Contribution to the Nonlinear Theory of Kinetic Instability of an Electron Beam in Plasma

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A nonlinear theory is developed for the excitation of a monochromatic wave during the interaction between an electron beam with a large thermal spread and plasma. By a suitable choice of the dimensionless variables, the set of equations describing the excitation of the wave by resonant beam particles and their motion under the action of this wave is reduced to a universal set, i.e., a set which is independent of the beam and plasma parameters. Numerical methods are used to solve the set of equations obtained in this way. The time dependence of the wave amplitude under nonlinear conditions is established and the characteristic trajectories of resonant particles on the phase plane are investigated.

1. It is known that the nonlinear restriction on the Landau damping of a plasma wave is connected with the trapping of resonant plasma particles in the potential well produced by the wave. The trapped particles execute phase oscillations relative to the wave. However, the mean energy transfer between the particles and the wave during one period is zero, and the wave amplitude becomes an oscillating function of time at a frequency of the order of the oscillation frequency of the particles in the wave, i.e. $\Omega = k\sqrt{e\varphi_0/m}$, where φ_0 is the amplitude of the potential and k the wave number. The phase mixing of the trapped particles due to the dependence of the oscillation period in the well on the particle energy leads to damping of the oscillations and to the establishment of a wave with time-independent amplitude. These qualitative features of the interaction between a monochromatic wave and plasma were elucidated by Mazitov,[1] Al'tshul' and Karpman,[2] and O'Neil,[3] who considered the nonlinear damping of a plasma wave of sufficiently large amplitude $\Omega \gg \gamma_L$, where γ_L is the linear Landau damping rate. It was found that, in this case, it was possible to obtain an approximate analytic solution of the problem by considering first the motion of the resonant particles in a wave of constant amplitude, and then taking into account the reaction of these particles on the wave.

Similar nonlinear effects should also occur during the excitation of a monochromatic plasma wave by an electron beam. In such problems, however, we do not have the small parameter $\gamma_L/\Omega \ll 1$. During the linear stage of the instability, when the wave amplitude is sufficiently small, we have $\gamma_L \gg 1$. Subsequently, however, the amplitude reaches values for which resonant particles with velocities $|v - \omega/k| \sim \gamma_L/k$, which excite the wave, are trapped by the potential well and $\gamma_L/\Omega \sim 1$. This is why the solution of the problem of the excitation of a monochromatic plasma wave can be obtained only by numerical methods. A problem of this kind was solved for the case of the instability of a monoenergetic beam in plasma in[4-6]. In the present paper we shall consider the kinetic instability which arises during the interaction between a beam with a large thermal spread and plasma.[1)]

2. The instability which we are considering is the inverse of the Landau damping effect during the presence of a beam in plasma. The linear growth rate during this instability is of the form

$$\gamma_L = \frac{2\pi^2 e^2}{mk^2}\omega_p \frac{\partial f_0}{\partial v} \quad (v = v_{ph}) \tag{1}$$

where $f_0(v)$ is the equilibrium distribution function for the beam in plasma, $v_{ph} = \omega_p/k$ is the phase velocity of the wave, and the wave frequency ω_p is equal to the Langmuir frequency. We shall suppose that the spectrum of the plasma oscillations is such that the frequency density is low and there is only one harmonic of the spectrum in the phase velocity interval for which $\partial f_0/\partial v_{ph} > 0$. It is found that the excitation of the monochromatic wave occurs under these conditions. This can also be achieved by the initial modulation of the beam with an amplitude exceeding the fluctuation amplitude. For low beam densities, when $\gamma_L \ll \omega_p$, the amplitude of the wave excited by the beam is also sufficiently small, $e\varphi_0 \ll mv_{ph}^2$. For such amplitudes the oscillations of the thermal plasma particles which determine the wave dispersion are found to remain linear. Essentially nonlinear effects appear only in the motion of the resonant particles with velocities close to the phase velocity of the wave, which determine the time dependence of the wave amplitude. The electric field in the wave can then be sought in the form

