Vacuum Polarization in Strong Inhomogeneous Fields

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It is shown that the motion of particles in strong fields is characterized by a correlation radius R_c, which is related to the electric field intensity E(r) by the equation $R_c^2(r) = 1/eE(r)$. In very strong fields R_c becomes much smaller than the distance over which the field varies, and this permits one to regard the field as homogenous even at small distances from a Coulomb center. This property of localizability enables us to determine the dielectric constant of the vacuum and the distribution of the vacuum polarization charge near an external charge of arbitrarily small radius R_0. The relation between the true charge Z_0 and the observable charge Z is derived. It is shown that, for a given value of Z, the charge Z_0 remains finite as $R_0 \to 0$. A possible mechanism for the removal of electrodynamic divergences is discussed.

1. THE PHYSICAL PICTURE

In the presence of sufficiently strong static (or slowly varying) fields single-particle bound states may appear with their energy levels so deep-lying that pair production becomes possible, and thus arises the question of the stability of the vacuum. We shall call such fields critical. A very well known example of this type is the nucleus with charge $Z = Z_c$ such that the energy of the K-electron reaches the value -1 ($m = \hbar = c = 1$). In the case of a point nucleus the critical value of the nuclear charge is given by $Z_c = 137$,[1] and for a nucleus having the usual radius $R = r_0 A^{1/3}$ (with $r_0 = 1.2 \times 10^{-13}$ cm) it is given by $Z_c = 170$.[2-7] When $Z = Z_c$ the energy of the nucleus with a vacant K-shell coincides with the energy of the state having two electrons in the K-shell and two positrons in the continuum with energy +1, that is, it is energetically possible to produce two pairs from the vacuum. If the K-shell is filled, then pair production is impossible due to the Pauli exclusion principle, and the vacuum is stable. Thus, in the case of fermions the stability of the vacuum is guaranteed by the Pauli exclusion principle even without taking the interaction between the particles into account.

The Dirac equation does not have a bound state in the K-shell for $Z > Z_c$; however, simultaneously with the disappearance of the K-state there is a change in the distribution of the vacuum polarization charge, that is, an additional negative charge density appears near the nucleus in the region $r \sim 1$. The volume integral of this additional charge density is equal to two electron charges.[5] Let us illustrate this phenomenon by another example, one which will play an important role in the following investigation.

Below an equation will be derived (formula (3.5)) between the true charge Z_0 of a nucleus having a small radius R_0 (the radius R_0 can be arbitrarily smaller than the radius of real nuclei) and the observable charge Z. The difference between Z_0 and Z is due to vacuum polarization. It follows from this formula that a negative charge, equal to $Z_0 - Z$, is distributed in the region $R_0 < r < 1$. For sufficiently small values of R_0, $Z \ll Z_0$, that is, the polarization charge almost completely screens the true charge of the nucleus. This polarization charge can be interpreted in the following way. Pairs appear in connection with the introduction of a nucleus into the vacuum; the positrons escape to infinity, but the electrons are distributed near the nucleus and screen its charge, thus reducing the observable charge (for $r \gtrsim 1$) to the value Z. This problem is a manifestation of the screening of the bare charge, which is a well-known effect in quantum electrodynamics.

Thus, the appearance of additional charges in a Coulomb field for $Z > Z_c$ is of exactly the same nature as the screening of the bare charge, the only difference being that for small Z the screening occurs over distances of the order of R_0 whereas for $Z > Z_c$ the additional screening charges are distributed in the region $r \sim 1$.

Let us return to the question of the stability of the vacuum. As has already been mentioned, in the case of fermions the stability of the vacuum is guaranteed by the Pauli exclusion principle, and taking the interaction between the electrons into account leads to unimportant changes in the value of the critical charge ($\delta Z_c \sim 1$) and to a small change in the distribution of the polarization charge for $Z > Z_c$ (a change of the order of 1/137). Since arguments to the contrary have appeared in the literature,[6,7] this point should be discussed.

The interaction between the electrons can be taken into consideration with a high degree of accuracy by utilizing the fact that the only states which play an important role are the states of an electron in the K-shell plus a positron of small energy in the continuum. In addition, one should take into consideration that the wave function of a positron with small momentum k can be represented in the form[8]

$$\Psi_k(\mathbf{r}) = \Delta^{1/2}(k)\,\Psi_0(r),$$

where $\Psi_0(r)$ denotes the wave function of the bound state with energy $\epsilon = -1$, and $\Delta(k)$ is expressed in terms of the amplitude for the scattering of a positron in a Coulomb field. Then it is possible to determine the change of the effective potential acting on the electron and to demonstrate that the associated corrections are unimportant. We shall limit ourselves to these sketchy remarks. A more detailed account will be presented in a separate article.

Much more important changes in the vacuum polarization arise in those fields which are critical for the production of bosons, for example, in fields in which the production of pion pairs is possible. In the case of bosons the Pauli exclusion principle does not prevent

the particles from accumulating in "dangerous" states, and the only factor leading to stability of the vacuum is the interaction between the mesons. The question of the stability of the vacuum with respect to boson production in the presence of a field has been treated in detail in[9]. The basic results of this work reduce to the following:

1. It is shown that in an electric field, having the shape of a potential well for negatively charged particles ($V(r) < 0$ for $r < R$), a bound state also appears for positively charged particles, for which $V(r) > 0$, when the depth of the well reaches a certain value which is smaller than the critical value. Upon a further increase in the depth of the well, both levels deepen, and when $V = V_c$ the sum of the energies of the positive and negative particles vanishes, that is, the vacuum becomes unstable within the framework of the single-particle problem.

2. The quantum field theory problem including the external field $V(r)$ (for V close to V_c) and with an interaction of the form $\lambda\hat{\varphi}^4$ between the mesons (here $\hat{\varphi}$ is the meson field operator and $\lambda > 0$) is solved. The problem can be solved because the production of meson pairs in the "dangerous state" plays the major role, and for this state the sum of the energies of the two kinds of mesons (positive and negative) is close to zero.

3. It is found that the energy of a pair does not vanish at any value of the external field, that is, taking the interaction between the mesons into account makes the vacuum stable. Polarization of the meson vacuum creates a screening field such that the effective field acting on a particle never reaches its critical value. Thus, an upper limit exists for the effective field. It is clear from the nature of the derivation that such limiting fields also emerge in the case of nonelectrical external fields.

4. The behavior of the mesons in the field created by the nucleons is investigated, and it is shown that a phase transition involving the formation of a meson condensate occurs at a definite value of the nucleon density. In addition to its usual equilibrium state having the usual dendensity, nuclear matter should have a metastable state with a density which is two to four times larger. Perhaps such superdense nuclei exist, together with ordinary nuclei, in the heavy component of cosmic rays.

