Construction of a General Cosmological Solution of the Einstein Equation with a Time Singularity

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It is shown how to describe the process of alternation of Kasner epochs in the oscillatory regime of approach to a singularity of the general (nonhomogeneous) solution to the Einstein equation. The universality of the previously established law of alternation of Kasner exponents is proved (for homogeneous models) and a general description of the rotations of the Kasner axes under the change of epochs is given. This achieves the existence proof of a general solution with a time singularity.

In our preceding papers (which are summarized in[1]) an oscillatory regime was discovered for the approach to a singularity of the cosmological solutions to the Einstein equation, and this regime was analyzed in detail for the example of the homogeneous model. The investigation of the homogeneous model was completed in the recent paper[2], where this model was discussed in its most general form, and a new effect was uncovered: the rotation of the Kasner axes in the successive alternation of the Kasner epochs.

The assertion was also made that it is just the oscillatory regime of approach to the singularity which is characteristic for the general (nonhomogeneous) solution of the Einstein equation, and the conjecture was made in[2] that such a general solution (both in empty space and in matter-filled space) exhibits all the traits which already appear in the homogeneous models.

In the asymptotic region of arbitrarily small times the evolution of the homogeneous models (of the types IX and VIII, according to Bianchi) consists of "Kasner epochs" which alternate according to a definite, regular rule. Accordingly the construction of the general solution in this region must also comprise: 1) the construction of the general solution for the individual Kasner epoch, and 2) a general description of the alternation of two successive epochs. The answer to the first question is given by the "generalized Kasner solution" which was earlier determined in[3]. The present communication is dedicated to answering the second question; we shall see that the alternation of epochs in the general solution does indeed occur in close analogy to the alternation in the homogeneous model. This completes the existence proof of the general cosmological solution of the Einstein equation with a time singularity.

We also remind the reader that the "regular" evolution of the homogeneous model may be violated by the occurrence of a series of small oscillations (cf.[1], Sec. 4). Although the likelihood of occurrence of such oscillations tends to zero as $t \to 0$,[4] a complete construction of the general solution must also include this case; this part of the problem was already solved earlier[5].

1. THE LAW OF ALTERNATION OF KASNER EXPONENTS

In the generalized Kasner solution the spatial metric (near the singularity) has the form

$$g_{\alpha\beta} = a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta, \tag{1.1}$$

where

$$a \sim t^{p_l}, \quad b \approx t^{p_m}, \quad c \sim t^{p_n},$$

and p_l, p_m, and p_n are functions of the coordinates related by the Kasner conditions[1)]

$$p_l + p_m + p_n = p_l^2 + p_m^2 + p_n^2 = 1.$$

The frame vectors **l**, **m**, and **n** are also functions of the coordinates; these vectors will be assumed here to be normalized so that $|\mathbf{l}| \sim |\mathbf{m}| \sim |\mathbf{n}| \sim 1$; in other words, the factors which determine the order of magnitude of the components $g_{\alpha\beta}$ are assumed included in the functions a, b, and c.

The (time) interval of applicability of the solution (1.1) is determined by conditions which follow from the Einstein equations. Near the singularity one may neglect the matter energy-momentum tensor in the 00- and $\alpha\beta$-components of these equations:

$$-R_0^0 = \frac{1}{2}\dot{\varkappa}_\alpha^\alpha + \frac{1}{4}\varkappa_\alpha^\beta \varkappa_\beta^\alpha = 0, \tag{1.2}$$

$$-R_\alpha^\beta = \frac{1}{2\sqrt{-g}}\frac{\partial}{\partial t}(\sqrt{-g}\,\varkappa_\alpha^\beta) + P_\alpha^\beta = 0. \tag{1.3}$$