$$E(t, x) = E(t)\sin(kx - \omega_p t). \tag{2}$$

To determine the function E(t) we shall use the following set of equations which describes the motion of resonant particles in the field of the wave and the change in the wave amplitude due to the interaction with these particles:

$$\frac{dv'}{dt} = -\frac{e}{m}E(t)\sin k\xi, \quad \frac{d\xi}{dt} = v', \tag{3}$$

$$\frac{1}{4\pi}E(t)\frac{dE}{dt} = -\overline{j^{res}E(t,x)} = eE(t)$$

$$\times \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} d\xi_0 \int_{-v_{0m}}^{v_{0m}} dv_0'(v_\Phi + v')\sin k\xi f_0(v_{ph} + v_0'). \tag{4}$$

These equations are referred to the frame attached to the wave, i.e. $v' = v - v_{ph}$, $\xi = x - v_{ph}t$; $j^{res} = -e\int dv'(v_{ph} + v')f$ is the current of resonant particles, the bar represent averaging over one wavelength λ, and ξ_0 and v_0' are the initial values of the coordinate and velocity of the particle which occupies the point ξ, v' at time t in phase space. The integral on the right-hand

1)The excitation of a monochromatic wave by an electron beam with a large thermal spread has also been treated numerically by Fried et al.[7]

side of Eq. (4) is evaluated within the finite interval of the resonant-particle velocities $-v_0^m < v' < v_0^m$. In deriving Eq. (4), we used the Liouville theorem on the conservation of the phase volume, $d\xi dv' = d\xi_0 dv_0'$, and the condition that the distribution function remains constant on the particle trajectories, $f(t, \xi, v) = f_0(v_{ph} + v_0')$ [the initial perturbation of the equilibrium distribution function in Eq. (4) can be neglected].

It is important to note that the change in the field phase due to the beam particles is neglected in Eqs. (3) and (4), and this is valid only for beams with sufficient thermal spread, so that the width Δv of the distribution function satisfies the condition

$$\Delta v \gg \sqrt{e\varphi_0/m}. \qquad (5)$$

It is well known that the main contribution to the change in the field phase is due to beam particles with sufficiently high velocities $v' \approx \Delta v$. Thus, in the linear theory, the phase change is proportional to

$$\int \frac{\partial f_0}{\partial v}(v' + v_{\rm ph})\frac{\partial v'}{v'}.$$

When condition (5) is satisfied the oscillations of particles with velocities $v' \approx \Delta v$ remain linear. The field phase due to the beam then varies linearly with time, and this leads merely to a negligibly small ($\sim\gamma_L/\omega$) change in the frequency and phase velocity.

When the beam spread is not too large, i.e. $\Delta v \sim \sqrt{e\varphi_0/m}$, so that the wave substantially disturbs the motion of all the beam particles, we must take into account not only the change in the amplitude but also in the field phase during the instability. This case is discussed in[7].

During linearization with respect to the amplitude of the oscillations, Eqs. (3) and (4) lead to an exponential increase in the field amplitude with time. In fact, if we integrate the equations of motion in this approximation, we obtain

$$\xi = \xi_0 + v_0't - \frac{e}{2im}E(t)\left[\frac{\exp ik(\xi_0 + v_0't)}{(ikv_0' + \gamma)^2} - \text{c.c.}\right]. \qquad (6)$$

This form is valid when $E(t) \gg E(0)$, in which case the first term which depends on the initial field amplitude can be neglected in $\xi(t)$.

