Kinetic Equations for a Classical Non-Ideal Plasma

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The Boltzmann, Landau and Balescu-Lenard kinetic equations for a plasma do not take full account of the interaction between the particles in the framework of the respective models. For example, these equations lead to the law of conservation of kinetic energy and hence do not take into account the contribution of the interaction to the internal energy of the plasma. Also not taken into account is the contribution of correlations to the entropy, etc. In[7] the author has considered a Boltzmann equation in which the interaction is completely taken into account in the framework of the binary-collision approximation. One of the problems of the present paper is to derive the analogous kinetic equation for a plasma in the first approximation in the plasma parameter. Because of the necessity of taking the polarization into account, this problem is considerably more difficult than that treated in [7], and a different approach is required to solve it. The resulting collision integral, like the collision integral in the Balescu-Lenard approximation, diverges at small distances. Various forms of the collision integrals (combinations of the Boltzmann, Landau and Balescu-Lenard collision integrals) have been proposed recently, converging at both large and small distances. However, the corresponding thermodynamic functions are not entirely correct. In this paper, the Boltzmann kinetic equation for a nonideal plasma is obtained taking into account the contribution, averaged over the velocities, of the dynamical polarizability of the plasma. The collision integral converges at small and large distances. At small distances the corresponding space correlation function goes over to the expression for the binary-collision approximation, while at large distances, it coincides, in the first approximation in the plasma parameter, with the Debye correlation function. A consequence of this is that correct expressions are obtained for the thermodynamic functions.

1. INTRODUCTION

In the present paper, kinetic equations for a spatially uniform nonideal plasma are considered.

The spatial uniformity condition means that in deriving the collision integrals we can use the zeroth approximation in the ratio of the Debye radius to the length over which an appreciable change in the first distribution functions $f_a(x, t)$ occurs. Here, a is the component index and $x = (\mathbf{r}, \mathbf{p})$.

The designation "nonideal plasma" means the following.

The form of the collision integral I_a depends on the model considered for the plasma. In the case when we can confine ourselves to taking binary collisions into account, the Boltzmann collision integral is used. We write it in the form proposed by Bogolyubov:[1-4]

$$I_a(\mathbf{p}_a, t) = \sum_b n_b \int \frac{\partial \Phi_{ab}}{\partial \mathbf{r}_a} \frac{\partial}{\partial \mathbf{p}_a} f_a(\mathbf{P}_a(-\infty), t) f_b(\mathbf{P}_b(-\infty), t)\, dx_b. \quad (1.1)$$

Here, $\mathbf{P}_a(-\infty)$ and $\mathbf{P}_b(-\infty)$ are the initial momenta of particles colliding at time t.

For a Coulomb plasma, the Landau and Balescu-Lenard collision integrals are used. We write the Balescu-Lenard collision integral in the form[4-6]

$$I_a(p_a, t) = \sum_b 2e_a^2 e_b^2 n_b \frac{\partial}{\partial p_{ai}} \int \frac{k_i k_j}{k^4} \frac{\delta(\mathbf{kv} - \mathbf{kv}')}{|\varepsilon(\mathbf{kv}, \mathbf{k})|^2} \times \left(\frac{\partial}{\partial p_{aj}} - \frac{\partial}{\partial p_{bj}}\right) f_a(\mathbf{p}_a, t) f_b(\mathbf{p}_b, t)\, d\mathbf{k}\, d\mathbf{p}_b. \quad (1.2)$$

Here, $\epsilon(\omega, \mathbf{k})$ is the dielectric permittivity of the Coulomb plasma. It is given by the expression

$$\varepsilon(\omega, \mathbf{k}) = 1 - i \sum_a \frac{4\pi e_a^2 n_a}{k^2} \int_0^\infty \int e^{-\Delta\tau + i(\omega - \mathbf{kv})\tau} \mathbf{k} \frac{\partial f_a(\mathbf{p}_a, t)}{\partial \mathbf{p}_a}\, d\tau\, d\mathbf{p}_a. \quad (1.3)$$

The Landau collision integral can also be written in the form (1.2), if we put $\epsilon(\omega, \mathbf{k}) = 1$ and limit the range of integration: $1/r_{min} \ge k \ge 1/r_D$, where r_D is the Debye radius.

It was noted in a previous paper by the author[7] that the Boltzmann kinetic equation in the binary-collision approximation does not take complete account of the interaction of the particles. This is manifested, in particular, in the fact that the interaction of the particles does not appear in the expression for the energy of the gas. In this sense, it may be said that the Boltzmann equation describes nonequilibrium processes only in an ideal gas. In [7], it was shown how the Boltzmann kinetic equation can be generalized so that one can use it to describe a nonideal gas.

An analogous situation also holds for a plasma. In the sense indicated above, the Landau and Balescu-Lenard equations are valid only for an ideal plasma.

One of the problems of the present paper consists in obtaining kinetic equations for a nonideal plasma in the first approximation in the plasma parameter. Because of the necessity of taking the polarization of the plasma into account, this problem is considerably more complicated than the corresponding problem for a gas. A special method has been developed for its solution.

