Two-Stream Instability in the Presence of Trapped Electrons

S. M. Korshunov
Moscow Physico-technical Institute
Submitted December 13, 1971
Zh. Eksp. Teor. Fiz. 62, 1764–1769 (May, 1972)

Oscillations of the space charge in an electron beam passing through a region with trapped electrons are considered in the geometrical optics approximation. It is shown that the presence of trapped particles leads to the appearance of instabilities in the beam. For perturbation frequencies such that $\omega \ll \Omega$ and $\omega \gg \Omega$, where Ω is the captured electron oscillation frequency, instability of the beam type develops. For perturbation frequencies $\omega = \pm n\Omega$, the instability is due to resonant interaction between the captured electrons and plane wave field.

1. It is known that the presence of trapped particles can exert an appreciable influence on plasma instability. Several investigations[1,2] have been devoted to the kinetic instability of potential oscillations, which arises in toroidal systems as a result of the appearance of particle traps, so that the particles are divided into untrapped and trapped. This instability is apparently the most dangerous in systems with rarefied plasma[1]. A similar situation takes place in adiabatic electron guns used in some microwave devices. The electron current is shaped in them by means of an inhomogeneous magnetic field. Such a gun contains a trap made up of a magnetic mirror and focusing electrodes. We consider in the present paper the hydrodynamic instability, produced as a result of the presence of trapped particles, of a nonrelativistic electron current in an adiabatic gun.

2. We consider the following simplified one-dimensional model. Captured electrons are contained in a specified potential well. Passing through them is a homogeneous monoenergetic beam of electrons. We assume that the energy of the untrapped electrons is high enough to be able to neglect the influence of the well potential on the motion. The shape of the potential well is chosen to be parabolic for simplicity. We assume also the presence of an immobile ionic background that cancels out the space charge. The perturbed motion of the untrapped and the trapped electrons is described by the following linearized system of equations:

$$\frac{\partial F}{\partial t}+v\frac{\partial F}{\partial x}=\frac{e}{m}\frac{\partial\varphi}{\partial x}\frac{\partial F_0}{\partial v}, \tag{1}$$

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}-\Omega^2x\frac{\partial f}{\partial v}=\frac{e}{m}\frac{\partial\varphi}{\partial x}\frac{\partial f_0}{\partial v}, \tag{2}$$

$$\frac{\partial^2\varphi}{\partial x^2}=-4\pi e\int_{-\infty}^{+\infty}f\,dv-4\pi e\int_{-\infty}^{+\infty}F\,dv. \tag{3}$$

Here F_0 and f_0 are the unperturbed distribution functions of the untrapped and the trapped electrons, respectively, F and f are the perturbations of the distribution functions, and Ω is the frequency of the oscillations of the trapped electrons.

We choose the unperturbed electron distribution function of the untrapped electrons in the form of a δ function

$$F_0=N\delta(v-V_1),$$

where N is the density of the untrapped electrons and V_1 is their velocity. The unperturbed distribution function of the trapped electrons, satisfying the equation

$$v\frac{\partial f_0}{\partial x}-\Omega^2x\frac{\partial f_0}{\partial v}=0,$$

should be a function of the integral of motion

$$\tfrac{1}{2}mv^2+\tfrac{1}{2}m\Omega^2x^2=\varepsilon,$$

where ϵ is the total energy of the electron in the well. The unperturbed distribution function of the trapped electrons is chosen therefore in the form

$$f_0=(n\Omega/L^2\pi)\delta(v^2+\Omega^2x^2-V_2^2).$$

Here n is the total number of electrons trapped in the well,

$$V_2=\left(\frac{2\varepsilon_0}{m}\right)^{1/2},\quad L=\frac{2}{\Omega}\left(\frac{2\varepsilon_0}{m}\right)^{1/2},$$

and ϵ_0 is the total energy of the trapped electrons.