The solution (1.1) was obtained neglecting the space components of the Ricci tensor P_α^β in Eq. (1.3). The condition for such a neglect to be valid are easily formulated in terms of the projections of these tensors along the directions **l**, **m**, and **n** (which were introduced in Sec. 3 of[3]). The diagonal projections of the Ricci tensor must satisfy the conditions

$$P_l^l, \quad P_m^m, \quad P_n^n \ll t^{-2} \tag{1.4}$$

(the second term in the diagonal projections of the equations (1.3) must be small compared to the first term). The off-diagonal projections of (1.3) determine the off-diagonal projections of the metric tensor (g_{lm}, g_{ln}, g_{mn}) which should only be small corrections to the leading terms of the metric, as given by (1.1). In the latter the only nonvanishing projections are the diagonal ones (g_{ll}, g_{mm}, g_{nn}) and the fact that the off-diagonal projections are small means that one should have:

$$g_{lm} \ll \sqrt{g_{ll}g_{mm}}, \quad g_{ln} \ll \sqrt{g_{ll}g_{nn}}, \quad g_{mn} \ll \sqrt{g_{mm}g_{nn}}. \tag{1.5}$$

1)We follow in this paper the notation introduced in[1].

This leads to the following conditions on the Ricci tensor

$$P_{lm} \ll ab / t^2, \quad P_{ln} \ll ac / t^2, \quad P_{mn} \ll bc / t^2. \tag{1.6}$$

If these conditions are satisfied one may disregard completely (in the leading order!) the off-diagonal components of the equation (1.3).

For the metric of the form (1.1) the Ricci tensor P_α^β is given by the formulas listed in Appendix C of[3]. The diagonal projections P_l^l, P_m^m, P_n^n contain the terms

$$\frac{1}{2}\left(\frac{\mathbf{a}\mathbf{l} \operatorname{rot} \mathbf{a}\mathbf{l}}{abc(\mathbf{l}[\mathbf{mn}])}\right)^2 \sim \frac{k^2a^2}{b^2c^2} = \frac{k^2a^4}{\Lambda^2t^2} \tag{1.7*}$$

and analogous terms with al replaced by bm and cn (1/k denotes the order of magnitude of spatial distances over which the metric changes substantially; Λ is the coefficient in the relation abc = Λt for the Kasner epoch under consideration). The requirement that these terms be small leads, according to (1.4), to the inequalities

$$a\sqrt{k/\Lambda} \ll 1, \quad b\sqrt{k/\Lambda} \ll 1, \quad c\sqrt{k/\Lambda} \ll 1. \tag{1.8}$$

It is remarkable that these inequalities are not only necessary, but also sufficient conditions for the existence of the solution (1.1). In other words, after the conditions (1.8) are satisfied all other terms in P_l^l, P_m^m, and P_n^n, as well as all terms in P_{lm}, P_{ln}, and P_{mn}, in general, satisfy the conditions (1.4) and (1.6) automatically. An estimate of these terms leads to the conditions[2)]

$$\frac{k^2}{\Lambda^2}(a^2b^2, \ldots, a^3b, \ldots, a^2bc, \ldots) \ll 1 \tag{1.9}$$

(the dots in the parentheses replace expressions which are obtained from the ones written out by a permutation of a, b, and c). All these inequalities contain on the left the products of powers of two or three of the quantities which enter into the inequalities (1.8), and therefore are a fortiori true if the latter are satisfied.

As t decreases there occurs eventually an instant (call it t_k) when one of the conditions (1.8) is violated[3)]. Thus, if during a given Kasner epoch the negative exponent refers to the function a(t) (i.e., to $p_l = p_1$), then at the instant t_k we will have

$$a(t_\text{к})\sqrt{k/\Lambda} \sim 1. \tag{1.10}$$

Since during that epoch the functions b(t) and c(t) decrease with the decrease of t, the other two inequalities (1.8) remain valid and at $t \sim t_k$ we shall have

$$b(t_\text{к}) \ll a(t_\text{к}), \quad c(t_\text{к}) \ll a(t_\text{к}). \tag{1.11}$$

It is remarkable that at the same time all the conditions (1.9) continue to hold. This means that all off-diagonal projections of the equations (1.3) may be disregarded, as before. In the diagonal projections only terms of one type, (1.7) become important, terms which contain a^4/t^2 (together with the factor $(\mathbf{l}\cdot \operatorname{curl} \mathbf{l})^2$).