The present article will investigate the polarization of the electron-positron vacuum in strong and inhomogeneous electric fields. It is found that a localization of the particle's Green's function arises in strong fields, that is, in the presence of the field the Green's function $G(\mathbf{r}, \mathbf{r}', \omega)$ oscillates rapidly for $R^2 = |\mathbf{r} - \mathbf{r}'|^2 \gg 1/eE$, where E denotes the electric field intensity. Thanks to this localizability property the polarization operator $\Pi(\mathbf{r}, \mathbf{r}', \omega) \to 0$ for $R^2 \gg 1/eE$. This property permits one to determine the dielectric constant of the vacuum in strong and inhomogeneous fields.

The problem of the distribution of the polarization charge around a nucleus of radius R_0 is solved, and the relation between the true charge Z_0 of the nucleus and the observable charge Z is determined for arbitrarily small values of R_0. In this connection, it is found that the true charge Z_0 associated with a fixed value of Z remains finite in the limit $R_0 \to 0$. In other words, taking the localizability of the Green's function into consideration eliminates the divergence in the renormalization of a static point charge. Arguments are presented concerning an analogous mechanism for eliminating the divergence in the D-function.

2. VACUUM POLARIZATION IN AN INHOMOGENEOUS FIELD

2.1 The Dielectric Constant of the Vacuum in a Strong Field

Let us derive the expression relating the dielectric constant to the polarization operator. For simplicity we consider the case of a static field. The generalization to nonstatic fields is obvious.

As usual the dielectric constant is defined in terms of the polarization $\mathbf{P}$:

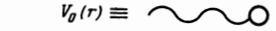
$$\varepsilon_{ik} = \delta_{ik} + 4\pi\alpha_{ik}, \quad P_i = \alpha_{ik}E_k.$$

The polarization satisfies the relation

$$\operatorname{div}\mathbf{P} = -\rho_1, \tag{2.1}$$

where ρ_1 denotes the density of the charges which are induced by the external field. In order to obtain ϵ_{ik} it will be sufficient to express ρ_1 in terms of the polarization operator. In order to do this, let us determine the potential acting on the electron in the presence of the external charges. We obtain the potential $V(r)$ in the form of a series in powers of e^2, without assuming the external field to be weak. In the zero-order approximation in e^2 we have

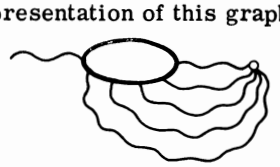
$$V(r) = e\int \frac{\rho_0(r')\,d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = V_0(r), \tag{2.2}$$

$V_0(r) \equiv$ 

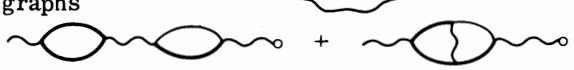
Here $\rho_0(r)$ is the density of the external charges, and $V_0(r)$ is the potential of the external field (more precisely, the potential multiplied by the charge e). Formula (2.2) can be symbolically written in the form $V_0(r) = eD_0\rho_0$, where $D_0 = |\mathbf{r} - \mathbf{r}'|^{-1}$.

To the first approximation in e^2, the field due to the induced charges is given by the graph

The heavy line in the loop indicates that one of the Green's functions must include the effect of the field $V(r)$, that is, all of the photon lines running from the external charges must be taken into consideration. A more detailed representation of this graph would be the following:



The graphs



$$= D_0\Pi D_0\Pi_0' V_0 + D_0\Pi_1' V_0.$$

appear in the next order in e^2. In the first loop of the first one of these graphs, the photon lines running from the external charges are joined to each of the lines in the loop, that is, both of the Green's functions of this loop are evaluated with the presence of the field $V(r)$

taken into account. The complete series in powers of e^2 is written in the following way:

$$V = V_0 + D_0\Pi' V_0 + D_0\Pi D_0\Pi' V_0 + \ldots, \qquad (2.3)$$

where

$\Pi' =$  $\Pi =$  (2.3′)

One can write the expression for V in the form

$$V = V_0 + D_0(\Pi' - \Pi)V_0 + D_0\Pi\{V_0 + D_0\Pi' V_0 + D_0\Pi D_0\Pi' V_0 + \ldots\}$$
$$= V_0 + D_0(\Pi' - \Pi)V_0 + D_0\Pi V, \qquad (2.4)$$

or, by multiplying by $D_0^{-1} = -\Delta/4\pi$ we obtain

$$\Delta V = -4\pi[e\rho_0(r) + (\Pi' - \Pi)V_0 + \Pi V]. \qquad (2.5)$$

Since we have introduced the observable charge e, the polarization operator Π must be regularized, that is, the expression ΠV stands for

$$\Pi_{Reg}V = \int \Pi(\mathbf{r}, \mathbf{r}')V(\mathbf{r}')d\mathbf{r}' - \frac{1}{6}\int \Pi^0(\mathbf{r} - \mathbf{r}')(\mathbf{r} - \mathbf{r}')^2 d\mathbf{r}'\Delta V, \qquad (2.6)$$

where Π^0 is the polarization operator without any external field. Formula (2.6) is analogous to the well-known expression in the momentum representation

$$\Pi_{Reg}(k^2) = \Pi(k^2) - k^2(d\Pi/dk^2)_{k^2=0}.$$

By changing to the coordinate representation with respect to $\mathbf{r} - \mathbf{r}'$, we obtain Eq. (2.6). However, charge renormalization can be carried out directly in expression (2.5). In order to do this, let us rewrite (2.5) in the form

$$\Delta V = -4\pi e_0^2(n_0(r) + n_1(r)), \qquad (2.5')$$

where e_0^2 is the bare charge, $n_0(r)$ is the density of external particles, and $n_1(r)$ is the density of polarization electrons. By definition the observable charge, i.e., the renormalized charge, is the charge in a weak, slowly varying field. Then according to Eq. (2.5) we have

$$e_0^2 n_1 = \Pi^0 V = \int \Pi^0(\mathbf{R})d\mathbf{R}\, V(\mathbf{r})$$
$$+ \int \Pi^0(\mathbf{R})\mathbf{R}\, d\mathbf{R}\, \nabla V + \frac{1}{6}\int \Pi^0(\mathbf{R})\mathbf{R}^2 d\mathbf{R}\Delta V + \ldots$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. The first term vanishes due to gauge invariance, and the second vanishes because Π^0 is isotropic. Transferring the term containing ΔV to the left-hand side and dividing by the factor associated with ΔV, we obtain

$$\Delta V = -4\pi e^2 n_0(r), \quad e^2 = e_0^2 \Big/ \left[1 + \frac{4\pi}{6}\int \Pi^0 R^2 d\mathbf{R}\right].$$

Consequently, by adding and subtracting the quantity $(4/6)\pi \int \Pi^0 R^2 d\mathbf{R}$ to the right-hand side of (2.5) and by changing to the charge e instead of e_0, we obtain (2.5) with the operator Π_{Reg} appearing instead of Π. The quantity e^2 appears everywhere in Π_{Reg} instead of e_0^2. From Eq. (2.5) it follows that

$$e\rho_1 = (\Pi' - \Pi)V_0 + \Pi_{Reg}V. \qquad (2.7)$$

In the next section it will be shown that the polarization operator Π has the property of localizability in the presence of a strong electric field E, that is

$$\Pi(\mathbf{r}, \mathbf{R}) = \begin{cases} \Pi^0(R) & \text{for } R^2 \ll 1/eE \\ 0 & \text{for } R^2 \gg 1/eE \end{cases}; \qquad (2.8)$$

here Π^0 is the polarization operator without any field. Π' also possesses the same property. Similar results are also obtained in the presence of a magnetic field.

