Anomalous Skin Effect in a Plasma with a Diffuse Boundary

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Penetration of an electromagnetic wave into a plasma in which the electron concentration is a prescribed function of the coordinates is considered under anomalous skin effect conditions. An integral equation for the electromagnetic field in a plasma is derived for an arbitrary relation between the electron mean free path, field penetration depth and size of the transition region at the boundary. An exact solution of the integral equation is obtained for the case when the electron concentration depends on the coordinate exponentially and the mean free path is infinite. The surface resistance and reflection coefficient are calculated.

P. L. Kapitza[1] advanced the hypothesis that anomalous skin effect takes in a filamentary high-frequency high-pressure gas discharge. However, the boundary of the plasma in the gas discharge is not sharp, so that the usual theory of the anomalous skin effect in metals[2,3] is not directly applicable to a plasma.

We develop here a theory of the anomalous skin effect in a plasma in which the electron concentration n_e is a function of one coordinate x. Such a situation can occur in a gas-discharge plasma of any type, consisting of hot electrons and cold ions whose concentrations as functions of the coordinates are established in accordance with the processes of heat exchange and ambipolar diffusion.

The initial system of equations describing the penetration of the electromagnetic wave into the plasma consists of the wave equation

$$\frac{d^2E_y}{dx^2}=\frac{4\pi i\omega}{c^2}j_y, \tag{1}$$

in which we neglect the displacement current in comparison with the conduction current, and the kinetic equation for the electrons

$$\frac{\partial f}{\partial t}+v_x\frac{\partial f}{\partial x}+\frac{eE_0}{m}\frac{\partial f}{\partial v_x}-\frac{eE_ye^{i\omega t}}{m}\frac{\partial f}{\partial v_y}=-\nu_{\rm eff}(f-f_0). \tag{2}$$

Here f_0 is the electron distribution function in the absence of an alternating field E_y, which we shall assume for concreteness to be Maxwellian:

$$f_0=\left(\frac{m}{2\pi kT_e}\right)^{3/2}n_e(x)\exp\left(-\frac{mv^2}{2kT_e}\right). \tag{3}$$

$E_0(x)$ is the constant (in time) electric field acting on the electrons (including the action exerted by the ions) and leading to the establishment of a specified concentration distribution $n_e(x)$. The potential $\varphi(x)$ of this field is connected with $n_e(x)$ by the Boltzmann distribution formula

$$n_e(x)=n_0\exp(-e\varphi(x)/kT_e). \tag{4}$$

The electromagnetic field $E_y(x)$ will be assumed to be weak. Putting

$$f=f_0+f_1e^{i\omega t}, \tag{5}$$

and neglecting the term proportional to f_1E_y, we obtain

$$(i\omega+\nu_{\rm eff})f_1+v_x\frac{\partial f_1}{\partial x}+\frac{eE_0}{m}\frac{\partial f_1}{\partial v_x}=\frac{eE_y}{m}\frac{\partial f_0}{\partial v_y}. \tag{6}$$

It is expedient to seek a solution of the linearized kinetic equation (6) by the method of characteristics, just as in the case of objects with sharp boundaries[4].

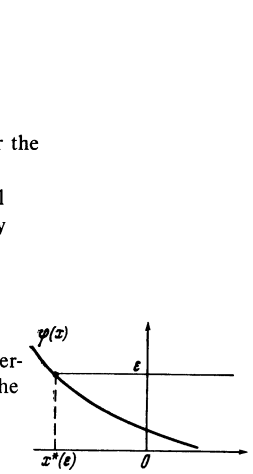

FIG. 1. Motion of electron in an external field with a potential $\varphi(x)$. $x^*(\epsilon)$ is the classical turning point.