As a result, the following equations are obtained for the process of alternation of two Kasner epochs

$$\begin{aligned} -R_l^l &= (\dot{a}bc)^\cdot / abc + \lambda^2a^2 / b^2c^2 = 0, \\ -R_m^m &= (a\dot{b}c)^\cdot / abc - \lambda^2a^2 / b^2c^2 = 0, \\ -R_n^n &= (ab\dot{c})^\cdot / abc - \lambda^2a^2/b^2c^2 = 0, \\ -R_0^0 &= \ddot{a} / a + \ddot{b} / b + \ddot{c} / c = 0, \end{aligned} \tag{1.12}$$

equations which differ from the corresponding equations of the homogeneous model only through the fact that the quantity

$$\lambda = (\mathbf{l} \operatorname{rot} \mathbf{l}) / (\mathbf{l}[\mathbf{mn}]) \tag{1.13}$$

is no longer a constant, but a function of the space coordinates. Since, however, (1.12) is a system of ordinary differential equations with respect to time, this difference does not affect at all the solutions of the equations and the law of alternation of Kasner exponents (the rule (3.16) in[1]) which follows from these solutions. Thus the law of alternation of exponents derived for the homogeneous model remains valid in the general case also[4)].

Below we shall require the explicit form of the solution of the equations (1.12). We write it out here in somewhat more detailed form than was done in Sec. 3 of[1]:

$$\begin{aligned} a^2 &= {}^1/{}_2 a_0^2 t_\text{к}^{2p_1} / \operatorname{ch}\varphi, \\ b^2 &= 2b_0^2 t_\text{к}^{2p_2} \exp\left(-\frac{p_2+p_1}{p_1}\varphi\right) \operatorname{ch}\varphi, \\ c^2 &= 2c_0^2 t_\text{к}^{2p_3} \exp\left(-\frac{p_3+p_1}{p_1}\varphi\right) \operatorname{ch}\varphi, \end{aligned} \tag{1.14}$$

where the function $\varphi(t)$ satisfies the equation[5)]

$$\frac{\partial\varphi}{\partial t} = \frac{\sqrt{2}|p_1|}{t_\text{к}} \exp\left(\frac{1+p_1}{2p_1}\varphi\right) \frac{1}{\sqrt{\operatorname{ch}\varphi}}. \tag{1.15}$$

As is usual, the notation for the Kasner exponents p_1, p_2, and p_3 (which here are coordinate functions!) assumes that $p_1 < p_2 < p_3$; $p_1 < 0$; a_0, b_0, c_0, and $\Lambda = a_0b_0c_0$ are constants (independent of the coordinates); the quantity

$$t_\text{к} = (4|p_1|\Lambda / |\lambda| a_0^2)^{1/2p_1} \tag{1.16}$$

corresponds to the instant of alternation of epochs. The "initial" epoch corresponds to times $t \gg t_k$ ($\varphi \to \infty$); the corresponding asymptotic expressions for the functions (1.14) are:

$$a = a_0 t^{p_1}, \quad b = b_0 t^{p_2}, \quad c = c_0 t^{p_3}. \tag{1.17}$$

where

$$a = a_0 A t^{p_1'}, \quad b = b_0 B t^{p_2'}, \quad c = c_0 C t^{p_3'}, \tag{1.18}$$

$$p_1' = \frac{p_2 + 2p_1}{1 + 2p_1} < 0, \quad p_2' = \frac{-p_1}{1+2p_1}, \quad p_3' = \frac{p_3 + 2p_1}{1 + 2p_1}, \tag{1.19}$$

$$\begin{aligned} A &= (1+2p_1)^{p_1'} t_\text{к}^{\frac{2p_1(1+p_1)}{1+2p_1}}, \quad B = (1+2p_1)^{p_2'} t_\text{к}^{\frac{-2p_1(1-p_2)}{1+2p_1}}, \\ C &= (1+2p_1)^{p_3'} t_\text{к}^{\frac{-2p_1(1-p_3)}{1+2p_1}}, \quad \Lambda' = ABC\Lambda = (1+2p_1)\Lambda. \end{aligned} \tag{1.20}$$

*[mn] ≡ m × n.