The Balescu-Lenard collision integral, like the Landau collision integral also, contains a divergence at small distances. In a number of papers,[8-10] different expressions for the collision integral have been considered, which coincide with the Boltzmann collision integral in the region of small distances and with the Balescu-Lenard collision integral in the region of large distances. In the present paper, we study the corresponding expression for the collision integral for a nonideal plasma.

The Balescu-Lenard kinetic equation is very complicated because of the inclusion of the dynamical polarization in it. It is therefore natural to attempt to study simpler model kinetic equations, which would reflect the fundamental properties of a nonideal plasma.

In Sec. 5, we study the Boltzmann equation for a nonideal plasma, with the averaged contribution of the dynamical polarization taken into account. The collision integral in this equation converges at small and large distances. The corresponding expression for the equilibrium correlation function at small distances coincides with the expression for the Boltzmann gas, and at large distances coincides with the Debye correlation function.

2. THE BALESCU-LENARD KINETIC EQUATION FOR A NONIDEAL PLASMA

We shall consider the kinetic equation for a nonideal plasma, taking the dynamical polarization into account; the equation is valid under the following two assumptions.

1. We are considering the approximation of second correlation functions, i.e., $g_3, g_4, \ldots$ are equal to zero and $f_a f_b + g_{ab} \approx f_a f_b$.
2. The collision integral I_a is determined by the spectral function of the rapid fluctuations of the phase density, δN_a, and of the field, $\delta \mathbf{E}$. This means that we are taking into account the contribution of fluctuations with correlation times shorter than the relaxation times of the functions f_a.

To derive the kinetic equation, we use the method described in [6,11,12]. We represent the collision integral for a spatially uniform plasma in the form

$$I_a = -\frac{e_a}{n_a}\frac{\partial}{\partial \mathbf{p}}\overline{\delta N_a(\mathbf{r},\mathbf{p},t)\,\delta \mathbf{E}(\mathbf{r},t)}. \tag{2.1}$$

The equations for the fluctuations δN_a ($N_a = \sum_i \delta(\mathbf{x}-\mathbf{x}_i(t))$) and $\delta \mathbf{E}$ in the approximation of second correlations (the first assumption) can be written in the form[11]

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)(\delta N_a - \delta N_a^{\text{ист}}) + e_a n_a \delta \mathbf{E}\frac{\partial f_a}{\partial \mathbf{p}} = 0,$$

$$\operatorname{div}\delta\mathbf{E} = 4\pi\sum_a e_a\int \delta N_a\,d\mathbf{p}. \tag{2.2}$$

The fluctuations δN_a^{sou} of the phase density of the source satisfy the equation

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\delta N_a^{\text{sou}} = 0. \tag{2.3}$$

This equation can be solved for the correlation $\overline{(\delta N_a \delta N_b)}^{\text{sou}}_{\mathbf{r},\mathbf{r}',\mathbf{p},\mathbf{p}',t,t'}$ with the initial condition

$$(\delta N_a \delta N_b)^{\text{sou}}_{\mathbf{r},\mathbf{r}',\mathbf{p},\mathbf{p}',t',t'} = \delta_{ab}\delta(\mathbf{r}-\mathbf{r}')\,\delta(\mathbf{p}-\mathbf{p}')f_a(\mathbf{r}',\mathbf{p}',t'). \tag{2.4}$$

The fluctuations δN_a and $\delta\mathbf{E}$ are functions of "fast" and "slow" times, i.e., $\delta N_a = \delta N_a(\mathbf{r}, \mathbf{p}, t, \mu t)$ and $\delta\mathbf{E} = \delta\mathbf{E}(\mathbf{r}, \mathbf{p}, t, \mu t)$. The dependence on the slow time μt in the functions δN_a and $\delta\mathbf{E}$ and their derivatives appears through the slowly varying functions $f_a(\mathbf{p}, \mu t)$ (the second assumption).

From Eqs. (2.2), for the Fourier components with respect to the coordinates and the fast time we obtain the expressions

$$\delta N_a(\omega,\mathbf{k},\mathbf{p},\mu t) = \delta N_a^{\text{sou}}(\omega,\mathbf{k},\mathbf{p},\mu t) - \frac{e_a n_a}{k^2}\int_0^\infty \mathscr{E}(\tau)\,\mathbf{k}\frac{\partial f_a(\mathbf{p},\mu(t-\tau))}{\partial \mathbf{p}}\,k\,\delta \mathbf{E}(\omega,\mathbf{k},\mu(t-\tau))\,d\tau, \tag{2.5}$$

$$\delta\mathbf{E}(\omega,\mathbf{k},\mu t) = \delta\mathbf{E}^{\text{sou}} + i\sum_a \frac{4\pi e_a^2 n_a}{k^2}\iint_0^\infty \mathscr{E}(\tau) \times \mathbf{k}\frac{\partial f_a(\mathbf{p},\mu(t-\tau))}{\partial \mathbf{p}}\,\delta\mathbf{E}(\omega,\mathbf{k},\mu(t-\tau))\,d\tau\,d\mathbf{p}.$$
$$\mathscr{E}(\tau) \equiv \mathscr{E}(\omega,k,\mathbf{v},\tau) = e^{-\Delta\tau + i(\omega-\mathbf{k}\mathbf{v})\tau}. \tag{2.5'}$$