Substituting the resulting expression in Eq. (4) and confining our attention to the approximation which is linear in $\delta\xi = \xi - \xi_0 - v_0't$, we have, after integration with respect to ξ_0,

$$\frac{dE}{dt} = \frac{4\pi e^2}{m}k\omega_p\gamma\int_{-v_0^m}^{v_0^m}\frac{dv_0'v_0'f_0(v_{\rm ph}+v_0')}{(k^2v_0'^2+\gamma^2)^2}E, \qquad (7)$$

i.e.

$$E(t) \sim e^{\gamma t}, \qquad (8)$$

and the growth rate is given by

$$\gamma = \gamma_L\frac{2}{\pi}\left\{\text{arctg}\frac{kv_0^m}{\gamma} - \frac{kv_0^m/\gamma}{1 + k^2v_0^{m2}/\gamma^2}\right\}. \qquad (8')$$

When $v_0^m \to \infty$, the growth rate γ tends to γ_L. Particles with velocities substantially greater than γ/k provide an appreciable contribution to the growth rate (the quantity $\gamma_L - \gamma$ falls relatively slowly, i.e., $\sim\gamma/kv_0^m$, as v_0^m increases). As a result, the interval Δv^{res} of resonant-particle velocities turns out to be numerically large in comparison with γ/k, i.e., $\Delta v^{res} \sim (3-5)\gamma/k$.

Equation (8), which follows from the linear theory, is valid for low enough amplitudes when the width of the region in which the particles are captured by the wave is much less than the resonant particle velocity interval, i.e. $\sqrt{e\varphi_0/m} \ll \Delta v^{res}$. As time increases, the amplitude reaches the value

$$\varphi_0 \sim \frac{m}{e}(\Delta v^{res})^2, \qquad (9)$$

for which a substantial part of the resonant particles is found to be trapped in the potential well produced by the wave. The oscillations then cease to grow and, owing to the presence of the captured particles, the amplitude exhibits oscillations which are damped as a result of the phase mixing of these particles. We note that a formula similar to that given by Eq. (9) was reported earlier in[8,9]. As noted above, the absence of a small parameter from Eqs. (3) and (4) means that they can be solved only by numerical methods. In terms of the dimensionless variables

$$\nu = \frac{1}{2\pi}\frac{kv'}{\gamma_L},\ \zeta = \frac{1}{2\pi}k\xi,\ \tau = \gamma_L t,\ \mathscr{E} = \frac{eEk}{m\gamma_L^2} \qquad (10)$$

(the last relation corresponds to $\varphi_0 = Em\gamma_L^2/ek^2$), we can rewrite Eqs. (3) and (4) in the form

$$\frac{d\nu}{d\tau} = -\frac{\mathscr{E}}{2\pi}\sin 2\pi\zeta,\quad \frac{d\zeta}{d\tau} = \nu, \qquad (11)$$

$$\frac{d\mathscr{E}}{d\tau} = 16\pi\int_0^{1/2}d\zeta_0\int_{-\nu_0^m}^{\nu_0^m}d\nu_0\,\nu_0\sin 2\pi\zeta(\tau, \zeta_0, \nu_0). \qquad (12)$$

In deriving this set of equations, we used the fact that, since $\Delta v \gg \Delta v^{res}$, the distribution for the resonant particles can be written in the form

$$f_0(v'+v_{\rm ph}) = f_0(v_{\rm ph}) + v'\partial f_0/\partial v_{\rm ph} \qquad (13)$$

and we have eliminated the integral with respect to $\zeta_0 < 0$ by using the conditions

$$\nu(-\zeta_0, -\nu_0, \tau) = -\nu(\zeta_0, \nu_0, \tau),$$
$$\zeta(-\zeta_0, -\nu_0, \tau) = -\zeta(\zeta_0, \nu_0, \tau). \qquad (14)$$

Accordingly, the set of equations given by Eqs. (11) and (12) has a useful form which does not depend on any of the parameters and, therefore, the solution of the problem of the excitation of a wave by an electron beam with large thermal spread reduces to the determination by numerical methods of the single dimensionless function $E(\tau)$.[2)]