2)The differentiation of exponential functions with coordinate-dependent exponents also leads to the appearance of terms involving factors like ln t; this circumstance does not change anything in the following line of reasoning, and for simplicity we omit such factors.

3)The case when two of the conditions (1.8) are violated (this can happen when the exponents p_1 and p_2 are close to zero) corresponds to the above-mentioned small oscillations, in which we are not interested here.

4)This assertion, as well as the equations (1.12) are already contained in[6].

5)This function is related to the variable τ used in[1] (which satisfied the relation dt = abcdτ) via the relation $A = 2|p_1|\Lambda\tau$.

2. ROTATION OF THE KASNER AXES

We now show that in addition to the variation of the functions a, b, c in the process of alternation of epochs there also occurs a rotation of the Kasner axes. We recall that we have designated (in[2]) as Kasner axes those directions along which the spatial scales of length vary according to the power laws t^{p_1}, t^{p_2}, and t^{p_3}. In the "initial" epoch, when the functions a, b, and c have the expressions (1.17), the spatial metric

$$g_{\alpha\beta} = a_0^2 t^{2p_1} l_\alpha l_\beta + b_0^2 t^{2p_2} m_\alpha m_\beta + c_0^2 t^{2p_3} n_\alpha n_\beta \qquad (2.1)$$

and the Kasner axes are determined by the vectors **l**, **m**, and **n**. In the "final" epoch, when the functions a, b, and c are given by the expressions (1.18), let the Kasner axes be directed along some new vectors **l**′, **m**′, and **n**′, so that the metric becomes

$$g_{\alpha\beta} = a_0^2 A^2 t^{2p_2'} l_\alpha' l_\beta' + b_0^2 B^2 t^{2p_1'} m_\alpha' m_\beta' + c_0^2 C^2 t^{2p_3'} n_\alpha' n_\beta'. \qquad (2.2)$$

The vectors **l**′, **m**′, and **n**′ are linear combinations of the original vectors **l**, **m**, and **n**. If one agrees to project all tensors (including $g_{\alpha\beta}$) as before onto the vectors **l**, **m**, and **n** it is clear that the rotation of the Kasner axes can be described as the appearance of off-diagonal projections g_{lm}, g_{ln}, and g_{mn}, which behave in time like linear combinations of the functions a^2, b^2, and c^2. We show how such projections do indeed appear.

For this we turn to the off-diagonal projections of the equations (1.3), which were not considered until now. As long as the required g_{lm}, g_{ln}, and g_{mn} are small (in the sense (1.15)) corrections to the first approximation metric (this happens, in any case in the initial epoch and for $t \sim t_k$), the equations can be linearized with respect to these quantities. A simple solution yields

$$\ddot{g}_{lm} + \dot{g}_{lm}\left(\frac{\dot{c}}{c} - \frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right) + g_{lm}\frac{4\dot{a}\dot{b}}{ab} = -2P_{lm} \qquad (2.3)$$

and two more equations, obtained by cyclic permutation of a, b, and c and the subscripts **l**, **m**, and **n**. The right-hand sides of these equations are determined by Eqs. (C.2) in[3], retaining only the largest terms in them; thus, for P_{lm} these will be the terms

$$P_{lm} \approx \frac{ab}{\Delta^2}(a\mathbf{l}\operatorname{rot} a\mathbf{l})(b\mathbf{m}\operatorname{rot} a\mathbf{l}) + \frac{ab}{2c}\left(\frac{a\mathbf{l}\operatorname{rot} a\mathbf{l}}{\Delta}\right)_{,n},$$

where $\Delta = abc(\mathbf{l}\cdot(\mathbf{m}\times\mathbf{n}))$. Estimating the orders of magnitude of these terms we find that

$$P_{lm} \sim k^2a^2/c^2, \quad P_{ln} \sim k^2a^2/b^2, \quad P_{mn} \sim k^2. \qquad (2.4)$$

It is essential that in this approximation all components of the matter tensor $T_{\alpha\beta}$ turn out to be small (this is shown by the estimates), compared to the components $P_{\alpha\beta}$ and can be omitted from the Einstein equations.