Here, Δ is a time parameter separating the fast and slow processes,

$$\omega_{min},\ 1/\tau_{\text{cor}} \gg \Delta \gg 1/\tau_{\text{coll}}.$$

Below, for brevity, $\mu t = t$. In the first approximation in $\tau_{\text{col}}\,\partial/\partial t$, from (2.5) we have

$$\delta N_a = \delta N_a^{\text{sou}} - \frac{e_a n_a}{k^2}\int_0^\infty \mathscr{E}(\tau)\,\mathbf{k}\frac{\partial f_a(\mathbf{p},t-\tau)}{\partial \mathbf{p}}\left(1-\tau\frac{\partial}{\partial t}\right)\delta\mathbf{E}(\omega,k,t)\,d\tau, \tag{2.6}$$

$$\varepsilon(\omega,\mathbf{k},t)\,\delta\mathbf{E}(\omega,\mathbf{k},t) = \delta\mathbf{E}^{\text{sou}} - i\frac{\partial\varepsilon}{\partial\omega}\frac{\partial\delta\mathbf{E}}{\partial t}. \tag{2.7}$$

Here,

$$\varepsilon(\omega,\mathbf{k},t) = 1 - i\sum_a \frac{4\pi e_a^2 n_a}{k^2}\iint_0^\infty \mathscr{E}(\tau)\,\mathbf{k}\frac{\partial f_a(\mathbf{p},t-\tau)}{\partial \mathbf{p}}\,d\tau\,d\mathbf{p}. \tag{2.8}$$

This expression for the dielectric permittivity differs from the expression (1.3) in that here the retardation of the function f_a is taken into account.

In the same approximation, we find from (2.7) an expression for the derivative $\partial\delta\mathbf{E}/\partial t$:

$$\varepsilon\frac{\partial\delta\mathbf{E}}{\partial t} = \frac{\partial\delta\mathbf{E}^{\text{sou}}}{\partial t} - \frac{\partial\varepsilon}{\partial t}\delta\mathbf{E}. \tag{2.9}$$

Hence,

$$\frac{\partial}{\partial t}(\delta\mathbf{E}\delta\mathbf{E})_{\omega,\mathbf{k},t} = \frac{\partial}{\partial t}\frac{(\delta\mathbf{E}\delta\mathbf{E})^{\text{sou}}_{\omega,\mathbf{k},t}}{|\varepsilon(\omega,\mathbf{k},t)|^2} \tag{2.10}$$

and consequently, in this approximation, the energy density of the electric field is given by the expression

$$\overline{(\delta\mathbf{E}\delta\mathbf{E})} = \frac{1}{(2\pi)^4}\int\frac{(\delta\mathbf{E}\delta\mathbf{E})^{\text{sou}}_{\omega,\mathbf{k},t}}{|\varepsilon(\omega,\mathbf{k},t)|^2}\,d\omega d\mathbf{k}. \tag{2.11}$$

We shall write an expression for the spectral functions of the fluctuation sources. They follow from (2.3) and (2.4) and have the form

$$(\delta N_a\,\delta N_b)_{\omega,\mathbf{k},\mathbf{p},\mathbf{p}',t} = \delta_{ab}\delta(\mathbf{p}-\mathbf{p}')\,n_a\cdot 2\,\mathrm{Re}\int_0^\infty \mathscr{E}(\tau)f_a(\mathbf{p},t-\tau)\,d\tau, \tag{2.12}$$

$$(\delta N_a\delta\mathbf{E})_{\omega,\mathbf{k},\mathbf{p},t} = \frac{i\mathbf{k}}{k^2}4\pi e_a n_a\cdot 2\,\mathrm{Re}\int_0^\infty \mathscr{E}(\tau)f_a(\mathbf{p},t-\tau)\,d\tau, \tag{2.13}$$

$$(\delta\mathbf{E}\delta\mathbf{E})^{\text{sou}}_{\omega,\mathbf{k},t} = \sum_a\frac{(4\pi)^2 e_a^2 n_a}{k^2}\,2\,\mathrm{Re}\int\int_0^\infty \mathscr{E}(\tau)f_a(\mathbf{p},t-\tau)\,d\tau\,d\mathbf{p}. \tag{2.14}$$

We represent the collision integral (2.1) in the form

$$I_a = -\frac{e_a}{n_a}\frac{1}{(2\pi)^4}\frac{\partial}{\partial \mathbf{p}}\int \mathrm{Re}(\delta N_a\delta\mathbf{E})_{\omega,\mathbf{k},t}\,d\omega\,d\mathbf{k}. \tag{2.15}$$