Let E(r) vary noticeably over a distance l satisfying the condition

$$l^2 \gg 1/eE. \qquad (2.9)$$

Then one can neglect the variation of the field over the distance $(eE)^{-1/2}$ which is characteristic for the polarization operator, and thus the polarization operator and consequently the dielectric constant as well can be calculated in a homogeneous field. But for a strong homogeneous field there exists the expression for the Lagrangian density $\mathcal{L}'$ which was derived as long ago as 1936 (by Heisenberg and Euler, and by Weisskopf):[10-12]

$$\mathcal{L}' = -\frac{e^2E^2}{24\pi^2}\ln eE, \quad \ln eE \gg 1, \qquad (2.10)$$

and from this expression we obtain

$$\mathbf{P} = \frac{\partial \mathcal{L}'}{\partial \mathbf{E}} = -\frac{e^2}{12\pi^2}\mathbf{E}\ln eE - \frac{e^2}{24\pi^2}\mathbf{E}.$$

Correct to within logarithmic terms, the dielectric constant is given by

$$\varepsilon_{ik} = \delta_{ik}\left(1 - \frac{e^2}{3\pi}\ln eE\right). \qquad (2.11)$$

Thus, the criterion for the applicability of formulas (2.10) and (2.11) is by no means the condition $E'/E \ll 1$ which was assumed during the derivation of (2.10), but rather it is the condition (2.9) which can be written in the form

$$(E'/E)^2 \ll eE. \qquad (2.9')$$

From Eqs. (2.11) and (2.1) we obtain

$$e\rho_1 = \frac{e^2}{12\pi^2}\ln(eE)\operatorname{div}\mathbf{E} + \frac{e^2}{12\pi^2}\frac{\mathbf{E}}{E}\nabla E. \qquad (2.12)$$

Expression (2.10) is obtained to first order in e^2. Meanwhile, as is clear from Eq. (2.11), the expansion is in powers of the quantity $e^2 \ln eE$, which becomes of the order of unity at small distances from a point charge (see below). In order to obtain more general results, it is necessary to return to expression (2.7). In the following calculations we shall discard the terms which do not contain $\ln eE \gg 1$ as a factor.

First of all let us simplify expression (2.7). Under condition (2.9) one can expand $V(\mathbf{r}')$ in the first term of (2.6) in a series around the point r and restrict our attention to the first nonvanishing term. We obtain the following result:

$$\Pi_{Reg}V = \frac{1}{2}\left\{\int \Pi(\mathbf{r}, \mathbf{R})R_iR_k\, d\mathbf{R} - \frac{\delta_{ik}}{3}\int \Pi^0(R)R^2 d\mathbf{R}\right\}\frac{\partial^2 V}{\partial x_i\, \partial x_k}. \qquad (2.13)$$

Here the equations

$$\int \Pi(\mathbf{r}, \mathbf{R})d\mathbf{R} = 0, \quad \int \Pi(\mathbf{r}, \mathbf{R})\mathbf{R}\, d\mathbf{R} = 0.$$

have been used. The first follows from gauge invariance, and the second follows from the symmetry of $\Pi(\mathbf{r}, \mathbf{r}')$ with respect to the interchange $\mathbf{r} \rightleftarrows \mathbf{r}'$. The tensor inside the curly brackets in (2.13) can be referred to its principal axes ($z \parallel \mathbf{E}$). Introducing the notation $\rho^2 = (x - x')^2 + (y - y')^2$ and $\zeta^2 = (z - z')^2$, instead of expression (2.13) we obtain

$$\Pi_{Reg}V = \frac{1}{4}\int (\Pi - \Pi^0)\rho^2\, d\mathbf{R}\Delta V + \frac{1}{2}\int \Pi\left(\zeta^2 - \frac{\rho^2}{2}\right)d\mathbf{R}\frac{\partial^2 V}{\partial z^2}. \qquad (2.14)$$

Due to the isotropic nature of Π^0 we have $\int \Pi^0(\zeta^2 - \rho^2/2)d\mathbf{R} = 0$; therefore it follows from (2.8) that the coefficient associated with $\partial^2 V/\partial z^2$ in the second term of (2.14) does not contain ln eE as a factor, just as is true for the second term on the right-hand side of (2.12).

In similar fashion, by using (2.8) we can verify that to within logarithmic accuracy the coefficient associated with ΔV in (2.14) can be reduced to the form

$$\frac{1}{4}\int (\Pi - \Pi^0)\rho^2 d\mathbf{R} \approx -\frac{1}{4}\int\limits_{R^2>1/eE} \Pi^0 \rho^2 d\mathbf{R} = -\frac{1}{6}\int\limits_{R^2>1/eE} \Pi^0 R^2 d\mathbf{R}. \quad (2.15)$$

The region of integration with $R^2 \sim 1/eE$ introduces a contribution which does not contain any dependence on ln eE. Finally, for the same reasons the first term in (2.7) also does not contain any dependence on ln eE. These assertions will be discussed in more detail below.

Thus, to within logarithmic accuracy it follows from expressions (2.14), (2.15), and (2.7) that

$$e\rho_1 = \frac{1}{6}\int\limits_{R^2>1/eE} \Pi^0 R^2 d\mathbf{R}\, \Delta V - \int \Pi\left(\zeta^2 - \frac{1}{2}\rho^2\right) d\mathbf{R}\frac{\partial^2 V}{\partial z^2}.$$

By comparing this expression with the expression for div $\mathbf{P}$ we find that, to the approximation we are using, the dielectric permittivity tensor reduces to the identity tensor and is given by

$$\varepsilon_{ik} = \left(1 - \frac{4\pi}{6}\int\limits_{R^2>1/eE} \Pi^0 R^2 d\mathbf{R}\right)\delta_{ik}. \quad (2.16)$$

Formula (2.16) is a generalization of expression (2.11) and, as will be shown below, it goes over into (2.11) for $1 - (e^2/3\pi)\ln eE \gg e^2$.

2.2. The Localizability of the Green's Functions in Strong Fields

In this section it will be shown that, in the presence of strong fields the Green's function $G(\mathbf{r}, \mathbf{r}', \omega)$ of a p particle is localizable, that is, it falls off abruptly for $|\mathbf{r} - \mathbf{r}'|^2 = R^2 \gg 1/eE$. This property of the Green's functions is very important in connection with deducing the asymptotic behavior of electrodynamical quantities at small distances, and it seems to us that this property may lead to the elimination of the divergences. For simplicity we shall assume that the external field has a frequency $\omega_0 \ll |\mathbf{r} - \mathbf{r}'|^{-1}$. Since the frequency $\omega \sim |\mathbf{r} - \mathbf{r}'|^{-1}$ of the Green's function turns out to be important in all of the calculations, one can regard the field as static.