Integrating (1) and (2) along the trajectories of the unperturbed motion and taking the constants of the motion $\partial F_0/\partial v$ and $\partial f_0/\partial w$ (here $w = v^2 + \Omega^2x^2$) outside the integral sign, we obtain

$$F=\frac{e}{m}\frac{\partial F_0}{\partial v}\int_{-\infty}^{t}\frac{\partial\varphi}{\partial x}dt',$$

$$f=2\frac{e}{m}\frac{\partial f_0}{\partial w}\int_{-\infty}^{t}v\frac{\partial\varphi}{\partial x}dt'.$$

The unperturbed trajectories of the untrapped and trapped particles are described by the respective equations

$$x(t)-x(t')=v(t-t'),$$

$$x(t)-x(t')=\frac{w^{1/2}}{\Omega}\sin\Omega t-\frac{w^{1/2}}{\Omega}\sin\Omega t'.$$

Representing the potential φ in the form

$$\varphi=\sum_{n=-\infty}^{+\infty}\varphi_n\exp\{-i(\omega+n\Omega)t+ikx\}$$

we obtain in the geometrical-optics approximation ($kL \gg 1$) expressions for the perturbations of the distribution functions

$$F=-\frac{ke}{m}\frac{\partial F_0}{\partial v}\sum_{n=-\infty}^{+\infty}\varphi_n\frac{\exp\{-i(\omega+n\Omega)t+ikx\}}{\omega-kv+n\Omega} \tag{4}$$

$$f=-\frac{2kve}{m}\frac{\partial f_0}{\partial w}\sum_{nrs=-\infty}^{+\infty}\varphi_n\frac{r\xi^{-1}J_s(\xi)J_r(\xi)}{\omega-(r-n)\Omega}\times\exp\{-i(\omega+n\Omega+s\Omega-r\Omega)t+ikx\}, \tag{5}$$

where $\xi = kV_2/\Omega \gg 1$.

Integrating (4) and (5) over the velocities and substituting the obtained expressions in (3), we obtain an infinite system of equations with respect to φ_n:

$$\sum_{n=-\infty}^{+\infty}\varphi_n\left\{\left[1-\frac{\omega_{p1}^2}{(\omega+n\Omega-kV_1)^2}\right]\delta_{nl}-\frac{\omega_{p2}^2}{\Omega}[1+(-1)^{l+n}]\right.$$
$$\left.\times\sum_{s=-\infty}^{+\infty}\frac{s(\xi^{-1}J_s(\xi)J_{s+l-n}(\xi))'}{\omega+n\Omega-s\Omega}\right\}=0,\qquad(6)$$

where $J_s(\xi)$ is a Bessel function, δ_{nl} is the Kronecker symbol, $\omega_{p1}^2 = 4\pi e^2N/m$, $\omega_{p2}^2 = 4e^2n/mL^3$, and the prime denotes differentiation with respect to ξ. In the derivation of (6) it was assumed that $v^2 \gg \Omega^2x^2$. The convergence of the determinant of a system of this type was proved by Ivanov and Murav'ev[3]. This makes it possible to confine ourselves in the derivation of the dispersion equation to a finite number of columns and lines in the determinant. Equating to zero the third-order determinant, we obtain

$$1-\frac{\omega_{p1}^2}{(\omega-kV_1)^2}-\frac{2\omega_{p2}^2}{\Omega}\sum_{s=-\infty}\left(\frac{J_s^2}{\xi}\right)'\frac{s}{\omega-s\Omega}=0.\qquad(7)$$

3. For oscillation frequencies $\omega = n\Omega + \omega_1$, where $|\omega_1| \ll \Omega$, we neglect all the terms of the sum with the exception of $s = n$, and obtain in place of (7) the simpler dispersion equation

$$1-\frac{\omega_{p1}^2}{(kV_1-n\Omega-\omega_1)^2}-\frac{2\omega_{p2}^2}{\Omega}\frac{n}{\omega_1}\left(\frac{J_n^2}{\xi}\right)'=0.\qquad(8)$$

We consider the case when kV_1 is not too close to $n\Omega$. It is easy to solve (8) approximately by using the condition $\Omega \gg \omega_1$. Expanding for this purpose the second term in powers of $\omega/(kV_1 - n\Omega)$ and discarding all terms of order $\omega_1^2/(kV_1 - n\Omega)^2$, we obtain the following quadratic equation for ω_1

$$\frac{2\omega_{p1}^2}{(kV_1-n\Omega)^3}\omega_1^2-\left[1-\frac{\omega_{p1}^2}{(kV_1-n\Omega)^2}\right]\omega_1+\frac{2\omega_{p2}^2}{\omega}n\left(\frac{J_n^2}{\xi}\right)'=0.\qquad(9)$$

The roots of this equation become complex when

$$\left[1-\frac{\omega_{p1}^2}{(kV_1-n\Omega)^2}\right]^2<\frac{16\omega_{p2}^2\omega_{p1}^2}{\Omega(kV_1-n\Omega)^3}n\left(\frac{J_n^2}{\xi}\right)'.\qquad(10)$$