It follows from the structure of the expression (2.6) for δN_a that the collision integral I_a can be represented in the form of a sum of two parts

$$I_a = I_a^{\text{ind}} + I_a^{\text{sou}}. \tag{2.16}$$

The first part is proportional to the spectral density $(\delta E \cdot \delta E)_{\omega,k,t}$ and may therefore be called the induced part. We write an expression for I_a^{ind} in general form, without immediately separating out the first derivative with respect to μt:

$$I_a^{ind} = \frac{e_a^2}{(2\pi)^4}\frac{\partial}{\partial \mathbf{p}}\,\mathrm{Re}\int_0 \int \frac{\mathbf{k}}{k^2}\mathscr{E}(\tau)\,(\delta E\delta E)_{\omega,\mathbf{k},(t-\tau),t}\,\mathbf{k}\frac{\partial f_a(\mathbf{p}, t-\tau)}{\partial \mathbf{p}}\,d\tau\, d\omega\, d\mathbf{k}. \quad (2.17)$$

In the spectral function of the field, the argument $(t-\tau)$ refers to the first factor, and t to the second factor.

The second term in (2.16)

$$I_a^{sou} = -\frac{e_a}{n_a}\frac{1}{(2\pi)^4}\frac{\partial}{\partial \mathbf{p}}\int \mathrm{Re}(\delta N_a^{sou}\,\delta E)_{\omega,\mathbf{k},t}\,d\omega\, d\mathbf{k} \quad (2.18)$$

is determined by the spectral function of the fluctuations δN_a^{sou} of the phase density of the source and of the fluctuations δE of the field.

By means of (2.9), we shall express δE in terms of δE^{sou} and the derivative $\partial\delta E/\partial t$:

$$\delta E = \frac{\delta E^{sou}}{\varepsilon}\left(1 - i\frac{\partial \varepsilon}{\partial \omega}\frac{\partial}{\partial t}\frac{1}{\varepsilon}\right) - i\frac{\partial\varepsilon}{\partial\omega}\frac{1}{\varepsilon^2}\frac{\partial \delta E^{sou}}{\partial t}. \quad (2.19)$$

We shall substitute (2.19) into (2.18) and use the formulas (2.12)–(2.14) for the spectral functions of the source fluctuations. As a result, we obtain the following expression:

$$I_a^{sou} = -\frac{e_a^2}{2\pi^3}\frac{\partial}{\partial \mathbf{p}}\int_0^\infty\int \frac{\mathbf{k}}{k^2}\Big\{\mathrm{Re}\left[\frac{i}{\varepsilon^*}\left(1 + i\frac{\partial \varepsilon^*}{\partial\omega}\frac{\partial}{\partial t}\frac{1}{\varepsilon^*}\right)\right]$$
$$\times \cos(\omega - \mathbf{kv})\tau f_a(\mathbf{p}, t-\tau)$$
$$- i\frac{\partial}{\partial\omega}\frac{1}{\varepsilon^*}\sin(\omega - \mathbf{kv})\tau\frac{\partial}{\partial t}f_a(\mathbf{p}, t-\tau)\Big\}\,d\tau\, d\omega\, d\mathbf{k}. \quad (2.20a)$$

The expression (2.20a) can be written in a simpler form:

$$I_a^{sou} = -\frac{e_a^2}{2\pi^2}\frac{\partial}{\partial \mathbf{p}}\int \mathrm{Re}\left[\frac{i}{\varepsilon^*}\left(1 + i\frac{\partial\varepsilon^*}{\partial\omega}\frac{\partial}{\partial t}\frac{1}{\varepsilon^*}\right)\right]\frac{\mathbf{k}}{k^2}\,\delta(\omega - \mathbf{kv})\,d\omega\, d\mathbf{k}\, f_a(\mathbf{p}, t). \quad (2.20b)$$

Formula (2.20a), despite its more complicated form, turns out to be more convenient in many cases, for example, in the derivation of the conservation laws.

The above decomposition of the collision integral I_a into a sum of two parts I_a^{ind} and I_a^{sou} is not, of course, absolutely necessary. By means of the formulas (2.9) and (2.19), the collision integral can be completely expressed in terms of the first distribution functions f_a, but with allowance for the retardation. If we neglect the retardation in the collision integral (2.17), (2.20), we obtain the Balescu-Lenard collision integral (1.2).

The kinetic equation with the collision integral (2.16), (2.17), (2.20) leads to the energy conservation law in the form

$$\frac{\partial}{\partial t}\left\{\sum_a n_a\int \frac{\mathbf{p}^2}{2m_a}f_a d\mathbf{p} + \frac{\overline{\delta E\delta E}}{8\pi}\right\} = 0. \quad (2.21)$$

The energy of the electric field is

$$\frac{\overline{\delta E\delta E}}{8\pi} = \frac{1}{(2\pi)^4}\int \frac{(\delta E\delta E)_{\omega,\mathbf{k}}}{8\pi}\,d\omega\, d\mathbf{k},$$

where

$$(\delta E\delta E)_{\omega,\mathbf{k}} = (\delta E\delta E)^{sou}_{\omega,\mathbf{k},t}/|\varepsilon(\omega, \mathbf{k}, t)|^2. \quad (2.22)$$

The spectral function $(\delta E\cdot\delta E)^{sou}_{\omega,\mathbf{k},t}$ is given by the expression (2.14).