Equations (11) and (12) were integrated on a computer using the Runge-Kutta method. We have processed 4000 resonant-particle trajectories for which the initial coordinates were varied within the range $0 < \zeta_0 < 0.5$ with a step of $\Delta\zeta_0 = 1/14$ and velocities within the range $-2 < \nu_0 < 2$ with a step of $\Delta\nu_0 = 1/125$ (for a given total number of particles this initial ensemble was found to be optimal and ensured that the amplitude was calculated to better than 5%). Figures 1 and 2 show $\mathscr{E}$ and $\gamma = \mathscr{E}^{-1}d\mathscr{E}/dt$ as functions of τ for the case $\mathscr{E}(0) = 0.01$, which were obtained as a result of the numerical integration of Eqs. (11) and (12). For small τ, when the field amplitude is comparable with the initial amplitude, the growth rate oscillates rapidly with time. This is followed by the exponential increase in the amplitude with the growth rate $\gamma = 0.90\gamma_L$, which corresponds to the linear theory (for comparison, we note that, for the chosen value of the maximum velocity of the resonant particles $v_0^m = 4\pi\gamma_L/k$, the growth rate given by Eq. (8)

2)The universal form was obtained only for $\Delta v \gg \Delta v^{res}$, and if this is valid we can use the representation given by Eq. (13).

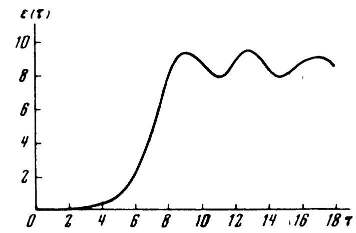

FIG. 1

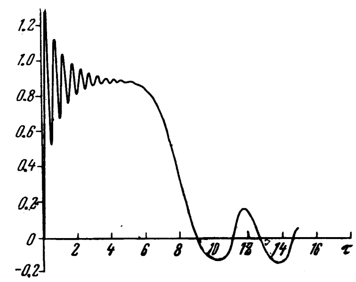

FIG. 2. $\gamma(\tau)/\gamma_L$ as a function of τ.

is $\gamma = 0.89\gamma_L$). The increase in the amplitude is then slowed down and, when $\tau = 8.8$, the field amplitude reaches the maximum value $\mathscr{E}^{(1)}_{max} = 9.4$. Oscillations in $\mathscr{E}(\tau)$ appear for large τ and the swing of the oscillations in the case of a beam with a large thermal spread is small: $\Delta\mathscr{E} \sim (1/10 - 1/7)\mathscr{E}_{max}$.[3)]

As the amplitude increases, a substantial part of the resonant beam particles is trapped in the potential well produced by the wave. Figures 3 and 4 show characteristic trajectories of the resonant particles on the phase plane. The numbers marked against these trajectories correspond to values of time τ at which the particle is located at the given point on the phase plane, and the initial positions are indicated by the asterisks. Figure 3 shows the phase trajectories of particles with $\nu_0 = 0$. For sufficiently large τ these particles assume considerable velocities in the wave field and travel on closed trajectories corresponding to trapped particles. Particles with large ν_0, whose trajectories are shown in Fig. 4, are found to escape for low τ and move on trajectories close to the unperturbed trajectories $\zeta = \zeta_0 + \nu_0\tau$. However, as the field amplitude increases, they are also trapped by the wave, provided only that $|\nu_0| \lesssim 0.6$, which corresponds to $|v_0| \lesssim 3.5\gamma_L/k$. Thus, for example, a particle with initial coordinates $\nu_0 = -0.2$, $\zeta_0 = 0.1$, which moves along the unperturbed trajectory for $\tau = 3$, cuts the boundary of the region which we are considering, $\zeta = -0.5$, and the subsequent trajectory, shown in Fig. 4, corresponds to a symmetrically located particle with $\zeta_0 = 1.1$ (because of the periodicity of the trajectories of escaping particles, we have shown in Fig. 4 only the part of these trajectories corresponding to $-0.5 < \zeta < 0.5$). When $\tau \geq 5$, the trajectory is greatly disturbed by the wave field and is found to be closed. A particle with initial coordinates $\zeta_0 = 0.26$, $\nu_0 = -0.6$ travels in a similar way with the only difference that, in this case, the trajectory cuts the boundary of the region $\zeta = -0.5$ four times (for $\tau = 1.4$, 2.8, 4.5, and 6.8) before it is trapped by the wave field. Subsequently, when $\tau = 10.5$ and when the field amplitude is at a minimum, these particles leave the potential well, but are then again trapped by the wave and execute motions on a closed trajectory.