We see that when $n > 0$ and $n < kV_1/\Omega$ the oscillations are unstable for all wavelengths at which $(\xi^{-1}J_n^2)' > 0$. If $n > 0$ and $n > kV_1/\Omega$, the oscillations are unstable for wavelengths satisfying the inequality $(\xi^{-1}J_n^2)' < 0$. At $n < 0$, the oscillations are unstable for wavelengths satisfying the inequality $(\xi^{-1}J_n^2)' < 0$. The instability increment at values $n \sim 1$ is maximal for wavelengths of the order of V_1/ω_{p1} and is equal to

$$\gamma=\left[\frac{(kV_1)^3}{\Omega}\frac{\omega_{p2}^2}{\omega_{p1}^2}n\left(\frac{J_n^2}{\xi}\right)'\right]^{1/3}.\qquad(11)$$

We consider the case $kV_1 - n\Omega \sim \omega_1$. Taking into account the smallness of the ratio ω_1/Ω, we neglect the unity term in (8) and obtain the following quadratic equation for ω_1

$$\frac{2\omega_{p2}^2}{\Omega}n\left(\frac{J_n^2}{\xi}\right)'\omega_1^2-\left[\frac{4\omega_{p2}^2}{\Omega}n\left(\frac{J_n^2}{\xi}\right)'(kV_1-n\Omega)-\omega_{p1}^2\right]\omega_1$$
$$+2\frac{\omega_{p2}^2}{\Omega}n\left(\frac{J_n^2}{\xi}\right)'(kV_1-n\Omega)^2=0.\qquad(12)$$

Equation (12) is valid if $\omega_{p1}/\omega_{p2} \sim 1/kL \ll 1$. The roots of this equation are complex under the condition

$$\omega_{p1}^2<(8\omega_{p2}^2/\Omega)n(J_n^2/\xi)'(kV_1-n\Omega).$$

The maximum instability increment reaches in this case a value on the order of

$$\gamma\sim\frac{\omega_{p1}^2}{\omega_{p2}^2}kV_2\exp\frac{kV_1}{\Omega}.$$

4. For frequencies that are not multiples of Ω, the dispersion equation (7) can be solved approximately in two limiting cases, $\omega \gg \Omega$ and $\omega \ll \Omega$. If $\omega \gg \Omega$, then we expand the denominator under the summation sign in a series, and retain terms up to $(\Omega/\omega)^3$. In the retained terms of the series, the infinite sums can be easily evaluated and the dispersion equation takes the form

$$1-\frac{\omega_{p1}^2}{(\omega-kV_1)^2}-\frac{\omega_{p2}^2}{\omega^2}-\frac{9}{4}\frac{\omega_{p2}^2(kV_2)^2}{\omega^4}=0.\qquad(13)$$

If the density of the trapped particles is low in comparison with the density of the untrapped particles, which is usually the case in an adiabatic gun, then, following Mikhaĭlovskiĭ[4], we can obtain a solution of (13). The roots of this equation are complex when $kV_1 < \omega_{p1}$. The maximum instability increment is reached at $kV_1 \approx \omega_{p1}$ and is equal to

$$\gamma=\omega_{p1}2^{-4/3}3^{1/2}\alpha^{1/3},\qquad(14)$$

where $\alpha = \omega_{p2}^2/\omega_{p1}^2$.

For frequencies $\omega \ll \Omega$, expanding in (7) the denominator under the summation sign in a series and discarding all terms with $(\omega/\Omega)^3$ and higher, we obtain

$$1-\frac{\omega_{p1}^2}{(kV_1)^2}+\frac{2\omega_{p2}^2}{\Omega^2}\left(\frac{1-J_0^2}{\xi}\right)'+\frac{4\omega_{p2}^2}{\Omega^4}\omega^2\left[\frac{1}{\xi}\sum_{s=1}^{\infty}\left(\frac{J_s}{s}\right)^2\right]'=0.\qquad(15)$$