It is convenient to introduce in place of f_1 two functions f_+ and f_-, describing electrons moving towards larger and smaller values of the coordinate x, respectively:

$$f_1=\begin{cases}f_+, & v_x>0,\\ f_-, & v_x<0.\end{cases}$$

The functions f_+ and f_- satisfy the equations

$$\frac{\partial f_\pm}{\partial x}-\frac{e}{m|v_x|}\frac{\partial\varphi}{\partial x}\frac{\partial f_\pm}{\partial|v_x|}\pm\frac{i\omega+\nu_{\rm eff}}{|v_x|}f_\pm=\pm\frac{eE_y(x)}{m|v_x|}\frac{\partial f_0}{\partial v_y}. \tag{7}$$

The solution of the characteristic equation

$$dx=-\frac{m|v_x|}{e\partial\varphi/\partial x}d|v_x|$$

is expressed in the form

$$\tfrac{1}{2}mv_x^2+e\varphi(x)=\varepsilon. \tag{8}$$

The characteristic (8) is an integral of the motion (in this case, the law of conservation of the total energy ϵ) of the electron in a field with a potential $\varphi(x)$. The general solution of (7) can now be represented in the form

$$f_\pm(\varepsilon,x)=\exp[\mp\Phi(x,x_0)]\Big\{A_\pm(\varepsilon)\pm\frac{e}{m}\frac{\partial f_0}{\partial v_y}\int_{x_0}^{x}\frac{E_y(x')\exp[\pm\Phi(x',x_0)]dx'}{\sqrt{2(\varepsilon-e\varphi(x'))/m}}\Big\},$$

where

$$\Phi(x_1,x_2)=\int_{x_2}^{x_1}\frac{i\omega+\nu_{\rm eff}}{\sqrt{2(\varepsilon-e\varphi(x))/m}}dx;$$

$A_+(\epsilon)$ and $A_-(\epsilon)$ are arbitrary integration functions, which are determined by the boundary conditions. Assume, for concreteness, that $n_e(x)$ is a monotonic function that tends to zero as $x\to-\infty$. Then, in accordance with formula (4), the potential $\varphi(x)$ is also a monotonic function, and $\varphi(x)$ increases without limit as $x\to-\infty$. In this case the electron motion is infinite only in the $x\to+\infty$ direction (see Fig. 1). Since the electromagnetic field should attenuate in the interior of the plasma, we can state that the distribution function of the electrons moving towards $x\to-\infty$ should tend to Maxwellian as $x\to\infty$, i.e.,

$$f_-=0,\quad x=\infty. \tag{9}$$

The electrons moving towards decreasing x are reflected backwards at the turning point $x^*(\epsilon)$, at which their velocity v_x becomes equal to zero. Thus, we arrive at a condition

$$f_+ = f_-, \quad x = x^*. \tag{10}$$

The conditions (9) and (10) determine uniquely the functions $A_\pm(\epsilon)$. In accordance with formula (8), we obtain the relation

$$e\varphi(x^*) = \varepsilon = e\varphi(x) + {}^1/_2 m v_x^2,$$

which determines the turning point $x^*(x, v_x)$ of the electron having a velocity v_x at the point x. By determining the functions $A_\pm(\epsilon)$ from the boundary conditions (9) and (10), we obtain f_+ and f_-:

$$f_+ = \frac{e}{m}\frac{\partial f_0}{\partial v_y}\left\{\int_{x^*}^{x} \exp[\Phi(x', x)] + \int_{x^*}^{\infty} \exp[-\Phi(x', x^*) - \Phi(x, x^*)]\right\} \times \frac{E_y(x')\,dx'}{[v_x^2 + 2e(\varphi(x) - \varphi(x'))/m]^{1/2}}, \quad v_x > 0; \tag{11}$$

$$f_- = \frac{e}{m}\frac{\partial f_0}{\partial v_y}\int_x^\infty \frac{E_y(x')\exp[-\Phi(x', x)]\,dx'}{[v_x^2 + 2e(\varphi(x) - \varphi(x'))/m]^{1/2}}, \quad v_x < 0. \tag{12}$$