The phase mixing of resonant particles is illustrated in Figs. 5 and 6. These figures show lines corresponding to $v_0' = \text{const}$, i.e. $f = \text{const}$, on the phase plane at different instants of time: i.e. $\tau = 5.8$ in Fig. 5 (linear stage) and $\tau = 8.8$ in Fig. 6 (maximum field). The rotation of the particles on the phase plane ensures that the $v' = 0$ line (thick line in Figs. 5 and 6) eventually takes the form of a complicated spiral. The multiply-connected region bounded by this spiral contains spirals corresponding to $v_0' > 0$ on which $f > f(0)$ (thin lines) and spirals on which $v_0' < 0$ and $f < f(0)$ (broken lines). It is clear that for large τ the distribution function for $\zeta = \text{const}$ is a highly oscillating function of velocity. The mixing of resonant particles on the phase plane should lead to the damping of the oscillations of $\mathscr{E}(\tau)$ shown in Fig. 1. However, this process cannot be in-

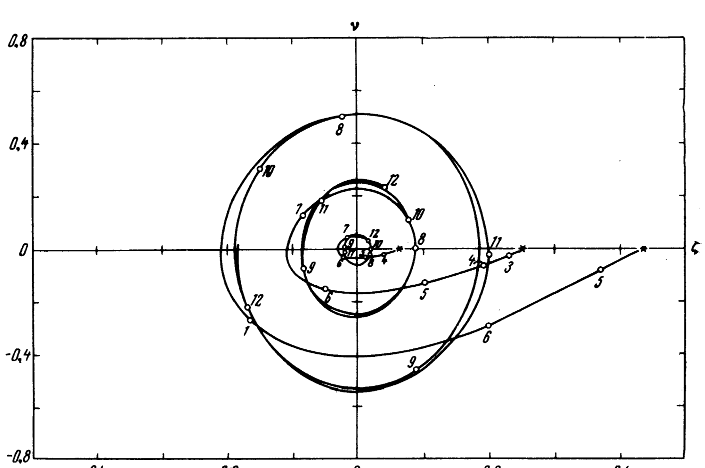

FIG. 3

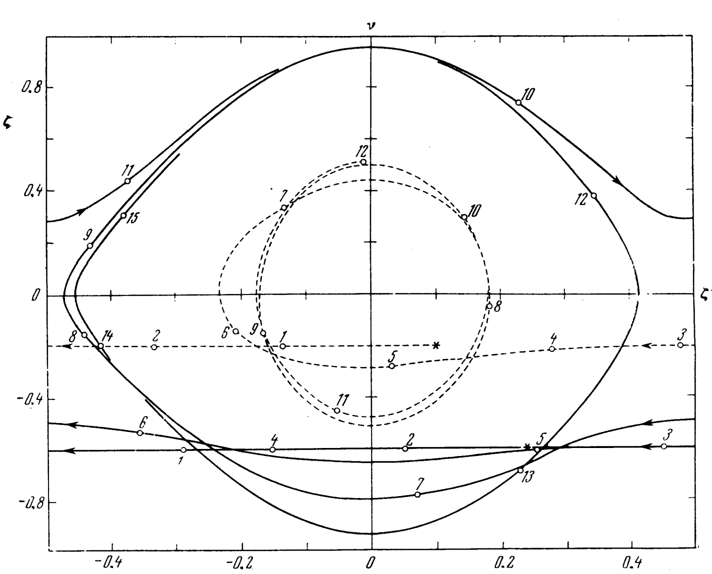

FIG. 4

[3)]Preliminary results of the integration of Eqs. (11) and (12) were reported earlier in[4]. The use of the Euler method in[4] during the integration of Eqs. (11) and (12) led to considerable uncertainties. As a result, the function $\xi(\tau)$ reported in[4] was close to the true function only for $\tau \leqslant 10$. For large τ the discrepancy became very substantial, and this led in[4] to the erroneous conclusion that the mean value about which the amplitude oscillated decreased with time. This was brought to our attention by R. Z. Sagdeev.

