The First Integral of the Kinetic Equation for Dissipative Systems in the Presence of a Magnetic Field

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As is well known, the phase density is not a first integral of the motion in dissipative systems, i.e., it is not conserved along the characteristics. The goal of the present work was to find the first integral of the kinetic equation for dissipative systems. The first integral of the kinetic equation was found for a system of noninteracting particles in an arbitrary magnetic field, where the system is losing energy by radiation, ionization of the medium, etc. If the energy is expended in ionizing the medium, then it is also possible to take account of collisions with nuclei leading to the disappearance of particles.

It is well known that a wide circle of physical problems reduce to the solution of the kinetic equation. Here the determination of the first integral of this equation plays an essential role, i.e., the determination of a quantity which is conserved along the characteristics. If the energy of the system is conserved (not being radiated and not being dissipated in the medium), then according to Liouville's theorem the phase density is such a quantity, and the differential equations of the characteristics can be represented in the form of Hamilton's equations, which is a necessary condition for the validity of Liouville's theorem. Thus, by determining the equations of the characteristics and knowing the boundary conditions, we can easily determine the phase density at any arbitrary point in phase space. Such an approach does not encounter any fundamental difficulties if the system consists of a stream of noninteracting and noncolliding particles in an external stationary field. (For example, the flux of galactic cosmic rays in the magnetosphere.) However, in many problems the energy of the system is either being dissipated to the surrounding medium or else is being radiated. As an example one can cite high-energy test particles in a plasma, the corpuscular streams in the upper atmospheres of planets, the streams of relativistic electrons in the (expanding) envelopes of novas and supernovas, etc. In the present work we have been able to obtain a first integral of the kinetic equation, having generalized Liouville's theorem to certain types of dissipative (and radiating) systems in the presence of a stationary external magnetic field.

Let us consider a system of noninteracting charged particles in a stationary magnetic field. Let a certain retarding force **f** act on each particle, where the force depends on the given particle's momentum p and on its spatial position **r** in such a way that it can be represented in the form of a product

$$\mathbf{f}(p, \mathbf{r}) = -\alpha(p)\beta(\mathbf{r})\mathbf{p}/p. \tag{1}$$

This assumption is satisfied when the deceleration of the particles is due to synchrotron radiation or due to ionization losses in the medium. We neglect the scattering by the atoms of the medium which occurs at angles not equal to zero. In this case the particles' equations of motion will differ from Hamilton's equations by the presence of a nonconservative retarding force on the right-hand side

$$\dot{q}_i = \partial H/\partial P_i, \quad \dot{P}_i = -\partial H/\partial q_i + f_i. \tag{2}$$

Here $P_i = p_i + (e/c)A_i$ are the components of the generalized momentum, and **A** is the vector potential of the magnetic field.

We shall assume that the Hamiltonian H, the resistive force **f**, and the phase density $\rho(\mathbf{r}, \mathbf{P})$ do not explicitly depend on the time. It is obvious that under the influence of the retarding forces ρ can have a stationary value only in the presence of a stationary source. In phase space the equation of continuity has the form

$$\sum_{j=1}^{6} \frac{\partial(\rho\dot{x}_j)}{\partial x_j} = 0. \tag{3}$$

The quantities $x_{1,2,3}$ correspond to the coordinates q_i, and the quantities $x_{4,5,6}$ correspond to the momenta P_i. Using these variables one can rewrite Eq. (3) in the following form:

$$\sum_{i=1}^{3}\left(\dot{q}_i\frac{\partial\rho}{\partial q_i} + \dot{P}_i\frac{\partial\rho}{\partial P_i}\right) = -\rho\sum_{i=1}^{3}\left(\frac{\partial\dot{q}_i}{\partial q_i} + \frac{\partial\dot{P}_i}{\partial P_i}\right). \tag{4}$$

Here the total time derivative of the phase density appears on the left-hand side. By substituting $\dot{q}_i$ and $\dot{P}_i$ from Eqs. (2) into the right-hand side, we obtain

$$\frac{d\rho}{dt} = -\rho\sum_{i=1}^{3}\frac{\partial f_i}{\partial P_i}. \tag{5}$$

Since the generalized momentum P_i is linearly related to the usual momentum p_i, one can write down

$$\partial f_i/\partial P_i = \partial f_i/\partial p_i. \tag{6}$$

Substituting expressions (1) and (6) into Eq. (5) we obtain

$$\frac{d\rho}{dt} = \rho\beta(\mathbf{r})\left[\frac{d\alpha}{dp} + (N-1)\frac{\alpha(p)}{p}\right], \tag{7}$$

where N is the number of dimensions of the momentum space under consideration.