Let us write down the equation for the Green's function in a mixed representation, that is, in a spatial representation with respect to the spatial coordinates and in a Fourier representation with respect to $t - t'$:

$$G = G(\mathbf{r}, \mathbf{r}', \omega).$$

The Green's function has a simpler form in this representation. In addition we shall utilize the fact that spin effects are unimportant at sufficiently high energies ($\omega \gg 1$), and so all the results can be derived by using the Klein-Gordon-Fock equation instead of the Dirac equation. This assertion is clarified in the Appendix. The equation for the Green's function of bosons in an electric field has the form

$$\Delta G + [(\omega - V)^2 - 1]G = \delta(\mathbf{r} - \mathbf{r}'). \quad (2.17)$$

The Green's function of a free particle is given by

$$G_0 = -\exp[i(\omega^2 - 1)^{1/2}R]/4\pi R. \quad (2.17')$$

The choice of the plus sign in the argument of the exponential replaces the usual condition for going around singular points in the momentum representation.

If $\omega \gg 1$ then G_0 oscillates rapidly for $R \gg 1/\omega$. Let us consider G for $R \ll l$, where l is the distance characterizing a substantial variation of the field. Then in Eq. (2.17) one can replace V(r) by

$$V(r) = V(r') + \frac{dV}{dr'}\frac{\mathbf{r}'}{r'}(\mathbf{r} - \mathbf{r}') = eE\zeta.$$

We have measured the potential V(r) from the point r', and the problem thus reduces to determining the Green's function in a homogeneous field. As we shall demonstrate, distances $R^2 \lesssim 1/eE$ are important in the Green's function, and the condition $R^2 \ll l^2$ is equivalent to condition (2.9) (or (2.9')), which is satisfied in the case we are interested in, namely, the case of rather strong fields. The solution of Eq. (2.17) in a homogeneous field reduces to the solution of the Schrödinger equation for an oscillator potential, and is determined by parabolic cylinder functions. This solution is given in the Appendix. Here we shall confine our attention to making a simple estimate of the significant distances which are adequate in order to calculate the polarization operator correct to terms of logarithmic order. In connection with our choice of the reference point of the potential, the Green's function $G(\mathbf{r}, \mathbf{r}')$ depends on the difference vector $\mathbf{R}$, but the vector $\mathbf{r}$ only enters as a parameter due to the dependence of E on r.

It follows from Eq. (2.17) that if $eER \ll \omega$, then the function G(R) goes over into $G_0(R)$. But, as is quite clear from Eq. (2.17) and from the subsequent integration over ω, the important values of ω are determined by the relation $\omega \lesssim R^{-1}$. Therefore we obtain

$$G(R) \to G_0(R), \quad R^2 \ll 1/eE. \quad (2.18)$$

On the other hand, for $\omega \ll eE\rho$, $eE\zeta$ it follows from (2.17) that G_0 oscillates strongly for $R^2 \gg 1/eE$, that is, $\overline{G} \to 0$ for $R^2 \gg 1/eE$, where the bar indicates the average over an interval ΔR,

$$1/eE \ll (\Delta R)^2 \ll R^2.$$

It follows from the solution of Eq. (2.17) with $V = eE\zeta$ that

$$G \to -\exp\{iCR^2eE\}/4\pi R, \quad R^2 \gg 1/eE,$$

where C is a number of the order of unity, which confirms our qualitative estimate.

Since in what follows G appears in the integrals of smooth functions over R, in practice one can get

$$G(R) \to 0 \quad \text{for} \quad R^2 \gg 1/eE. \quad (2.19)$$

In order to clarify this important assertion, let us consider G in the quasiclassical approximation:

$$G \sim \exp\{i(S(\mathbf{r}) - S(\mathbf{r}'))\},$$

where the action S satisfies the Hamilton-Jacobi equation

$$(\partial S/\partial\xi)^2 + (\partial S/\partial\rho)^2 = (\omega - V)^2 - 1. \quad (2.20)$$

The argument of the exponential will be minimal if a

classical trajectory exists connecting the points $\mathbf{r}$ and $\mathbf{r}'$.

Let us demonstrate that the classical trajectory exists only under the condition $\omega \gg eER$, and consequently in the opposite case the difference $S(\mathbf{r}) - S(\mathbf{r}')$ is large and the Green's function oscillates strongly. Let us denote the transverse momentum by $\partial S/\partial\rho = \kappa$. Then, neglecting the one on the right-hand side of Eq. (2.20), we obtain

$$(\partial S/\partial\zeta)^2 = (\omega - V)^2 - \varkappa^2. \tag{2.21}$$

The classical action function is obtained from (2.21) with the additional condition

$$\partial S(\zeta, \varkappa)/\partial\varkappa = 0. \tag{2.22}$$

Since we are interested in values of $\omega \gtrsim eE\zeta$, then in our estimates we can neglect the term V^2 on the right-hand side of (2.21), and the problem reduces to the classical problem of the motion of a particle with energy ω^2 in a gravitational field with acceleration $g = eE\omega$. Let us consider points $\mathbf{r}$ and $\mathbf{r}'$ located at the same height ($\zeta = 0$, $\rho = R$). Then we obtain the following result from the formulas for the distance of flight (range) in a gravitational field:

$$\rho = \frac{\omega^2 \sin 2\varphi}{2g} = \frac{\omega^2 \sin 2\varphi}{2\omega eE}$$

from here it follows that

$$\omega > eE\rho.$$

For $\rho = 0$ and $R = \zeta$ we obtain the condition

$$\omega^2 > g\zeta = eE\omega\zeta.$$

Thus, the condition for the existence of a classical trajectory joining the points $\mathbf{r}$ and $\mathbf{r}'$ is given by

$$\omega > eER.$$

We note that in the quasiclassical approximation, by using Eqs. (2.21) and (2.22) the Green's function can be determined for an arbitrary $V(\mathbf{r})$ without making the restriction (2.9). In our case the criterion for the validity of the quasiclassical treatment, $dp^{-1}/d\zeta \ll 1$, has the form

$$\frac{d(\omega - V)^{-1}}{d\zeta} = \frac{V'}{(\omega - V)^2} \sim \frac{eE}{\omega^2} \sim eER^2 \ll 1,$$

while the important values of R are determined by the condition

$$R^2 eE \sim 1.$$

Nevertheless, it can be anticipated that, as usually happens,[13] the quasiclassical approximation gives a good quantitative estimate.