In a state of local statistical equilibrium, an expression for the spatial spectral function of the field follows from (2.22):

$$(\delta E\delta E)_{\mathbf{k}} = 4\pi kT/(1 + r_D^2 k^2).$$

The spatial correlation function corresponding to this expression is

$$g_{ab}(r) = -\frac{e_a e_b}{kT}\frac{e^{-r/r_D}}{r}, \quad (2.23)$$

which describes the Debye correlation of particles in the plasma.

Thus, the kinetic equation with the collision integral (2.16), (2.17), (2.20) is valid for a nonideal plasma in the first approximation in the plasma parameter.

3. THE LANDAU COLLISION INTEGRAL FOR A NONIDEAL PLASMA

If the integral equations (2.5) are solved by perturbation theory, then in place of (2.16), (2.17), (2.20), we obtain the following expression for the collision integral:

$$I_a = \sum_b \frac{2}{\pi}e_a^2 e_b^2 n_b\frac{\partial}{\partial p_{ai}}\mathrm{Re}\int_0^\infty\int \frac{k_i k_j}{k^4}\exp\{-\Delta\tau - i(\mathbf{kv}_a - \mathbf{kv}_b)\tau\}$$
$$\times\left(\frac{\partial}{\partial p_{aj}} - \frac{\partial}{\partial p_{bj}}\right)f_a(\mathbf{p}_a, t-\tau)f_b(\mathbf{p}_b', t-\tau)\,d\tau\, d\mathbf{p}_b\, d\mathbf{k}. \quad (3.1)$$

The range of integration over k is bounded by the condition $1/r_{min} \geq k \geq 1/r_D$. The expression (3.1) differs from the Landau integral only by taking into account the time retardation.

The kinetic equation with the collision integral (3.1) also leads to the conservation law (2.21). Now, however,

$$\frac{\overline{\delta E\delta E}}{8\pi} = \frac{1}{(2\pi)^3}\sum_{ab}(4\pi)^2\frac{e_a^2 e_b^2 n_a n_b}{2}\int_0^\infty\int e^{-\Delta\tau}\sin(\mathbf{kv} - \mathbf{kv}')\tau\frac{1}{k^4}$$
$$\times\mathbf{k}\left(\frac{\partial}{\partial \mathbf{p}} - \frac{\partial}{\partial \mathbf{p}'}\right)f_a(\mathbf{p}, t-\tau)f_b(\mathbf{p}', t-\tau)\,d\mathbf{p}\, d\mathbf{p}'\, d\tau\, d\mathbf{k}. \quad (3.2)$$

This expression follows from (2.22) in the perturbation-theory approximation.

A kinetic equation with the collision integral (3.1) was used by Silin[13,4] to describe fast processes in a plasma. In the presence of fast processes, the distribution functions in expression (3.1) have the form $f_a = f_a(p, t, t)$, i.e., depend on the fast and slow times. In [13,4], retardation was taken into account only with respect to the fast time, and the contribution of the interaction to the energy of the plasma was thus not taken into account.

4. PLASMA NONIDEALITY DUE TO INTERACTION OF CHARGED PARTICLES WITH PLASMONS

In the expressions (2.17), (2.20) for the collision integral of a nonideal plasma and in the expression (2.22) for the energy of a nonideal plasma, the contribution of all the rapid fluctuations δN_a and δE, i.e., those whose correlation times are shorter than the time between collisions, is taken into account.

Among the rapid fluctuations, we can separate out the contribution of the plasmons, which are excitations for which ω and $\mathbf{k}$ are related by the dispersion equation

$$\operatorname{Re}\varepsilon(\omega, \mathbf{k}) = 0$$

and the damping constants are such that $\omega_k \gg \gamma_k \gg 1/\tau_{col}$.

Kinetic equations for a nonideal plasma, taking into account only the contribution of the plasmons, were considered in the paper.[6] The corresponding expression for the collision integral in the first approximation in $\gamma_k\,\partial/\partial t$ follows from (2.17) and (2.20). We write it in the form

$$I_a = \frac{e_a^2}{(2\pi)^4}\frac{\partial}{\partial \mathbf{p}}\operatorname{Re}\int_0^\infty\int\frac{\mathbf{k}}{k^2}\left\{\left(1-\frac{\tau}{2}\frac{\partial}{\partial t}\right)(\delta\mathbf{E}\delta\mathbf{E})_{\omega,\mathbf{k},t}\,\mathbf{k}\frac{\partial f_a(\mathbf{p}, t-\tau)}{\partial \mathbf{p}} + \frac{8\pi\varepsilon''(\omega,\mathbf{k})}{|\varepsilon(\omega,\mathbf{k})|^2}f_a(\mathbf{p},t)\right\}\mathscr{E}(\tau)\,d\tau\,d\omega\,d\mathbf{k}. \quad (4.1)$$

This expression coincides with that given in [6].