We shall solve (2.3) for the initial and final epochs and then "match" the solutions at $t \sim t_k$. During the initial epoch, with a, b, and c taken from (1.17) we have

$$g_{lm} = C_{lm}a^2 + D_{lm}b^2 + \frac{b^2}{p_1 - p_2}\int P_{lm}\frac{tdt}{b^2} - \frac{a^2}{p_1 - p_2}\int P_{lm}\frac{tdt}{a^2}. \qquad (2.5)$$

By assumption, in this epoch the Kasner axes coincide with the directions **l**, **m**, **n**; this means that g_{lm} cannot contain terms proportional to a^2 or to b^2, i.e., we must have $C_{lm} = D_{lm} = 0$. With P_{lm} taken from (2.4) we then find the order of magnitude

$$g_{lm} \sim a^4b^2k^2/\Lambda^2 \sim a^4b^2/a_\kappa^4, \qquad (2.6)$$

where $a_k = a(t_k)$ is taken from (1.10) (we note that owing to the conditions (1.8) we have $g_{lm} \ll ab$, as it should be).

For the final epoch the solution has the same form (2.5) with p_2', p_1', and p_3' in place of p_1, p_2, and p_3 and with other constants C_{lm}' and D_{lm}'. The latter should be determined from the condition of matching to the solution (2.6) for $t \sim t_k$; this implies $C_{lm}' \sim (b_k/a_k)^2 \ll 1$ and $D_{lm}' \sim 1$. Thus, for the final epoch we have

$$g_{lm} \sim a^2b_\kappa^2/a_\kappa^2 + b^2 + a^4b^2/a_\kappa^4$$

(the sum here should, of course, be understood only as a way of enumerating the types of terms that occur in g_{lm}). Since, however, for $t \sim t_k$ the function a decreases with the decrease of t, the last term will become small compared to b^2 (after leaving the intermediate period $t \sim t_k$, between the two epochs), so that we remain only with

$$g_{lm} \sim a^2b_\kappa^2/a_\kappa^2 + b^2 \qquad (2.7)$$

We thus obtain from the two equations

$$g_{ln} \sim a^2c_\kappa^2/a_\kappa^2 + c^2, \qquad g_{mn} \sim b^2c_\kappa^2/a_\kappa^2 + c^2b_\kappa^2/a_\kappa^2. \qquad (2.8)$$

The expression (2.7) signifies a rotation (in the **l**, **m** "plane") of the second Kasner axis by a large "angle" (~ 1), and a rotation of the second Kasner axis by a small "angle" ($\sim b_k^2/a_k^2 \ll 1$). The small rotations are, however, outside the approximation considered. Taking only the big rotations into account, we find from all the expressions (2.7—8), that the new Kasner axes are related to the old ones by relations of the form

$$\mathbf{l}' = \mathbf{l}, \quad \mathbf{m}' = \mathbf{m} + \sigma_m\mathbf{l}, \quad \mathbf{n}' = \mathbf{n} + \sigma_n\mathbf{l}, \qquad (2.9)$$

where the coefficients are $\sigma_m, \sigma_n \sim 1$.

We stress the fact that the use of the linearized equations (2.3), which led to the expressions (2.7—8), is legitimate only as long as all g_{lm}, g_{ln}, g_{mn} are small (in the sense of (1.5)). This condition is obviously violated when during the evolution of the metric towards a new epoch the function b stops being small compared to a. But by this time all the components $P_{\alpha\beta}$ will be damped out to such an extent that they will disappear from the Einstein equations, and after that these equations will be satisfied by the generalized Kasner solution with arbitrary orientation of the axes[6)].