Thus, the Green's function G possesses the properties of localizability (2.18) and (2.19). From Eqs. (2.21) and (2.22) one can estimate the characteristic values of R which are suitable even in the case when E varies abruptly in some region, that is

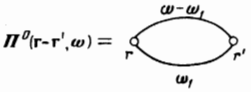
$$\omega > |V(\mathbf{r}) - V(\mathbf{r}')| = e\left|\int_{r'}^{r} E(\xi)\,d\xi\right|.$$

Together with the condition $\omega < 1/R$, we obtain the result

$$G(\mathbf{r}, \mathbf{R}) = \begin{cases} G_0(\mathbf{r}, \mathbf{R}), & R \ll |V(\mathbf{r}') - V(\mathbf{r})|^{-1}, \\ 0, & R \gg |V(\mathbf{r}') - V(\mathbf{r})|^{-1}. \end{cases} \tag{2.23}$$

2.3. The Polarization Operator in the Coordinate Representation

First let us determine the polarization operator when no field is present. To the lowest order approximation in e^2 we have

$\Pi^0(\mathbf{r}-\mathbf{r}', \omega) =$ 

$$= e^2 \int \Gamma_0 G_0(\mathbf{r} - \mathbf{r}', \omega - \omega_1)\,\Gamma_0 G_0(\mathbf{r} - \mathbf{r}', \omega_1)\frac{d\omega_1}{2\pi i}. \tag{2.24}$$

Since we are interested in the case of a static electric field ($\omega \ll \omega_1$), it is sufficient to confine our attention to the calculation of Π^0_{44}. Therefore, one should take the matrix γ_0 as the vertex Γ_0 if the calculation is done with the aid of Dirac Green's functions. As we have already mentioned, the same result is obtained to within logarithmic accuracy when the calculation is carried out with the aid of the Green's functions for bosons, but in this case as usual the fourth component of the difference of the particle's momenta should be taken as the vertex in the graph (2.24), that is, $\Gamma_0 = 2\omega_1 - \omega$. Neglecting ω and using (2.17′) we obtain the following result for $R \ll 1$

$$\Pi^0(R) = 4e^2 \int G_0^2(R, \omega_1)\,\omega_1^2 \frac{d\omega_1}{2\pi i}$$

$$= \frac{e^2}{2\pi^3}\frac{i}{R^2}\int_{-\infty}^{+\infty} \exp(2i|\omega_1|R)\,\omega_1^2 d\omega_1.$$

In order to make the integrals convergent, it is necessary to assume that R has a small, positive, imaginary part. Then, by omitting the integral over the infinite semicircle in the upper half-plane of ω_1 and by introducing a branch cut along the positive imaginary axis, we obtain

$$\Pi^0(R) = \frac{e^2}{\pi^3 R^2}\int_0^{\infty} e^{-2R\xi}\xi^2 d\xi = \frac{e^2}{4\pi^3 R^5}. \tag{2.25}$$

Expression (2.25) is valid for $R \ll 1$; for $R \gg 1$ one will obtain $\Pi^0 \sim e^{-2R}$. Substituting (2.25) into formula (2.16) we obtain

$$\varepsilon_{ik} = \delta_{ik}\left(1 - \frac{e^2}{3\pi}\ln eE\right),$$

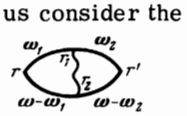
that is, a result which is identical to expression (2.11), which was obtained from the Lagrangian (2.10).

Now let us demonstrate that the polarization operator in the presence of the field actually does satisfy relations (2.8), which we utilized in order to obtain expression (2.16) for the dielectric constant. To first-order in e^2, the validity of these relations follows at once from the localizability of the Green's functions in the field. In fact, one finds

$$\Pi(\mathbf{r}, \mathbf{R}, \omega) = 4e^2 \int_{-\infty}^{+\infty} G(\mathbf{r}, \mathbf{R}, \omega_1)\,G(\mathbf{r}, \mathbf{R}, \omega - \omega_1)\,\omega_1^2 \frac{d\omega_1}{2\pi i}$$

and therefore expressions (2.18) and (2.19) lead to (2.8). It is not difficult to verify that the properties (2.8) are preserved to any order in e^2.

As an example, let us consider the graph



Here the heavy lines indicate that the Green's functions are to include the presence of the external field. With regard to the integrals over $\mathbf{r}_1$ and $\mathbf{r}_2$, it is easy to see that the only regions of integration which are important

are those in which all coordinate differences are smaller than $R = |\mathbf{r} - \mathbf{r}'|$. One can also easily verify this point in the momentum representation, with which the readers of this article are probably more accustomed. In the momentum representation, only the momenta $p_{int} \gtrsim p_R$ of the internal lines are important, where p_R is the Fourier transform of R. Therefore, if $R^2 \ll 1/eE$ then one also has $R^2_{int} \ll 1/eE$ for all internal lines, and according to (2.18) all of the Green's functions are replaced by free-particle Green's functions; hence

$$\Pi(\mathbf{r}, \mathbf{R}) \to \Pi^0(\mathbf{R}), \quad R^2 \ll 1/eE.$$

In the opposite case, when $R^2 \gg 1/eE$, there necessarily exist at least two Green's functions with distances $R^2_{int} \gg 1/eE$ and according to (2.19) we see that

$$\Pi(r, R) \to 0, \quad R^2 \gg 1/eE.$$

One can also utilize the more general conditions for R from expression (2.23).

Let us introduce the logarithmic variable $\xi = \ln r^{-2}$. The quantity

$$\chi = \frac{4\pi}{6}\int_{r_2}^{1} \Pi^0 R^2 dR$$

can be represented in the form

$$\chi = \frac{e^2}{3\pi}\int_0^{\xi} d\xi\, \varphi(\xi). \tag{2.26}$$

To first order in e^2 the function $\varphi(\xi)$ is identical to unity.

According to Eq. (2.16) the dielectric constant has the form

$$\varepsilon = 1 - \gamma(\ln eE). \tag{2.27}$$

Expression (2.27) determines the polarizability in an external field to all orders in e^2. It is necessary, however, to remember that (2.27) has been derived in the logarithmic approximation. Therefore this expression becomes incorrect in regions where $1 - \chi \sim e^2$.

3. THE ELECTRIC FIELD AT SMALL DISTANCES FROM A CHARGE

3.1. Distortion of the Coulomb Field at Small Distances

The expression derived above for the dielectric constant immediately enables us to solve the problem of the distribution of the charge density near an arbitrarily small nucleus. Let us represent the electric field at the point r in the form

$$eE = Q(r)/r^2.$$

Then Q(r) is the charge inside a sphere of radius r, multiplied by e. Let the observable charge of the nucleus, i.e., the charge at large distances, be equal to Ze; or alternatively

$$Q(r) \to Ze^2, \quad r \to \infty. \tag{3.1}$$

The screening charge due to vacuum polarization decreases at small distances, and $Q(r) \gg Ze^2$. We shall be interested in those distances where $Q(r) \gg 1$. Then the characteristic length $R_c^2 = 1/eE$ satisfies the localizability condition (2.9):

$$R_c^2/r^2 = 1/Q(r) \ll 1 \tag{3.2}$$

and the variation of the field over the distance R_c can be neglected. The problem reduces to the problem of a homogeneous field.