If the energy losses occur continuously, then for each particle the momentum is linked with the time t; therefore, in Eq. (7) one can change from the independent variable t to the variable p:

$$\frac{dp}{dt}\frac{d\rho}{dp} = \rho\beta(\mathbf{r})\left[\frac{d\alpha}{dp} + (N-1)\frac{\alpha(p)}{p}\right]. \tag{8}$$

Since the magnetic field does not change the particle energy, no external electric field is present, and the self-consistent field is negligible, one can write down

$$dp / dt = -f(p, \mathbf{r}). \tag{9}$$

Substituting this expression into Eq. (8) and using formula (1), we obtain

$$\frac{d \ln \rho}{dp} = -\left(\frac{d \ln \alpha}{dp} + \frac{N-1}{p}\right). \tag{10}$$

This equation can be integrated by elementary methods:

$$\frac{\rho(\mathbf{p}_1)}{\rho(\mathbf{p}_2)} = \left(\frac{p_2}{p_1}\right)^{N-1} \frac{\alpha(p_2)}{\alpha(p_1)}. \tag{11}$$

The obtained relation also gives a connection between the phase densities at two arbitrary points of the phase trajectory. Thus, under the indicated assumptions the first integral of the kinetic equation is given by the quantity

$$\rho(\mathbf{p}) p^{N-1} \alpha(p).$$

This quantity is conserved along the phase trajectory of a dissipative system in the presence of a magnetic field. In deriving this result, it was assumed that the energy losses occur along all of the particle trajectories. In actual fact this restriction is unessential, since the phase density is conserved on those parts where the retarding force vanishes. Therefore, if two phase points (at each of which $f(p, \mathbf{r}) \neq 0$) are connected by a phase trajectory, then the phase densities at these two points are related, as usual, by relation (11).

Liouville's theorem can also be generalized to that case when the scattered particles, which are losing energy to the medium, disappear as a result of collisions with the atoms of this medium. Similar situations arise in connection with the fragmentation of nuclei, and also during the collisions of electrons with the nuclei of the medium when the electrons lose all of their energy by bremsstrahlung. Here the right-hand side of Eq. (3) will not be equal to zero, but it will be equal to the decrease in the number of particles from a unit volume of phase space per unit time:

$$n(\mathbf{r}) v \sigma,$$

where $n(\mathbf{r})$ is the number of the medium's atoms per unit volume, v is the particle's velocity, and σ is the cross section of the interaction.

In what follows it is important that the energy losses should also be proportional to the density of the medium. This is true if they have the nature of ionization losses. In this case one can write

$$\beta(\mathbf{r}) = m_{\text{med}} n(\mathbf{r}), \tag{12}$$

where m_{med} denotes the mass of an atom of the medium. Then, instead of Eq. (10) we will have

$$\frac{d \ln \rho}{dp} = -\left(\frac{d \ln \alpha}{dp} + \frac{N-1}{p}\right) + \frac{p \sigma(p)}{m_{\text{med}} (m^2 + p^2/c^2)^{1/2} \alpha(p)}. \tag{13}$$

Integrating (13) we obtain

$$\ln \frac{\rho(\mathbf{p}_1) p_1^{N-1} \alpha(p_1)}{\rho(\mathbf{p}_2) p_2^{N-1} \alpha(p_2)} = -\frac{1}{m_{\text{cp}}} \int_{p_1}^{p_2} \frac{p \sigma(p)\, dp}{(m^2 + p^2/c^2)^{1/2} \alpha(p)}. \tag{14}$$

Relation (14) is the analogue of Liouville's theorem for the case when the current is not only retarded by the medium but it also loses particles due to collisions.

One can simplify this formula if σ and α weakly depend on p. Such an assumption is valid for a stream of nuclei having energies greater than 300 to 400 MeV per nucleon. Here the integral on the right-hand side of Eq. (14) can easily be evaluated:

$$\ln \frac{\rho(\mathbf{p}_1) p_1^{N-1} \alpha(p_1)}{\rho(\mathbf{p}_2) p_2^{N-1} \alpha(p_2)} = -\frac{\varepsilon_2 - \varepsilon_1}{\lambda \alpha}. \tag{15}$$

Here ϵ is the relativistic energy of the particles, and $\lambda = A/\sigma$ is the mean free path of the particles. The analog of Liouville's theorem, which we have derived, may turn out to be useful in solving various problems related to the motion of streams of particles in magnetic fields. In particular, the spectra of the heavy nuclei, which are captured from the galactic cosmic radiation as a consequence of ionization losses in the upper atmosphere, was derived[1] with the aid of Eq. (15).

[1]D. Kh. Morozov, Paper at the All-union Conference on the Physics of Cosmic Rays, Tbilisi, 1971 (to be published).

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