For a quantitative determination of the coefficients (2.9) one can get away without an exact solution of (2.3), making use of the existence of the following exact first integral of the equations (1.2—3) (without matter!):

$$\tfrac{1}{2}(\varkappa_{\alpha;\beta}^{\beta} - \varkappa_{\beta;\alpha}^{\beta}) = C_\alpha/\sqrt{-g}, \qquad (2.10)$$

where C_α are arbitrary functions of the spatial coordinates. This relation is a generalization of the Bianchi identity for the three-dimensional Ricci tensor:

$$P_{\alpha;\beta}^{\beta} = \tfrac{1}{2}P_{;\alpha}$$

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6)In this reasoning it is understood that the oscillation amplitudes (i.e., the ratios a_k/b_k, a_k/c_k) are sufficiently large) so that there is room for all required conditions to be true. We recall that in the asymptotic region of arbitrarily close approach to the singularity, the amplitudes increase without bound.

and is obtained by substituting into it the expressions of P_α^β and P in terms of κ_α^β according to (1.3), followed by some transformations which take into account (1.2); we do not dwell on the details of this transformation.

The expression in the left-hand side of (2.10) is nothing other than R_α^0. If the matter energy-momentum tensor would be exactly zero, the Einstein Equations $R_\alpha^0 = T_\alpha^0$ would imply that one has to set the functions C_α equal to zero; the integral (2.10) would then express nothing beyond the known fact that the various four-dimensional components of the Einstein equations are related to each other. The nontrivial circumstance expressed by the relation (2.10) is in this case related to the fact that the absence of T_0^0 and T_α^β in the right-hand sides of the equations (1.2) and (1.3) is only approximate, and admissible only in the approximation under discussion. Owing to the equations $R_\alpha^0 = T_\alpha^0$ the relation (2.10) shows that in this approximation, throughout the whole evolution of the metric, including the transition period between the epochs, the quantities $\sqrt{-g}\,T_\alpha^0$ remain constant in time:

$$\sqrt{-g}\,T_\alpha{}^0 = \text{const.} \qquad (2.11)$$

We recall that it is just from relations of this type in Sec. 3 of[3] that the distribution and motion of matter were derived in the generalized Kasner solution. We now see that the quantities $\sqrt{-g}\,T_\alpha^0$ remain the same also during the different Kasner epochs in the oscillatory regime of approach to the singularity.

Computing the expression $\sqrt{-g}\,R_\alpha^0$ for two successive epochs, with the metrics (2.1) and (2.2) (the computation is described in Sec. 3 of[3]) and comparing the results we obtain

$$\sum \mathbf{l}\{[\mathbf{mn}]\nabla p_l + (p_n - p_l)\mathbf{m}\operatorname{rot}\mathbf{n} + (p_l - p_m)\mathbf{n}\operatorname{rot}\mathbf{m}\}$$
$$= \sum A\mathbf{l}'\{[B\mathbf{m}'C\mathbf{n}']\nabla p_l' + (p_n' - p_l')B\mathbf{m}'\operatorname{rot}C\mathbf{n}' + (p_l' - p_m')C\mathbf{n}'\operatorname{rot}B\mathbf{m}'\}, \qquad (2.12)$$

where the summation is over simultaneous cyclic permutations of the vectors **l**, **m**, and **n**, the coefficients A, B, and C and the subscripts l, m, and n; according to (2.1) and (2.2) the Kasner exponents are: $(p_l, p_m, p_n) = (p_1, p_2, p_3)$ and $(p_l', p_m', p_n') = (p_2', p_1', p_3')$.

Equation (2.12) establishes the required relation between $\mathbf{l}'$, $\mathbf{m}'$, $\mathbf{n}'$ and **l**, **m**, **n**. Substituting into this relation the expressions (2.9) for $\mathbf{l}'$, $\mathbf{m}'$, $\mathbf{n}'$ and projecting the equation on the directions **m** and **n** (i.e., taking the scalar products with the appropriate reciprocal vectors $\tilde{\mathbf{m}}$ and $\tilde{\mathbf{n}}$ (cf.[3] Sec. 3), we obtain the following definitive expressions:

$$\sigma_m = -\frac{2}{p_2 + 3p_1}\left([\mathbf{lm}]\nabla\frac{p_1}{\lambda} + \frac{2p_1}{\lambda}\mathbf{m}\operatorname{rot}\mathbf{l}\right)\frac{1}{(\mathbf{l}[\mathbf{mn}])},$$
$$\sigma_n = \frac{2}{p_3 + 3p_1}\left([\mathbf{nl}]\nabla\frac{p_1}{\lambda} - \frac{2p_1}{\lambda}\mathbf{n}\operatorname{rot}\mathbf{l}\right)\frac{1}{(\mathbf{l}[\mathbf{mn}])} \qquad (2.13)$$

(the third projection of (2.12) then becomes an identity, as can be shown easily).