Substituting the distribution functions (11) and (12) in the expression for the current density

$$j_y(x) = -e\int v_y f_1 dv = -e\int_{-\infty}^{+\infty} dv_z \int_{-\infty}^{+\infty} v_y dv_y \int_0^\infty dv_x (f_+ + f_-),$$

and reversing the order of integration with respect to v_x and x′, we obtain the connection between the current density of the plasma and the electromagnetic field

$$j_y(x) = \frac{e^2 n_0}{\sqrt{\pi}\, m\bar{v}}\left\{\int_{-\infty}^{x} E_y(x') G(x, x')\,dx' + \int_x^\infty E_y(x') G(x', x)\,dx'\right\}, \tag{13}$$

where

$$G(x', x) = \exp(-e\varphi(x)/kT_e) \times \int_0^\infty \exp\left\{-\frac{m v_x^2}{2kT_e}\right\} \frac{e^{-\Phi(x', x)} + e^{-\Phi(x', x^*) - \Phi(x, x^*)}}{[v_x^2 + 2e(\varphi(x) - \varphi(x'))/m]^{1/2}}\, dv_x \tag{14}$$

is the conductivity kernel; $\bar{v} = \sqrt{2kT_e/m}$ is the average thermal velocity of the electrons. Substituting the current density (13) into the wave equation (1), we obtain an integro-differential equation for the electromagnetic field in the plasma:

$$\frac{d^2E_y(x)}{dx^2} = i\frac{\omega_0^2\omega}{\sqrt{\pi}\, c^2\bar{v}}\left\{\int_{-\infty}^{x} E_y(x') G(x, x')\,dx' + \int_x^\infty E_y(x') G(x', x)\,dx'\right\}. \tag{15}$$

Here $\omega_0 = \sqrt{4\pi e^2 n_0/m}$ is the plasma frequency of the electrons. Equation (15) makes it possible, in principle, to find the field in the plasma for an arbitrary ratio of the electron mean free path l, the dimension a of the region in which the electron concentration changes significantly, and the depth of penetration δ of the field.

If the width of the transition region on the boundary is small compared with the wavelength outside the plasma, $ka = \omega a/c \ll 1$, then the fields outside the plasma and inside the plasma can be determined separately, just as in the case of a sharp boundary. Outside the plasma, at distances that are small compared with the wavelength c/ω but large in comparison with the width of the transition region a, the electron concentration tends to zero, and Eq. (15) goes over into

$$d^2E_y(x)/dx^2 = 0,$$

whence

$$E_y(x) = A(x + B), \quad a \ll |x| \ll c/\omega. \tag{16}$$

The constant A in (16) is a normalization constant and is arbitrary by virtue of the linearity and homogeneity of (15). To solve the external problem it is thus necessary to know only one complex constant B, which is obtained by determining the distribution of the field inside the plasma.

Let us establish a connection between the constant B and the coefficient for the reflection of the wave from the plasma. Outside the plasma, in the region $|x| \gg a$, the electromagnetic field $E_y(x)$ satisfies the wave equation

$$\frac{d^2E_y}{dx^2} + k^2E_y = 0, \quad k = \frac{\omega}{c}.$$

Its general solution can be represented in the form

$$E_y(x) = c(e^{-ikx} - re^{ikx}), \quad |x| \gg a. \tag{17}$$

The coefficient r in (17) is, by definition, the coefficient of reflection of the wave from the plasma. In the region $a \ll |x| \ll 1/k$ the exponentials in (17) can be expanded in series:

$$E_y(x) = -ik(1+r)c\left(x - \frac{1}{ik}\frac{1-r}{1+r}\right), \quad a \ll |x| \ll \frac{1}{k}.$$

Comparing this expression with (16), we get

$$A = -ik(1+r)c, \quad B = -\frac{1}{ik}\frac{1-r}{1+r},$$

whence

$$r = (1 + ikB)/(1 - ikB). \tag{18}$$

To calculate the surface resistance of the plasma under the condition $ka \ll 1$, we can start, as in the case of an equilibrium low-temperature plasma[5], from the usual definition[6] of the surface impedance:

$$Z = -\frac{4\pi i\omega}{c^2}\frac{E_y(x)}{E_y'(x)}, \quad a \ll |x| \ll \frac{1}{k}.$$