The kinetic equation with the collision integral (4.1) also leads to the energy conservation law (2.21), but in the integral over ω and $\mathbf{k}$ in (2.22) only the contribution of the plasmons is now taken into account (cf. Sec. 17 in [6]).

In the expression (4.1), the contribution of plasmons with damping constants greater than the collision frequency is taken into account. This approximation is closely related to the so-called quasi-linear approximation for a plasma. It has been examined in detail in the work of Vedenov,[14] Kadomtsev,[15] Zavoiskiĭ, and Rudakov,[16] in Sec. 58 of Silin's book,[4] and in Tsytovich's book.[17]

5. INCLUSION OF THE AVERAGED DYNAMICAL POLARIZATION IN THE BOLTZMANN KINETIC EQUATION FOR A NONIDEAL PLASMA

The Boltzmann collision integral for a Coulomb plasma contains a divergence at large distances, and the Balescu-Lenard integral diverges at small distances. In the papers by Hubbard,[8] Aono,[9] Alyamovskiĭ,[10] and others, different forms of the collision integrals, simultaneously taking into account binary-collision processes and polarization processes, have been proposed.

The simplest solution of this problem lies in using for the collision integral a combination of three integrals: the Boltzmann I_a^B, Landau I_a^L and Balescu-Lenard I_a^{BL} integrals:

$$I_a = I_a^B - I_a^L + I_a^{BL}. \quad (5.1)$$

In this expression, the integral I_a^L compensates the divergence of the Boltzmann integral at large distances and the divergence of the integral I_a^{BL} at small distances. Such a generalization, although attractive by virtue of its relative simplicity, is nevertheless not completely satisfactory, for the following reason.

All three collision integrals in the right-hand side of (5.1) are known to us for a nonideal plasma. The integrals I_a^L and I_a^{BL} for a nonideal plasma are given by the expressions (3.1) and (2.17), (2.20). For the integral I_a^B, we shall use the expression obtained in [7]:

$$I_a^B = \sum_b n_b\int\frac{\partial\Phi_{ab}}{\partial\mathbf{r}_a}\frac{\partial}{\partial\mathbf{p}_a}f_a(\mathbf{P}_a(-\infty),t)f_b(\mathbf{P}_b(-\infty),t)\,dx_b$$

$$-\sum_b n_b\int_0^\infty\int\frac{\partial\Phi_{ab}}{\partial\mathbf{r}_a}\frac{\partial}{\partial\mathbf{p}_a}\tau\frac{\partial^2}{\partial t\,\partial\tau}f_a(\mathbf{P}_a(-\tau),t)f_b(\mathbf{P}_b(-\tau),t)\,d\tau\,dx_b. \quad (5.2)$$

The first term in (5.2) coincides with the Boltzmann collision integral (1.1). The second term allows for the nonideality in the binary-collision approximation.

We shall study the correction to the plasma energy due to allowance for the nonideality. This is determined by the correlation function g_{ab}. In the local-equilibrium approximation, the expression for g_{ab} corresponding to (5.1) has the form

$$g_{ab} = \left(\exp\left\{-\frac{e_ae_b}{rkT}\right\} - 1 + \frac{e_ae_b}{rkT} - \frac{e_ae_b}{rkT}e^{-r/r_D}\right)f_a^M f_b^M, \quad (5.3)$$

where f_a^M is the Maxwell distribution. The first term in the right-hand side of (5.3) is the correlation function of the Boltzmann gas, and the third term is the Debye correlation function for the plasma. The second term (with the opposite sign) is the correlation function in the perturbation-theory approximation.

It follows from the expression (5.3) that at large distances the second term cancels the first only in the first approximation in e_ae_b/rkT. Because of this, the correlation function at large distances falls off not exponentially, but only like $1/r^2$. Even alone, this shows that the approximation (5.1) is not completely satisfactory.

Other approaches have also been developed toward solving the problem of defining a plasma collision integral that is finite at all distances. Some of these consist in replacing the Coulomb interaction potential by the Debye potential, i.e.,

$$\frac{e_ae_b}{r} \to \frac{e_ae_b}{r}e^{-r/r_D}. \quad (5.4)$$

This approximation, however, also leads to incorrect expressions for the thermodynamic functions. Thus, for example, in the calculation of the internal energy density in the Debye approximation with the potential (5.4), in place of the usual expression

$$\sum_{ab}\frac{n_an_b}{2}\int\frac{e_ae_b}{r}g_{ab}\,d\mathbf{r}\,d\mathbf{p}_a\,d\mathbf{p}_b$$

we shall have the expression

$$\sum_{ab}\frac{n_an_b}{2}\int g_{ab}g_{ab}\,d\mathbf{r}\,d\mathbf{p}_a\,d\mathbf{p}_b.$$

This discrepancy is completely natural, since the choice of the potential in the form (5.4) presupposes that the equilibrium spatial correlation of the particles is established more rapidly than the distribution in the momenta. There is, however, insufficient justification for this assumption, i.e., there are no reasons to assume that the polarizability of the plasma is static.