Finally, we show how the general results obtained here fit into the properties of homogeneous models discussed earlier[2] as a special case.

In the homogeneous models the selection of the frame vectors is determined by choosing definite values for the structure constants of the group of motions of the space; these vectors, which in[2] were denoted by $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$, were therefore not related to the directions of the Kasner axes. The latter were defined by the vectors

$$\mathbf{l} = \mathbf{e}^1,$$
$$\mathbf{m} = \mathbf{e}^2 + \frac{M_1}{M_2}\mathbf{e}^1 = \mathbf{e}^2 + \operatorname{ctg}\theta_m\mathbf{e}^1, \qquad (2.14)$$
$$\mathbf{n} = \mathbf{e}^3 + \frac{N_2}{N_3}\mathbf{e}^2 + \frac{N_1}{N_3}\mathbf{e}^1 = \mathbf{e}^3 + \operatorname{ctg}\varphi_n\mathbf{e}^2 + \frac{\operatorname{ctg}\theta_n}{\sin\varphi_n}\mathbf{e}^1,$$

where M_1, N_1, ... are defined according to (2.17) in[2].[7] Making use of Eqs. (3.6) in[2] we find a relation between the Kasner axes, for the alternation of two Kasner epochs, in the form (2.9), with the coefficients

$$\sigma_m = -\frac{4p_1}{p_2 + 3p_1}\operatorname{ctg}\theta_m, \qquad \sigma_n = -\frac{4p_1}{p_3 + 3p_1}\frac{\operatorname{ctg}\theta_n}{\sin\varphi_n}. \qquad (2.15)$$

The same result can be obtained from the general equations (2.13) by setting there p_1 = const, $\lambda = 1$ and noting that owing to the properties of the frame vectors $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ in type IX homogeneous spaces in the Bianchi classification, we have

$$\frac{\mathbf{m}\operatorname{rot}\mathbf{l}}{\mathbf{l}[\mathbf{mn}]} = \operatorname{ctg}\theta_m\frac{\mathbf{e}^1\operatorname{rot}\mathbf{e}^1}{\mathbf{e}^1[\mathbf{e}^2\mathbf{e}^3]} = \operatorname{ctg}\theta_m, \qquad \frac{\mathbf{n}\operatorname{rot}\mathbf{l}}{\mathbf{l}[\mathbf{mn}]} = \frac{\operatorname{ctg}\theta_n}{\sin\varphi_n}.$$

This proves the correspondence between the general and the homogeneous models.

We note, in this connection, that we have indicated the inequalities $a_k^2 \gg b_k^2 \gg c_k^2$ as a condition of applicability of the results in[2]. The more detailed analysis carried out here shows that the weaker conditions $a_k^2 \gg b_k^2$, $a_k^2 \gg c_k^2$ already suffice.[8]

[7]The vectors (2.14) differ from the vectors $\mathbf{l}_K$, $\mathbf{m}_K$, and $\mathbf{n}_K$ introduced in[2] by the factors 1, $1/M_2$ and $1/N_3$, respectively. The necessity of introducing this change is related to the somewhat changed definitions of the functions a, b, and c in[2] compared to the present paper; the functions a, b, and c here correspond to the functions a, M_2b, and N_3c in[2] (cf. the reasoning in[2], Sec. 3). Corresponding to this the vectors **l**, **m**, and **n** are defined in such a manner that the metric (2.13) in[2] would take the form

[8]We use this occasion to correct a misprint in[2]: The expression (A.7) for P_1^2 should read

[1]V. A. Belinskiĭ, E. M. Lifshitz, and I. M. Khalatnikov, Usp. Fiz. Nauk **102,** 463 (1970) [Sov. Phys.-Usp. **13,** 745 (1971)]; Adv. Phys. **19,** 525 (1970).

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