The active part of this expression has the physical meaning of the surface resistance. In accordance with (16) we get

$$R = \mathrm{Re}\, Z = \frac{4\pi\omega}{c^2}\,\mathrm{Im}\, B. \tag{19}$$

In the limiting case of a small electron path $l = \bar{v}/|i\omega + \nu_{\mathrm{eff}}|$, the variation of the potential $\varphi(x)$ over the path can be neglected, and we can also put $E_y(x') = E_y(x)$. After calculating the integral of the conductivity kernel

$$\int_{-\infty}^{x} G(x, x')\,dx' + \int_x^\infty G(x', x)\,dx' = \sqrt{\pi}\frac{\bar{v}\exp\{-e\varphi(x)/kT_e\}}{i\omega + \nu_{\mathrm{eff}}}$$

expression (15) becomes an ordinary differential equation

$$\frac{d^2E_y}{dx^2} = \frac{\omega_0^2(x)E_y}{c^2(1 - i\nu_{\mathrm{eff}}/\omega)}, \quad l \ll a, \; l \ll \delta,$$

where $\omega_0(x) = \sqrt{4\pi e^2 n_e(x)/m}$.

In the opposite limiting case of an infinitely long electron path

$$l \gg a, \quad l \gg \delta$$

we can put in (14) $\Phi = 0$, we obtain the following expression for the conductivity kernel:

$$G(x, x') = \exp\left\{-\frac{e}{2kT_e}[\varphi(x) + \varphi(x')]\right\} K_0\left\{\frac{e}{2kT_e}|\varphi(x) - \varphi(x')|\right\}. \tag{20}$$

Here $K_0(x)$ is a Bessel function. Further investigation of Eq. (15) with the kernel (20) requires that a concrete form of the potential $\varphi(x)$ be specified.

Let us consider the case when the potential depends linearly on the coordinate, $\varphi(x) = -E_0 x$. Then the electron concentration depends on x exponentially[1]:

$$n_e(x) = n_0 \exp(x/a), \quad a = kT_e/eE_0.$$

Changing over to new variables ξ and $f(\xi)$ defined by

$$x = a\left(\xi - \ln\frac{\omega_0^2\,\omega a^3}{\sqrt{\pi}\,c^2\bar{v}}\right), \quad f(\xi) = E_y(x(\xi)), \tag{21}$$

we reduce Eq. (15) with kernel (20) to the dimensionless form

$$\frac{d^2 f(\xi)}{d\xi^2} = i\int_{-\infty}^{+\infty} f(\xi')\, e^{(\xi+\xi')/2} K_0\left(\frac{|\xi-\xi'|}{2}\right) d\xi'. \tag{22}$$

Its asymptotic solution outside the plasma (as $\xi \to -\infty$) is

$$f(\xi) = \alpha(\xi+\beta), \quad \xi \to -\infty, \tag{23}$$

where β is a constant on the order of unity and α is a normalization constant. Comparing (16), (21), and (23), we get

$$B = a\left(\beta + \ln\frac{\omega_0^2\,\omega a^3}{\sqrt{\pi}\,c^2\bar{v}}\right). \tag{24}$$

To calculate the constant β and to find the functions $f(\xi)$ in explicit form, it is necessary to solve Eq. (22) with a non-difference and non-degenerate kernel. There is no standard method for solving equations of this type in general form. In our case, however, we can find the solution of interest to us by a method first introduced by Hartman and Luttinger[7] and later by Kaner and Makarov[8].