From formula (2.27) for the dielectric constant ϵ in a homogeneous field, we obtain the following result in the region where there are no external charges (outside the nucleus):

$$D = \varepsilon E = E[1 - \chi(\ln eE)] = C_1/r^2. \tag{3.3}$$

The constant C_1 is determined from condition (3.1) for $r \to \infty$, where $\chi \to 0$. As a result we find

$$eE = Ze^2/r^2[1 - \chi(\ln eE)], \tag{3.4}$$

where $\chi(\xi)$ is given by formula (2.26) and $\ln eE = \xi + \ln Q(\xi)$. Formula (3.4) determines the field outside the nucleus. Let us assume for simplicity that the nucleus has its charge distributed over the surface of a sphere of radius R_0. Then we have $E = 0$ for $r < R_0$, that is, the solution is determined for all values of r. The true charge of the nucleus, Z_0, is related to the observable charge Z by the equation

$$Z = Z_0\left[1 - \chi\left(\ln \frac{Z_0 e^2}{R_0^2}\right)\right]. \tag{3.5}$$

Formulas (3.4) and (3.5) cease to be correct for small values of R_0, when $[1 - \chi(\ln Z_0 e^2 R_0^{-2})]/e^2$ approaches unity. The case of arbitrarily small values of R_0 is considered in the next section.

From Eqs. (2.26) and (3.5) we obtain the result

$$1 - \frac{e^2}{3\pi}\ln\frac{Z_0 e^2}{R_0^2} \geqslant e^2, \tag{3.5'}$$

for values of R_0^2 satisfying the condition

$$Z_0\left(1 - \frac{e^2}{3\pi}\ln\frac{Z_0 e^2}{R_0^2}\right) = Z.$$

The following relation is obtained for the determination of Q(r):

$$Q(\xi) = \frac{Ze^2}{1 - \chi(\xi + \ln Q(\xi))} \tag{3.6}$$

At large values of r

$$\chi(\ln eE) \to \frac{e^2}{3\pi}\ln\frac{Q(r)}{r^2}$$

and Eq. (3.4) gives

$$eE = \frac{Ze^2}{r^2}\left[1 - \frac{e^2}{3\pi}\ln\frac{Q(r)}{r^2}\right]^{-1} \approx \frac{Ze^2}{r^2}\left(1 + \frac{e^2}{3\pi}\ln\frac{Q(r)}{r^2}\right), \tag{3.4'}$$

This expression differs from the well-known formula describing the distortion of the Coulomb interaction[17,18] only by the appearance of Q(r) inside the logarithm instead of unity. One can obtain a simple interpolation formula by replacing Q(r) inside the logarithm sign by $1 + Q(r)$. Then expression (3.4′) will have the correct limit for $Q \ll 1$. For real nuclei having a radius $R_0 = r_0 A^{1/3}$, $Q(r) \approx Ze^2$, and formula (3.4′) with Q(r) replaced by $1 + Q(r)$ differs from the well-known result noticeably only for $Z > 137$.

3.2. The Distribution of the Polarization Charge at Ultra-small Distances and a Possible Mechanism for the Elimination of Divergences

By ultra-small distances we mean those distances at which $[1 - \chi(\xi + \ln[1 + Q(\xi)])]/e^2 \lesssim 1$. In this region the charge distribution is determined by the properties of electrodynamics at small distances, and in order to

determine $Q(\xi)$ it is necessary to make a conjecture about the nature of the function $\chi(\xi)$. Let us introduce the quantity

$$e^2(\xi) = e^2 / [1-\chi(\xi)].$$

This quantity determines the electrons' interaction over distances $r = e^{-\xi/2}$ in the absence of an external field. By virtue of the renormalizability of quantum electrodynamics,[14,15] the function $\varphi(\xi)$, which appears in the expression for χ (see formula (2.26)), only depends on ξ through its dependence on the quantity $e^2(\xi)$. We note that $e^2(\xi)$ differs from the "invariant charge" $e^2_{inv}(\ln k^2)$ which is usually introduced in quantum electrodynamics. The two expressions coincide for $1-\chi \gg e^2$. We have the following expression for $Q(\xi)$

$$Q(\xi) = Z\frac{e^2}{1-\chi(\eta)} = Ze^2(\eta), \quad \eta = \xi + \ln(1+Q). \tag{3.7}$$

Let us consider the two possibilities: 1) $e^2(\xi)$ tends to infinity as ξ approaches a certain value ξ_1, and 2) $e^2(\xi) \to e_0^2$ as $\xi \to \infty$. The first possibility corresponds to the case of "zero-charge" (i.e., the vanishing of the renormalized charge in field theories with a point interaction), a situation which has been discussed in a number of articles.[19,20] If instead of starting from the observable charge e we had started from the bare charge e_0, then in case 1) the observable charge would vanish for any finite value of e_0. This possibility means that quantum electrodynamics is not a logically closed theory, and in order to eliminate the infinity in $e^2(\xi)$ which appears as $\xi \to \xi_1$ nonelectrodynamic interactions must be included (for example, the gravitational interaction). The second possibility—corresponding to a finite bare charge—implies that the apparent divergence of the charge is the result of an incorrect use of the formula $\chi(\xi) = e^2\xi/3\pi$ for $e^2(\xi) \sim 1$, and it also indicates that the intuitive arguments which were made in favor of "zero-charge" are not confirmed.

Let us consider possibility 1). Let us assume for simplicity that $e^2(\xi)$ has a pole at $\xi = \xi_1$:

$$e^2(\xi) = a/(\xi_1 - \xi), \quad a > 0. \tag{3.8}$$

Substitution into (3.4) gives

$$Q(r^2) = Za\left[\xi_1 - \ln\frac{Q(r^2)}{r^2}\right]^{-1}. \tag{3.9}$$

From Eq. (3.9) we find

$$\xi = \ln\frac{1}{r^2} = -\frac{Za}{Q} - \ln Q + \xi_1. \tag{3.9'}$$

It follows from this expression that, as a function of Q r^2 has a minimum at $Q = Q_m = Za$. The minimum value is determined by the relation

$$\xi_1 - \ln\frac{Q_m}{r_m^2} = 1, \quad \left(\frac{dQ}{dr^2}\right)_{r=r_m} = -\infty. \tag{3.10}$$

This relation only determines the order of magnitude of r_m. Thus, to within the accuracy of our determination of χ, the quantity Q inside the logarithm sign can be multiplied by a quantity ~ 1, which is a functional of E.

Thus, Eq. (3.9) has a solution only for $r > r_m$. If the radius R_0 of the nucleus is greater than r_m, then we have $E = 0$ for $r < R_0$, and the solution is determined from Eq. (3.9) for $r > R_0$.

