Nonlinear Ion Sound in a Fully Ionized Current-Carrying Plasma

B. A. Al'terkop

Institute of High Temperatures, USSR Academy of Sciences

Submitted December 10, 1971

Zh. Eksp. Teor. Fiz. 62, 1760-1763 (May, 1972)

A nonlinear partial differential equation is obtained, under conditions of applicability of the hydrodynamic approximation, and describes the time development of ion-sound stability in a fully ionized plasma with a current. A stationary solution is obtained near the stability threshold. The characteristics of a stationary ion-sound wave are determined at both high and low supercriticalities of the system.

1. It is well known that a homogeneous plasma in a constant electric field is unstable relative to excitation of ion-sound oscillations (see, for example,[1]). In a rarefied plasma, the ion-sound instability, which is due to the Cerenkov effect on electrons, is stabilized as a result of the quasilinear action of the excited oscillations on the electron distribution function.[2,3] In a more dense plasma, the instabilities are excitations with wavelength greater than the free path length of the electrons ($\nu_e > kv_{Te}$). Here, in the case of a fully ionized plasma, the conversion of the energy of directed motion into oscillations is determined by the excitations of the temperature of the electrons, due to the finite electron thermal conductivity.

In the present work, we have studied the nonlinear stage of development of the ion instability of a dense, fully ionized plasma. It is shown that in this case the saturation of the ion-sound instability is determined by the competition between the linear buildup of the unstable mode and the nonlinear damping due to the generation of its higher harmonics, the stability of which is determined by the ion viscosity. The mechanism of generation of the harmonics in the given case can be assumed to be similar to the distortion of the profile of a sound wave of finite amplitude in ordinary gasdynamics.[4]

2. We shall describe the behavior of the nonisothermal ($T_e \gg T_i$), fully ionized plasma in a constant electric field E_0 on the basis of the following set of hydrodynamic equations:

$$\frac{d_i\mathbf{v}_i}{dt} = -\frac{1}{NM}\nabla NT + \eta\Delta\mathbf{v}_i, \qquad (1)$$
$$\frac{\partial N}{\partial t} + \operatorname{div} N\mathbf{v}_i = 0, \quad NT\frac{d_e s}{dt} = \operatorname{div}(\varkappa\nabla T).$$

Here $N = N_e = N_i$ is the density of the plasma; T is the temperature of the electrons, $\eta = (4/3)0.96\,T_i/M\nu_{ii}$ is the specific ion viscosity, $\kappa = N\chi = 3.16NT/m\nu_{ei}$ is the electron thermal conductivity, m and M are the masses of the electron and the ion, s is the specific electron entropy, $d_\alpha/dt = \partial/\partial t + v_\alpha\nabla$, and v_α is the hydrodynamic velocity of the electrons ($\alpha = e$) and ions ($\alpha = i$). The set of equations (1) corresponds to the model of single-liquid hydrodynamics with variable temperature, under conditions when the thermal conductivity of the plasma is determined by the electronic component and the viscosity by the ionic component. We assume for simplicity that the change in the electron temperature takes place mainly because of the high thermal conductivity, which is valid for[1)]

$$k^2\chi \gg \omega, \quad \mathbf{ku} \gg k^2\eta \qquad (2)$$

(ω and $\mathbf{k}$ are the characteristic frequency and wave vector of the excitations, $\mathbf{u} = -e\mathbf{E}_0/0.51m\nu_e$ is the current velocity of the electrons).

The temperature oscillations under conditions (2) can be regarded as small and need be taken into account in (1) only in the leading linear terms. Here the system (1) reduces to the form

$$(\partial/\partial t - \eta_0\Delta + \mathbf{v}\nabla)\mathbf{v} = -c^2(\nabla\tau + \nabla\rho), \qquad (3)$$
$$\partial\rho/\partial t + \mathbf{v}\nabla\rho + \operatorname{div}\mathbf{v} = 0, \quad \mathbf{u}\nabla\rho + \chi_0\Delta\tau = 0.$$

Here

$$\rho = \ln(N/N_0), \quad \tau = T/T_0, \quad c = (T/M)^{1/2};$$

N_0 and T_0 are the equilibrium density and temperature, v is the perturbation of the velocity of the ions, $\eta_0 = \eta(T_0)$, and $\chi_0 = \chi(T_0)$. Eliminating the variables τ and ρ from the system (3) (using the smallness of the dissipative terms), we obtain a nonlinear partial differential equation for the description of the ion-sound instability:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\mathbf{v} = \left[-\frac{c^2}{\chi_0}\mathbf{u}\nabla + \eta_0\frac{\partial}{\partial t}\Delta\right]\mathbf{v} - \frac{1}{2}\nabla\frac{\partial v^2}{\partial t} - \frac{\partial}{\partial t}(\mathbf{v}\nabla)\mathbf{v} - \frac{1}{2}\nabla(\mathbf{v}\nabla)v^2. \qquad (4)$$

The left side of this equation determines the propagation velocity of the wave, the components in the square brackets in the right sign determine the current jump and the damping due to viscosity, and the last three terms describe the nonlinear effects. In the linear approximation, for excitations of the form $e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}}$, the spectrum of ion-sound oscillations follows from (4) ($\omega \to \omega + i\gamma$)

$$\omega^2 = k^2c^2, \quad \gamma = \frac{\omega}{2k^2\chi_0}\mathbf{ku} - \frac{1}{2}k^2\eta_0. \qquad (5)$$

As should be expected,[4] upon satisfaction of conditions (2) the sound is propagated in the plasma at the isothermal velocity $(\partial P/\partial\rho)_T^{1/2} = (T/M)^{1/2}$. From the condition of instability of the mode with the given wave vector

$$\mathbf{ku} > k^2\eta_0 k^2\chi_0/\omega \qquad (6)$$

it follows that the higher harmonics, which appear because of the nonlinear interaction of the waves, will be damped. Thus, the effective mechanism for stabilization of the instability can be the transfer of energy from the linearly unstable harmonics into the higher harmonics, in which the energy is dissipated.