To solve (22) it is convenient to employ the bilateral Laplace transformation

$$F(k) = \int_{-\infty}^{+\infty} f(\xi)\, e^{-k\xi}\, d\xi, \quad f(\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(k)\, e^{k\xi}\, dk, \quad c = \operatorname{Re} k. \tag{25}$$

The real number c can be chosen arbitrarily inside the band in which $F(k)$ is regular. Substituting (25) in (22) and noting that the integral

$$i\int_{-\infty}^{+\infty} e^{(k+1/2)u} K_0\left(\frac{|u|}{2}\right) du = \frac{\pi}{\sqrt{k(k+1)}}$$

converges in the band $-1 < \operatorname{Re} k < 0$, we obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k^2 F(k)\, e^{k\xi}\, dk = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sqrt{k(k+1)}} F(k)\, e^{(k+1)\xi}\, dk \tag{26}$$

In order for the function $f(\xi)$ to have the asymptotic form (23) as $\xi \to -\infty$, it is necessary that the function $F(k)$ have a pole of second order at zero. We seek a function $F(k)$ which is regular in the band $-1 < \operatorname{Re} k < 1$, with the exception of the point $k = 0$, at which $F(k)$ has a pole of second order with a coefficient equal to α:

$$\lim_{k\to 0} k^2 F(k) = \alpha. \tag{27}$$

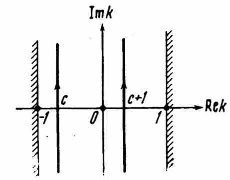

FIG. 2. Band where the function F(k) is regular.

The function $k^2F(k)$ thus has no singularities in the band $-1 < \operatorname{Re} k < 1$, and we can shift the contour in the left side of (26) to the right by unity (see Fig. 2), and then redesignate k by k + 1. As a result we obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{(k+1)^2 F(k+1) - \frac{\pi F(k)}{\sqrt{k(k+1)}}\right\} e^{(k+1)\xi}\, dk = 0. \tag{28}$$

Since Eq. (28) should hold for any ξ, the integrand vanishes identically:

$$(k+1)^2 F(k+1) = \frac{\pi}{\sqrt{k(k+1)}} F(k). \tag{29}$$

We thus obtain for $F(k)$ a homogeneous functional equation. Putting

$$u(k) = \ln(F(k)/\sqrt{k}),$$

we obtain for $u(k)$ the finite-difference equation

$$u(k+1) - u(k) = \ln\pi - 3\ln(k+1).$$

Noting that the right-hand side can be represented in the form

$$(k+1)\ln\pi - k\ln\pi - 3(\ln\Gamma(k+2) - \ln\Gamma(k+1)),$$

where $\Gamma(x)$ is the gamma function, we obtain the general solution of (29) in the form

$$F(k) = \frac{(\pi)^k\sqrt{k}}{\Gamma^3(1+k)}\, g(k).$$

Here $g(k)$ is an arbitrary periodic function with period unity. This function can be chosen such that $F(k)$ has the required analytic properties. It can be shown that accurate to a constant factor this choice is unique. Accurate to a constant factor, the sought function $g(k)$ is given by

$$g(k) = e^{-2\pi i k}\left(\frac{2\pi i}{1 - e^{-2\pi i k}}\right)^{5/2}.$$

Choosing the constant factor in accordance with the condition (27), we obtain the sought function $F(k)$:

$$F(k) = \alpha\frac{(\pi)^k\sqrt{k}}{\Gamma^3(1+k)}\, e^{-2\pi i k}\left(\frac{2\pi i}{1 - e^{-2\pi i k}}\right)^{5/2}. \tag{30}$$

The field distribution in the plasma can now be represented in the form of a contour integral

$$f(\xi) = \frac{\alpha}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(\pi)^k\sqrt{k}}{\Gamma^3(1+k)}\left(\frac{2\pi i}{1 - e^{-2\pi i k}}\right)^{5/2} e^{-2\pi i k + k\xi}\, dk. \tag{31}$$

As $\xi \to -\infty$, the main contribution to the integral (31) is made by the pole at the point $k = 0$. Expanding the function (30) in Laurent series in the vicinity of the point $k = 0$

$$F(k) = \alpha\left(\frac{1}{k^2} + \frac{\ln\pi + 3C + \pi i/2}{k} + \ldots\right)$$