Recognizing that the argument of the Bessel function $\xi \gg 1$ and that $(J_s/s)^2$ decreases very rapidly with increasing number s, we can neglect in (14) all the terms in the sum, starting with $s = 2$. We have in this case the equation

$$1-\frac{\omega_{p1}^2}{(kV_1)^2}+\frac{2\omega_{p2}^2}{\Omega}\left(\frac{1-J_0^2}{\xi}\right)'+\frac{4\omega_{p2}^2}{\Omega^4}\omega^2\left(\frac{J_1^2}{\xi}\right)'=0,$$

from which we obtain

$$\omega=\frac{\Omega^2}{2\omega_{p2}}\left[\left(\frac{J_1^2}{\xi}\right)'\right]^{-1/2}\left[\frac{\omega_{p1}^2}{(kV_1)^2}+\frac{2\omega_{p2}^2}{(kV_2)^2}-1\right]^{1/2}.$$

We have neglected here J_0^2 in comparison with unity, which is perfectly admissible, since $\xi \gg 1$. We see that if the condition

$$\omega_{p1}^2/(kV_1)^2+2\omega_{p2}^2/(kV_2)^2<1,$$

is satisfied, the perturbations will increase for wavelengths satisfying the relation $(\xi^{-1}J_1^2)' > 0$. On the other hand, if we satisfy the condition

$$\omega_{p1}^2/(kV_1)^2+2\omega_{p2}^2/(kV_2)^2>1,$$

the perturbations increase for wavelengths satisfying the inequality $(\xi^{-1}J_1^2) < 0$.

5. The results can be understood from the following physical considerations. For waves whose length is much shorter than the well dimension, we can neglect the inhomogeneity of the density of the trapped particles. Regarding the trapped particles as two opposing beams, we can readily see that the instability for frequencies that are not multiples of Ω has the character of two-stream instability. For frequencies that are multiples of Ω, the instability is due to the following cause. The trapped particles constitute a harmonic oscillator. The field of the untrapped electrons

$\sim \exp(ikx - i\omega t)$ can be regarded in this case as an external perturbation acting on the oscillator. It is known that if the external force depends on the spatial coordinate, then resonance will be observed if the external-force frequency is a multiple of the oscillator frequency. Consequently, the instability at the perturbation harmonics that are multiples of Ω is due to the resonant interaction between the trapped electrons and the inhomogeneous field of the untrapped ones.

6. Let us consider the conditions under which the obtained instabilities are decisive. The oscillation frequency of the trapped electrons depends, generally speaking, on their energy. The appearance of instability causes the trapped electrons to become redistributed with respect to energy, with a certain equilibrium temperature $T \sim mV_2^2/2$, where $mV_2^2/2$ is the energy of the trapped electrons. The results will be valid, however, if

$$\gamma / kV_T \gg 1, \qquad (16)$$

where $V_T \sim V_2$ is the thermal velocity of the trapped electrons.

In the case of frequencies that are not multiples of Ω, the maximum instability increment is given by (14), and the condition (16) will be satisfied if

$$(n_2/n_1)(V_1/V_2)^3 \gg 1,$$

which coincides with the usual condition for the validity of the hydrodynamic analysis of two-stream instability. For perturbation frequencies that are multiples of Ω, it is impossible to satisfy the inequality (16), since $\gamma \sim \omega_1 \ll \Omega$ and $kV_2/\Omega \gg 1$. However, if the initial distribution function of the trapped electrons is close to a δ-function, then the resonant instabilities do assume a role. The reason is that a certain time is required for the initial distribution to "spread out." This time is determined by the quantity $(\Delta\Omega)^{-1}$, where $\Delta\Omega$ is the correction to the fundamental frequency Ω and is necessitated by allowance for the anharmonic terms in the potential energy. If $1/\Delta\Omega \gg 1/\gamma$, then resonant instability sets in.

The author is sincerely grateful to A. D. Gladun, A. A. Ivanov, V. G. Leiman, and V. I. Ryzhiĭ for a discussion of the results.

[1]B. B. Kadomtsev and O. P. Pogutsě, in the coll. Voprosy teorii plazmy (Questions on the Theory of Plasma), No. 5, Atomizdat, 1967.
[2]R. Z. Sagdeev and A. A. Galeev, Dokl. Akad. Nauk SSSR **180,** 839 (1968) [Sov. Phys.-Dokl. **13,** 562 (1968)].
[3]A. A. Ivanov and V. F. Murav'ev, Prikl. Mat. Teor. Fiz. **2,** 10 (1971).
[4]A. B. Mikhaĭlovskiĭ, Teoriya plazmennykh neustoĭchivosteĭ (Theory of Plasma Instabilities), Vol. 1, Atomizdat, 1970, p. 19.

Translated by J. G. Adashko
203