)}{\omega}\right). \quad (25)$$

Let $\nu(p) = \nu_0(\beta^{1/2}p)^r$. Introducing the notation $\nu_0/\omega = \mu$, we have

$$\operatorname{Re} n - 1 = -\frac{16}{15\sqrt{\pi}}\frac{\omega_0^2}{\omega^2}\xi\Phi_r(\mu), \quad (26)$$

$$\operatorname{Im} n = \frac{8}{15\sqrt{\pi}}\frac{\omega_0^2}{\omega^2}\xi\mu\Psi_r(\mu).$$

Here

$$\Phi_r(\mu) = \int_0^\infty \frac{\eta^{5/2}e^{-\eta}d\eta}{1+\mu^2\eta^r}, \quad \Psi_r(\mu) = \int_0^\infty \frac{\eta^{5/2+r}e^{-\eta}d\eta}{1+\mu^2\eta^r}. \quad (27)$$

It is clear that the refractive index has the asymptotic behavior (1) independently of the value of r.

The same asymptotic behavior is also valid in the ultrarelativistic isotropic case, if one sets $\xi = 1$. Indeed, in this case

$$n - 1 = -\frac{2\gamma\pi}{\omega^2}\int_0^\infty d\varepsilon\varepsilon^3 f_0(\varepsilon)I(\varepsilon), \quad (28)$$

$$I(\varepsilon) = \int_{-1}^{1}\frac{(1-\varkappa^2)^2 d\varkappa}{1-\varkappa+i\nu(\varepsilon)/\omega}$$

The computation yields for ν independent of ϵ: (29)

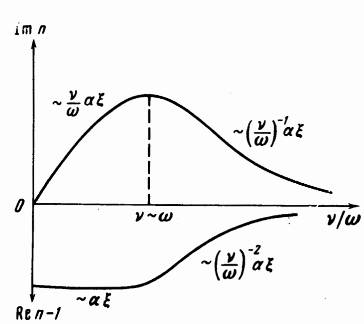
$$\operatorname{Re} I = \frac{4}{3} + \frac{\nu^2}{\omega^2}\left[6 + \frac{\nu^2-4\omega^2}{2\omega^2}\ln\frac{\nu^2+4\omega^2}{\nu^2} - \frac{4\nu}{\omega}\operatorname{arctg}\frac{2\omega}{\nu}\right],$$

$$\operatorname{Im} I = -\frac{\nu}{\omega}\left\{\frac{20}{3} + \frac{\nu^2}{\omega^2}\left[-2 + \ln\frac{\nu^2+4\omega^2}{\nu^2} + \left(\frac{\nu}{\omega} - \frac{4\omega}{\nu}\right)\operatorname{arctg}\frac{2\omega}{\nu}\right]\right\}.$$

A direct expansion of these clumsy expressions for $\nu/\omega \ll 1$ and $\nu/\omega \gg 1$ leads again to the asymptotic behavior (1).

The behavior of $n(\omega/\nu)$ is illustrated in the figure. We see that for large collision frequencies, i.e., for small mean free paths, the GW does not interact at all with matter, in agreement with[1,2].

As expected, the absorption exhibits a nonmonotone dependence on ν. In the nonrelativistic case the absorption attains a maximum for a value of ν satisfying the equation



The dependence of the real and imaginary parts of the refractive index on the frequency of collisions.

$$\int_0^\infty \eta^{5/2+r} e^{-\eta} \frac{1-\mu^2\eta^r}{(1+\mu^2\eta^r)^2} d\eta = 0. \tag{30}$$

The integral can only be computed for p-independent ν. Then, for r = 0 we obtain $\mu_{max} = 1$, i.e., $\nu = \omega$.

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