($C = 0.577\ldots$ is Euler's constant), we obtain in accordance with the residue theorem

$$f(\xi) = \alpha(\xi + \ln\pi + 3C + {}^1\!/_2 i\pi), \quad \xi \to -\infty,$$

and thus

$$\beta = \ln\pi + 3C + i\pi/2. \tag{32}$$

[1]Of course, the electron density in all of space cannot be described by the indicated formula. It is natural to assume that this formula gives the asymptotic behavior of $n_e(x)$ as $x \to -\infty$, and $n_e(x)$ tends to a constant limit n as $x \to +\infty$. In this case the formulas obtained by us are valid if $a \gg \delta$, where $\delta = (c^2 vm/4\pi e^2\tilde{n}\omega)^{1/3}$.

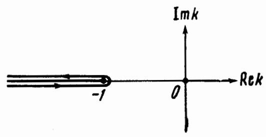

FIG. 3. Integration contour suitable for the calculation of the integral (31) as $\xi \to \infty$.

Comparing (18), (19), (24), and (32) we obtain the reflection coefficient r and the surface resistance R:

$$r = 1 + 2\frac{\omega a}{c}\left\{-\frac{\pi}{2} + i\left(\ln\frac{\sqrt{\pi}\,\omega_0^2\omega a^3}{c^2\bar{v}} + 3C\right)\right\}, \quad \frac{\omega a}{c} \ll 1;$$
$$R = 2\pi^2\omega a / c^2 = 2\pi^2\cdot 10^{-9}\,\omega a \text{ [ohms]}.$$

Thus, the effective depth of penetration of the field in the case of an exponentially increasing electron concentration is of the order of a, which is the reciprocal of the argument of the exponential, and does not depend on the frequency.

We obtain now the asymptotic expression for the field in the plasma as $\xi \to \infty$. The main contribution to the integral (31) is made in this case by the small vicinity of the branch point k = −1. Shifting the integration contour, as shown in Fig. 3, we obtain

$$f(\xi) \approx -\frac{a}{4\pi^{3/2}}\frac{e^{-\xi}}{\xi^{3/2}}, \quad \xi \to \infty.$$

We call attention to the fact that in the case of the strongly anomalous skin effect, when the electron mean free path is infinitely large, the electrons effectively "transport" the field as they move into the interior of the plasma. This leads to a slower damping of the field than in the ordinary skin effect. (In the ordinary skin effect, the field attenuates more rapidly than any exponential in a plasma with an exponentially increasing electron density.)

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[1]P. L. Kapitza, Zh. Eksp. Teor. Fiz. **57**, 1801 (1969) [Sov. Phys.-JETP **30**, 973 (1970)].
[2]G. E. H. Reuter and E. H. Sondheimer, Proc. R. Soc. A **195**, 336 (1948).
[3]V. P. Silin and A. A. Rukhadze, Élektromagnitnye svoǐstva plazmy i plazmopodobnykh sred (Electromagnetic properties of Plasma and Plasma-like Media), Gosatomizdat, 1961, p. 116.
[4]B. É. Meǐerovich, Zh. Eksp. Teor. Fiz. **58**, 324 (1970) [Sov. Phys.-JETP **31**, 175 (1970)]; Zh. Eksp. Teor. Fiz. **59**, 276 (1970) [Sov. Phys.-JETP **32**, 149 (1971)].
[5]B. É. Meǐerovich and L. P. Pitaevskiǐ, Zh. Eksp. Teor. Fiz. **61**, 235 (1971) [Sov. Phys.-JETP **34**, 121 (1972)].
[6]L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred (Electrodynamics of Continuous media), Fizmatgiz, 1957, p. 355 [Addison-Wesley, 1959].
[7]L. E. Hartmann and J. M. Luttinger, Phys. Rev. **151**, 430 (1966).
[8]É. A. Kaner and N. M. Makarov, Zh. Eksp. Teor. Fiz. **57**, 1435 (1969) [Sov. Phys.-JETP **30**, 777 (1970)].

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