3. We now consider oscillations propagating along the field E_0, which, according to (6) are excited before

1) We note that in ordinary gasdynamics, the condition $\chi \gg \eta$ is satisfied only in a medium with an anomalously high thermal conductivity.[4]

the others. For the description of their development, we use the one-dimensional equation (4), in which we omit the cubic nonlinear component for simplicity. Here we obtain the equation

$$\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} = -\frac{c^2}{\chi_0}u\frac{\partial v}{\partial x} + \eta_0\frac{\partial^3 v}{\partial x^2\,\partial t} - \frac{\partial^2 v^2}{\partial x\,\partial t}, \tag{7}$$

which is similar to that investigated in[5] for the case of ion-sound instability of a weakly ionized plasma. For the determination of the criticality of the system relative to excitation of the ion-sound mode of given wavelength, we introduce the parameter

$$\varepsilon = \gamma_+/\gamma_- - 1, \tag{8}$$

where $\gamma_+ = cu/2\chi_0$, $\gamma_- = k^2\eta_0/2$, the instabilities correspond to $\epsilon > 0$. as is well known,[5] (see also [6]), near the stability threshold, when $\epsilon \ll 1$, the solution of Eq. (7) determines the establishment in the plasma of an ion-sound wave of almost harmonic profile, the square of the amplitude of which changes according to the law

$$A(\tau) = 3\varepsilon A_0 e^{2\tau}[3\varepsilon + A_0(e^{2\tau} - 1)]^{-1}, \tag{9}$$

where $\tau = \gamma t$, $A = aa^*$, $a = kv/\gamma_+$, v is a slowly changing complex amplitude of the unstable harmonic, $A_0 = A(\tau = 0)$. According to Eq. (9), the instability is characterized by a soft excitation regime and, inasmuch as the stationary value of $A(\infty) = 3\epsilon$, the amplitude of the established wave is proportional to the square root of the supercriticality.

In the established state, all the quantities can be regarded as functions of the variable $\xi = kx - \omega t$. Here Eq. (4) reduces to the form

$$\frac{1}{\varepsilon+1}\frac{d^2w}{d\xi^2} - \left(\delta + w - \frac{1}{2}\frac{\gamma_+}{kc}w^2\right)\frac{dw}{d\xi} + w = 0. \tag{10}$$

In this equation, $w = kv/\gamma_+$, $\delta = (kc = \omega)/\gamma_+$. It is seen that the cubic nonlinear term contains the small parameter $\gamma_+/kc \ll 1$ and therefore leads to small, unimportant effects. For example, if $\epsilon \ll 1$, setting

$$w = w_1e^{i\xi} + w_2e^{2i\xi} + \text{c.c.},$$

we get from (10)

$$w_1w_1^* = 3\varepsilon\left(1 + \frac{1}{2}\frac{\gamma_+}{kc}\right), \quad \frac{kc-\omega}{\omega} = \frac{3}{2}\varepsilon\left(\frac{\gamma_+}{kc}\right)^2.$$

The cubic nonlinearity in (10) thus leads to a small shift in the frequency and an insignificant contribution to the amplitude of the wave. If we neglect these small effects, we can reduce Eq. (10) to the form

$$\frac{1}{\varepsilon+1}\frac{d^2w}{d\xi^2} - w\frac{dw}{d\xi} + w = 0, \tag{11}$$

in which periodic solutions correspond to $\delta = 0$[5] or

$$\omega/k = c. \tag{12}$$

The condition (12) determines the propagation velocity of the wave for arbitrary ϵ. Analysis of Eq. (11) shows[5,6] that the wave front becomes steeper with increase in the supercriticality ϵ and for $\epsilon \gg 1$ the stationary wave is characterized by an almost sawtooth profile and the amplitude

$$kv_{max} = \pi\gamma_+ = \pi uc/2\chi_0 = eE_0/(T_0M)^{1/2}. \tag{13}$$

An estimate of the amplitude of the wave for $\epsilon \gg 1$ can easily be obtained if we consider that the rate of growth of the amplitude in this case is $dw/d\xi \sim 1$. Assuming that the maximum value $w = w_{max}$ is obtained within a half period ($T_\xi = 2\pi$), i.e., $dw/d\xi \sim w_{max}/\pi$, whence we obtain $w_{max} \approx \pi$, which agrees with (13).

It should be noted that the result (13) is valid, strictly speaking, only in the one-dimensional case (magnetized plasma or small transverse parameters), since in the opposite case, the possible development of ion-sound turbulence[7] could significantly change the stationary state for $\epsilon > 1$.

The author is grateful to A. A. Galeev and A. A. Rukhadze for discussion of the work.

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Translated by R. T. Beyer
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