Let us consider the case when $R_0 < r_m$. As before we have $E = 0$ for $r < R_0$. For $R_0 < r < r_m$ the field E is determined by expression (3.3) with a constant C_1 which can easily be expressed in terms of the true charge of the nucleus

$$eE = \frac{Z_0e^2}{r^2}\frac{e^2(\ln eE)}{e^2(\ln eE_0)}, \quad eE_0 = \frac{Z_0e^2}{R_0^2}. \tag{3.11}$$

The quantity Q(r) is determined by the relation

$$Q(r^2) = Z_0e^2\frac{\ln(Q_0/R_0^2) - \xi_1}{\ln(Q/r^2) - \xi_1}. \tag{3.12}$$

A solution is possible for $r < r_m$ only under the conditions

$$\ln\frac{Q}{R_0^2} > \xi_1, \quad \ln\frac{Q}{r^2} > \xi_1.$$

The solution (3.12) is shown in the accompanying figure ($r < r_m$). In order to relate Z_0 to Z, it is necessary to equate the solution (3.12) and the solution (3.9) at the point r_m. However, to within the accuracy of our calculation of χ, this matching can be achieved only in order of magnitude, because for $r = r_m$ we have $1-\chi \sim e^2$. Therefore we obtain

$$Z_0e^2\left(\ln\frac{Z_0e^2}{R_0^2} - \xi_1\right) = \gamma Za, \tag{3.13}$$

where γ denotes a number of the order of unity. For $r > r_m$ the field E and the quantity Q are determined as usual by specifying the observable charge Z. The discontinuous behavior of the derivative dQ/dr^2 is a consequence of our approximate representation of the characteristic length by the formula $R_c^2 = 1/eE$. It is not difficult to verify that the more exact criterion

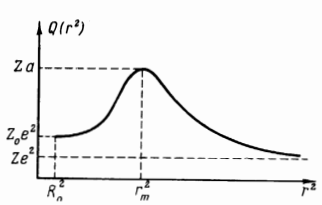
$$R_c\int_r^{r'} eEdl \sim 1$$

which follows from (2.23), leads to a smoothing-out of the curve Q(r) over an interval $\delta r \sim R_c \sim Q^{-1/2}$. An important result follows from formula (3.13): As $R_0 \to 0$ the true charge Z_0 not only does not tend to infinity, but it becomes arbitrarily smaller than the observable charge. (For $Z_0e^2 \lesssim 1$ one should replace Z_0e^2 in Eq. (3.13) by $1 + Z_0e^2$.) Thus, the localizability of the polarization operator eliminates the divergence in the vacuum polarization near an external, fixed, point charge even under the assumption that $e^2(\xi)$ tends to infinity.

Now let us consider the second possibility, when $e^2(\xi)$ tends to a finite limit as $\xi \to \infty$. In this case the relation between the charge Z_0 and the charge Z is determined from Eq. (3.7):

$$Z_0e^2 = Ze^2(\ln Z_0e^2R_0^{-2}). \tag{3.7'}$$

Finally, let us make a few remarks about a possible mechanism for the elimination of the divergence in $e^2(\xi)$. The possibility of a finite limit for $e^2(\xi)$ has been discussed in detail in a number of articles.[16,21,22] Such a possibility is still attractive in one respect: Here the entire mass of the electron is electromagnetic in origin.[23] In fact, the following relation exists[24] for



the mass contained inside a sphere of radius r or, alternatively, the mass inside the region $\xi > \ln r^{-2}$:

$$\frac{1}{m(\xi)}\frac{dm(\xi)}{d\xi} = -\frac{3}{4\pi}e^2(\xi)$$

or

$$\ln\frac{m(\xi)}{m} = -\frac{3}{4\pi}\int_0^{\xi} e^2(\xi)\,d\xi.$$

If $e^2(\xi) \to e_0^2$ as $\xi \to \infty$, then $m(\infty) \to 0$ or $m(r^2) \to 0$ as $r \to 0$, that is, the entire mass is electrodynamic.

In the case when $e^2(\xi)$ is finite, the quantity χ defined in Eq. (2.26),

$$\chi(\xi) = \frac{e^2}{3\pi}\int_0^{\xi} d\xi\,\varphi(q(\xi)),$$

must also tend to a finite limit. This implies that the function $\varphi(q)$ must vanish as $\xi \to \infty$ or $\varphi(q_0) = 0$, where $q_0 = e_0^2/3\pi$. However, by starting from the expansion of $\varphi(q)$ for small q it has not been possible to prove the existence of a zero for $\varphi(q)$.[22] The localizability of the polarization operator and the finiteness of the vacuum polarization, which have been proved above for an external, fixed, point charge, suggest that it may be possible to find a new method of grouping the diagrams, thus taking the localizability of the Green's functions in the presence of a strong field into account; this new method will lead to expressions similar to the ones cited above. In this connection an expression may be obtained for $\varphi(q)$ which is valid not only for small q, and from this expression it will follow that $\varphi(q_0) = 0$.

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APPENDIX

A. The Connection Between the Dirac Equation and the Klein-Gordon-Fock Equation

At large energies, when spin effects are unimportant, the solution of the Dirac equation in an external field can be expressed in terms of the solution of the Klein-Gordon-Fock equation. For simplicity we shall confine our attention to the case of a spherically symmetric electric field.

Let us write down the ψ-function of the Dirac equation in terms of two-component radial functions u(r) and v(r):

$$\Psi_{jlm\,\varepsilon} = \frac{1}{r}\Omega_{jlm}\begin{Bmatrix} u_\varepsilon(r) \\ v_\varepsilon(r)\end{Bmatrix}.$$

From the Dirac equation we have

$$u' + \varkappa r^{-1}u - (\varepsilon + 1 - V)v = 0,$$
$$v' - \varkappa r^{-1}v + (\varepsilon - 1 - V)u = 0.$$

By eliminating the function v(r) we find

$$u'' + \frac{(u' + \varkappa u/r)V'}{\varepsilon + 1 - V} + \left[(\varepsilon - V)^2 - 1 - \frac{\varkappa(\varkappa+1)}{r^2}\right]u = 0.$$

The analogous equation for v can be obtained by making the substitutions

$$V \to -V, \quad \varepsilon \to -\varepsilon, \quad \varkappa \to -\varkappa.$$

The substitution $\varphi = u(\epsilon - V + 1)^{-1/2}$ now gives[4,5]

$$\varphi'' + \left\{\left[(\varepsilon - V)^2 - 1 - \frac{\varkappa(\varkappa+1)}{r^2}\right] - \frac{3}{4}\frac{(V')^2}{(\varepsilon - V + 1)^2} - \frac{V'' - 2\varkappa V'/r}{2(\varepsilon - V + 1)}\right\}\varphi = 0.$$

In the case $\epsilon \sim V \gg 1$, $\kappa \gg 1$ which we are interested in, the last two terms inside the curly brackets can be neglected, and the equation for φ goes over into the Klein-Gordon-Fock equation. In addition, $u \approx v$, and therefore

$$u = v = (\varepsilon - V)^{1/2}\varphi.$$

If Ψ is normalized to unity, that is

$$\int_0^{\infty} (u^2 + v^2)\,dr = 1,$$

then the normalization of φ corresponds to a single boson.