We shall consider another model, in which the dynamical character of the plasma polarizability is taken into account approximately.

We note that in the derivation of the kinetic equation for an ideal plasma with allowance for the polarization, i.e., the expression (1.2), the initial expression for the collision integral can be written (in place of (2.15)) in the form

$$I_a = \frac{\partial}{\partial\mathbf{p}_a}\sum_b n_b\int\mathbf{k}\Phi_{ab}(k)\operatorname{Im}g_{ab}(\mathbf{k},\mathbf{p}_a,\mathbf{p}_b)\frac{d\mathbf{k}}{(2\pi)^3}d\mathbf{p}_b, \quad (5.5)$$

$$\Phi_{ab}(\mathbf{k}) = 4\pi e_ae_b/k^2.$$

If we substitute into this the solution of the equation for the correlation function in the Debye approximation, disregarding the retardation, we obtain the expression (1.2). The expression in the integrand in (1.2) is proportional to the square of $\Phi_{ab}(\mathbf{k})$. In this case, one of the factors, in the initial expression (5.5), remains unchanged, while the second changes when the polarization is taken into account:

$$\Phi_{ab}(\mathbf{k}) \to \Phi_{ab}(\mathbf{k}) \,/\, |\varepsilon(\mathbf{kv}, \mathbf{k})|^2. \tag{5.6}$$

Thus, when the polarization is taken into account, only the potential in the equation for the correlation function g_{ab} is screened.

We shall take the averaged effect of the dynamical polarization into account in the following way. In place of (5.6), we shall use the following potential

$$\tilde{\Phi}_{ab}(\mathbf{k}) = \Phi_{ab}(\mathbf{k}) \left(\sum_a e_a^2 n_a\right)^{-1} \sum_a \int \frac{e_a^2 n_a f_a}{|\varepsilon(\mathbf{kv}, \mathbf{k})|^2} d\mathbf{p}_a. \tag{5.7}$$

We shall study $\tilde{\Phi}_{ab}$ for a local-equilibrium state. Using the identity

$$\sum_a e_a^2 n_a \int \frac{f_a}{|\varepsilon(\mathbf{kv}, \mathbf{k})|^2} d\mathbf{p}_a = \sum_a e_a^2 n_a \frac{r_D^2 k^2}{1 + r_D^2 k^2},$$

we obtain from (5.7)

$$\tilde{\Phi}_{ab}(\mathbf{k}) = 4\pi e_a e_b \frac{r_D^2}{1 + r_D^2 k^2}. \tag{5.8}$$

Hence, it follows that

$$\tilde{\Phi}_{ab}(\mathbf{r}) = \frac{e_a e_b}{r} e^{-r/r_D}. \tag{5.9}$$

Thus, in the local-equilibrium approximation, the Coulomb potential Φ_{ab} in the equation for g_{ab} is replaced by the screened potential (5.9). This means that, to obtain a kinetic equation taking into account the averaged dynamical polarization, in the spatially uniform case we can use the following system of equations for the functions f_a and g_{ab}:

$$\begin{gathered}\frac{\partial f_a}{\partial t} = \sum_b n_b \int \frac{\partial \Phi_{ab}}{\partial \mathbf{r}_a} \frac{\partial}{\partial \mathbf{p}_a} g_{ab} \equiv I_a, \\ \left(\frac{\partial}{\partial t} + \mathbf{v}_a \frac{\partial}{\partial \mathbf{r}_a} + \mathbf{v}_b \frac{\partial}{\partial \mathbf{r}_b}\right) g_{ab} = \left(\frac{\partial \tilde{\Phi}_{ab}}{\partial \mathbf{r}_a} \frac{\partial}{\partial \mathbf{p}_a} + \frac{\partial \tilde{\Phi}_{ab}}{\partial \mathbf{r}_b} \frac{\partial}{\partial \mathbf{p}_b}\right) f_a f_b.\end{gathered} \tag{5.10}$$

Solving the second equation using (5.9), and substituting g_{ab} into I_a, we obtain the collision integral

$$I_a = \sum_b \frac{2}{\pi} e_a^2 e_b^2 n_b \frac{\partial}{\partial p_{ai}} \int_0^\infty \int \frac{k_i k_j}{k^2} \frac{r_D^2}{1 + r_D^2 k^2} e^{-\Delta\tau - i(\mathbf{kv} - \mathbf{kv}')\tau}$$
$$\times \left(\frac{\partial}{\partial p_{aj}} - \frac{\partial}{\partial p_{bj}}\right) f_a(\mathbf{p}_a, t - \tau) f_b(\mathbf{p}_b, t - \tau)\, d\mathbf{k}\, d\mathbf{p}_b\, d\tau. \tag{5.11}$$

As in the more exact expression (1.2), the integral over **k** in (5.11) converges at small k. In this case, as is easily seen, correct expressions are also obtained for the thermodynamic functions in the Debye approximation.

In Eq. (5.10) for the function g_{ab}, the pair interaction is taken into account by perturbation theory. We shall now study the Boltzmann equation with allowance for the averaged dynamical polarization.