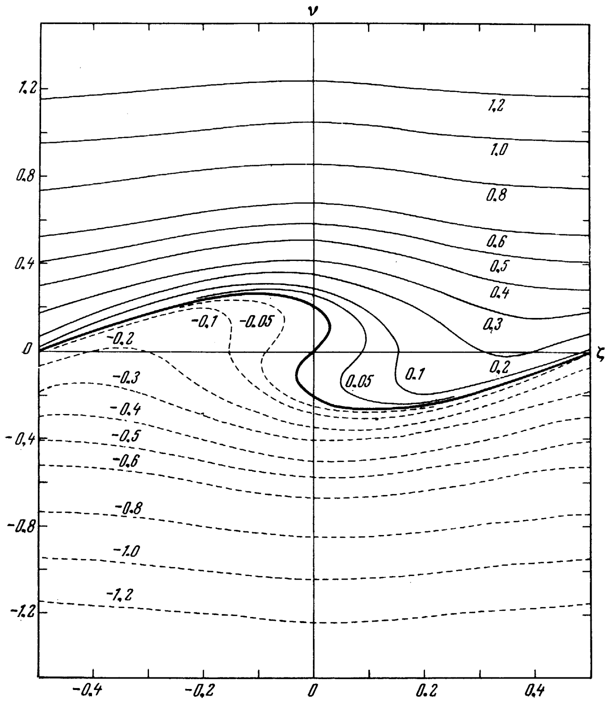

FIG. 5

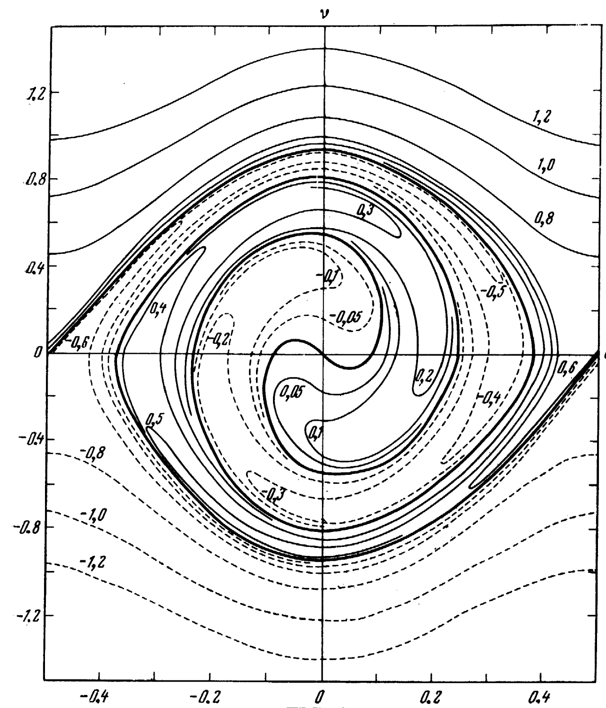

FIG. 6

vestigated on the computer because of the accumulation of relative errors.

We note in conclusion that the energy of the monochromatic plasma wave excited by a beam with a large thermal spread is

$$E^2 / 4\pi = \mathscr{E}^2 n_1 mv\Delta v(\gamma_L / k\Delta v)^3$$

and is usually much less than the beam energy ($\gamma_L/k\Delta v \ll 1$), since only a small part of the beam participates in the wave excitation.

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