B. The Green's Function and the Polarization Operator in the Coordinate Representation

For $x^2 \ll 1$ the Green's function of a free, spin-zero particle has the form

$$G^{(0)}(x) = \int \frac{e^{ipx}}{p^2}\frac{d^4p}{(2\pi)^4} = -\frac{1}{4\pi^2 i}\frac{1}{x^2}.$$

in the coordinate representation. For spin-1/2 particles the Green's function is given by

$$G^{(1/2)}(x) = \int \frac{\hat{p}}{p^2}e^{ipx}\frac{d^4p}{(2\pi)^4} = -i\gamma_\nu\frac{\partial G^{(0)}}{\partial x_\nu} = \frac{\hat{x}}{2\pi^2 x^4}.$$

In the mixed representation, we obtain the following expression, which was utilized above:

$$G^{(0)}(R,\omega) = \int G^{(0)}(R,t)\,e^{i\omega t}dt = -\frac{1}{4\pi R}e^{i|\omega|R}.$$

To first order in e^2 the polarization operator for spin-1/2 particles is given by

$$\Pi_{\mu\nu}^{(1/2)}(x) = 4\pi e^2\,\mathrm{Sp}\frac{\gamma_\mu\hat{x}\gamma_\nu\hat{x}}{4\pi^4 x^8} = \frac{4e^2}{\pi^3}\frac{2x_\mu x_\nu - x^2\delta_{\mu\nu}}{x^8}.$$

It is not difficult to verify that, in contrast to the momentum representation, $\Pi_{\mu\nu}^{(1/2)}(x)$ has a gauge-invariant form, that is,

$$\partial\Pi_{\mu\nu}^{(1/2)}/\partial x_\nu = 0.$$

In the case of spin-zero particles the vertices are $2p_\mu - k_\mu$ and $2p_\nu - k_\nu$, and by Fourier transformation of $\Pi_{\mu\nu}^{(0)}(k)$ one can easily obtain

$$\Pi_{\mu\nu}^{(0)} = 2\{G^{(0)}(x)\partial_\mu\partial_\nu G^{(0)}(x) - \partial_\mu G^{(0)}(x)\partial_\nu G^{(0)}(x)\}4\pi e^2 = \frac{e^2}{\pi^3}\frac{2x_\mu x_\nu - x^2\delta_{\mu\nu}}{x^8},$$

that is, we again obtain a gauge-invariant expression, which differs from the polarization operator for spin-1/2 particles by a multiplicative factor of 4. In the mixed representation we obtain

$$\Pi_{44}^{(0)} = \frac{e^2}{\pi^3}\int\frac{(t^2+R^2)e^{i\omega t}}{(t+R)^4(t-R)^4}dt = \frac{e^2}{4\pi^3}\frac{1}{R^5}, \quad \omega \to 0,$$

which agrees with the expression obtained in the text.

C. The Green's Function in a Homogeneous Electric Field

Let us consider the Green's function of the equation

$$\Delta\Psi + (\omega - eEz)^2\Psi = 0$$

in the two limiting cases: $R^2eE \gg 1$ and $R^2eE \ll 1$ (where $R = |\mathbf{r} - \mathbf{r}'|$). After making the substitution

$$G(\mathbf{r}, \mathbf{r}') = \int \frac{d^2\varkappa}{(2\pi)^2} e^{i\varkappa\rho} G_\varkappa(z, z').$$

where $\rho = \{x - x', y - y'\}$ and changing the variable z to $t = (eE)^{1/2}z - \omega/(eE)^{1/2}$, the equation for the Green's function takes the form of the differential equation for parabolic cylinder functions:

$$\frac{d^2G_\varkappa}{dt^2} + (t^2 - \lambda)G_\varkappa = \delta(t - t'), \quad \lambda = \frac{\varkappa^2}{eE}. \qquad (A.1)$$

The properties of the corresponding homogeneous equation are investigated in detail in[25]. The solution of Eq. (A.1) can be written down in terms of the two independent solutions of the homogeneous equation:

$$G_\varkappa(t, t') = f_1(t_>)f_2(t_<)/2i, \qquad (A.2)$$

where

$$t_{\gtrless} = {}^1/_2(t + t') \pm {}^1/_2|t - t'|,$$
$$f_{1,2} = (1 \mp i)^{-1/2 \pm i\lambda/2} D_{-1/2 \mp i\lambda/2}((1 \mp i)t) \approx$$
$$\approx t^{-1/2} \exp\{\pm {}^1/_2 i(t^2 - \lambda \ln t)\}, \quad t > |\lambda|,$$

and $D_p(z)$ is the parabolic cylinder function. Thus,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi}\int_0^\infty d\varkappa J_0(\varkappa\rho)\varkappa G_\varkappa(z, z')$$

$$= \frac{2^{1/2}}{4\pi i E^{1/2}}\int_0^\infty d\varkappa J_0(\varkappa\rho)\varkappa e^{\pi\lambda/4} D_{-1/2 - i\lambda/2}[(1 - i)t_>] D_{-1/2 + i\lambda/2}[(1 + i)t_<],$$

where J_0 is the Bessel function. Due to the presence of the factor $e^{\pi\lambda/4}$ in the last integral, large values of λ are important in the functions D. In addition, we shall be interested in the region $t^2 \gg 1$. (For $R^2eE \gg 1$, t^2 is very large compared to unity since $z \sim R$ in Eq. (A.1), but in the case when $R^2eE \ll 1$, t^2 is still large due to the additive term ω^2/eE).

Thus, we need to know the asymptotic form of the parabolic cylinder function $D_p(z)$ for $|p| \gg 1$ and $|z| \gg 1$. In our case this asymptotic behavior is described by the following expression:[25]

$$D_{-1/2 \pm i\lambda/2}((1 \pm i)t) = 2^{-1/4 \pm i\lambda/4}\left(\pm\frac{i\lambda}{4}\right)^{\pm i\lambda/4} e^{\mp i\pi/8 \mp i\lambda/4} \frac{e^{\mp i\xi}}{(t^2 - \lambda)^{1/4}},$$

$$\xi = \frac{t}{2}(t^2 - \lambda)^{1/2} - \frac{\lambda}{4}\ln\frac{[t + (t^2 - \lambda)^{1/2}]^2}{\lambda}.$$

By substituting this asymptotic expression into Eq. (A.2), we obtain the following result for $G_\varkappa(t, t')$:

$$G_\varkappa(t, t') = -\frac{1}{2i}\frac{\exp\{i(\xi(t_>) - \xi(t_<))\}}{(t_>^2 - \lambda)^{1/4}(t_<^2 - \lambda)^{1/4}}(eE)^{-1/2}.$$

With the aid of the mean-value theorem we obtain

$$G_\varkappa(z, z') = -\frac{1}{2i}\frac{\exp\{i(eEt_1^2 - \varkappa^2)^{1/2}|z - z'|\}}{(t^2eE - \varkappa^2)^{1/4}(t'^2eE - \varkappa^2)^{1/4}};$$

$$t_1 \in [t_>; t_<], \quad (eE)^{1/2}t_1 = eEz_1 - \omega = CeER - \omega, \quad C \sim 1$$

or

$$G(\mathbf{r}; \mathbf{r}') = -\frac{\exp\{i|CR^2Ee - \omega R|\}}{4\pi R}$$

both for $R^2eE \ll 1$ and for $R^2eE \gg 1$, which therefore confirms the localizability property of G in a strong field—the property which has been utilized in the main text of this article.

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186