In order to use the results of [7] immediately, in place of the equations for the functions f_a and g_{ab} we shall examine the equations for the functions f_a and f_{ab}. Taking (5.9) into account, in place of Eqs. (11) of [7], we must now use the equations

$$\frac{\partial f_a}{\partial t} = \sum_b n_b \int \frac{\partial \Phi_{ab}}{\partial \mathbf{r}_a} \frac{\partial}{\partial \mathbf{p}_a} f_{ab} dx_b \equiv I_a, \tag{5.12}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_a \frac{\partial}{\partial \mathbf{r}_a} + \mathbf{v}_b \frac{\partial}{\partial \mathbf{r}_b} - \frac{\partial \Phi_{ab}}{\partial \mathbf{r}_a} \frac{\partial}{\partial \mathbf{p}_a} - \frac{\partial \Phi_{ab}}{\partial \mathbf{r}_b} \frac{\partial}{\partial \mathbf{p}_b}\right) f_{ab} = \frac{\partial}{\partial t} f_a f_b. \tag{5.13}$$

Hence, using the expression (15) of [7], we obtain

$$f_{ab} = f_a(\tilde{\mathbf{P}}_a(-\infty), t) f_b(\tilde{\mathbf{P}}_b(-\infty), t)$$
$$- \frac{\partial}{\partial t} \int_0^\infty \tau \frac{\partial}{\partial \tau} f_a(\tilde{\mathbf{P}}_a(-\tau), t) f_b(\tilde{\mathbf{P}}_b(-\tau), t)\, d\tau. \tag{5.14}$$

Here, $\tilde{\mathbf{P}}_a$ and $\tilde{\mathbf{P}}_b$ are the initial momenta of the particles interacting according to the law $\tilde{\Phi}_{ab}$.

Substituting (5.14) into (5.12), we obtain the Boltzmann equation for a nonideal plasma with allowance for the averaged dynamical polarization. It can be written conveniently in the form

$$I_a = \sum_b n_b \int_0^\infty\!\!\int \left(1 - \tau \frac{\partial}{\partial t}\right) \frac{\partial \Phi_{ab}}{\partial \mathbf{r}_a} \frac{\partial}{\partial \mathbf{p}_a} f_a(\tilde{\mathbf{P}}_a(-\tau), t) f_b(\tilde{\mathbf{P}}_b(-\tau), t)\, d\tau\, dx_b. \tag{5.15}$$

From this kinetic equation follows the energy conservation law

$$\frac{\partial}{\partial t}\Big\{\sum_a n_a \int \frac{\mathbf{P}^2}{2m_a} f_a d\mathbf{p}$$
$$+ \sum_{ab} \frac{n_a n_b}{2} \int \Phi_{ab} f_a(\tilde{\mathbf{P}}_a(-\infty), t) f_b(\tilde{\mathbf{P}}_b(-\infty), t)\, d\mathbf{r}\, d\mathbf{p}_a\, d\mathbf{p}_b\Big\} = 0. \tag{5.16}$$

Hence, in the local-equilibrium approximation, we find the internal-energy density and the correlation function:

$$\sum_a n_a \frac{3}{2} kT + \frac{1}{2} \sum_{ab} n^a n^b \int \Phi_{ab} e^{-\tilde{\Phi}/kT} d\mathbf{r}, \tag{5.17}$$

$$g_{ab} = \exp\left\{-\frac{\tilde{\Phi}_{ab}}{kT}\right\} - 1 = \exp\left[-\frac{e_a e_b}{r} e^{-r/r_D}\right] - 1. \tag{5.18}$$

At small distances, an expression for g_{ab} follows from (5.18) in the binary-collision approximation, while at large distances in the first approximation in the plasma parameter, the expression (5.17) coincides with the Debye correlation function (2.23).

We note that allowing for the averaged polarizability in (5.11) changes the value of the "Coulomb logarithm" and, consequently, the values of all the kinetic coefficients. From (5.11), we find

$$L = \ln(a/\mu) \to \tilde{L} = \ln[1 + a^2/\mu^2]^{1/2} > L. \tag{5.19}$$

Here, μ is the plasma parameter. The quantity a depends on the method of cut-off at small distances: a = 1, if $r_{\min} = e_a e_b/kT$.

It follows from (5.18) that

$$\tilde{L} - L \approx \begin{cases} \mu^2/2a^2 & \text{for } \mu \ll 1 \\ 1 - \mu/a & \text{for } 1 - \mu/a \ll 1 \end{cases}. \tag{5.20}$$

Replacing L by $\tilde{L}$ leads, in particular, to a decrease of the electrical conductivity compared with that calculated from the Spitzer formula.

The calculation of the kinetic coefficients on the basis of the Boltzmann equation (5.12), (5.14) can be performed only numerically. In this connection, we note that, because of the allowance for the polarization, the collision integral (5.15) can no longer be reduced to the usual Boltzmann form, since the problem in this case does not reduce to the problem of two-particle collisions. Because of this, the calculation of the kinetic coefficients must be performed directly for the collision integral (5.15).

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