Thermo-Absorption and Nonlinear Absorption of Sound in Dielectrics in the Hydrodynamic Region

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The absorption of high-frequency sound in dielectrics in the presence of a stationary drift of thermal phonons is considered. The dependence of the sound-absorption coefficient on the drift velocity is found; it is shown that this effect can easily be found experimentally in those crystals in which phonon hydrodynamic phenomena are observed. Nonlinear attenuation of high-frequency sound in dielectrics, due to drag and the heating of the phonons by the sound wave, is also studied. Phenomenological equations are derived which describe, within the limits of the hydrodynamic region, the self-consistent propagation of sound and heat flow for arbitrary phonon drift velocities. In addition, a nonlinear correction to the sound-wave intensity as a function of the coordinate is calculated. Numerical estimates are presented for crystalline He^4. In particular, it is shown that the nonlinear absorption should be appreciable at a temperature of 0.6°K and sound frequency of the order of 100 MHz, provided that the intensity of the sound input to the sample is about 0.03 W/cm^2.

1. INTRODUCTION

At high frequencies Ω, when $\Omega\tau_p \gg 1$ (τ_p is the relaxation time of the thermal phonons), the sound absorption can be regarded as the process of the scattering of sound phonons by thermal phonons.[1-3] The value of the absorption coefficient Γ depends here both on the parameters of the sound wave (frequency and polarization), and on the state of the phonon gas in the crystal. In the calculation of Γ it is usually assumed that the phonons are in a state of thermodynamic equilibrium at the temperature T,[1,2] or the possibility of heating of the sound wave of a group of phonon modes that are weakly connected with the remaining modes of the lattice[3] is taken into account. In either case it is assumed that the phonon gas as a whole is at rest relative to the crystal lattice. Such a statement of the problem is undoubtedly correct if the crystal has an effective mechanism for the dissipation of the total quasimomentum of the phonon system. However, in extremely pure dielectrics at $T \lesssim 0.02$–$0.04\ \Theta$, where Θ is the Debye temperature, phonon-phonon U-processes and impurity scattering are too weak to produce any appreciable friction between the phonon gas and the crystal lattice.[4,5] In this case, even a comparatively small temperature gradient can produce a phonon drift of V commensurate with the sound velocity. This should lead to a change in the sound absorption coefficient Γ, i.e., to thermo-absorption of the sound.

In addition to the temperature gradient, drift of the phonon gas can also be produced by the sound wave itself, as a result of the drag effect.[6,7] Phonons are dragged when the normal collisions are sufficiently intense;[4] the state of the phonon gas remains practically in equilibrium and is described by a biased Bose-Einstein distribution. The local drift velocity and the temperature are determined in this case by the intensity of the sound wave and the value of the absorption coefficient.[6,7] Inasmuch as the sound absorption coefficient depends on the drift velocity and on the temperature of the phonon gas, it is clear that the drag and heating of phonons by the sound should lead to nonlinear sound absorption, i.e., to a non-exponential dependence of the energy of the sound wave on the spatial coordinate.

In the present research, we consider thermo-absorption and nonlinear sound absorption in the hydrodynamic region, when two hydrodynamic parameters—the drift velocity V and the temperature T—completely characterize the state of the phonon gas. Here we shall assume that the dissipation of the quasimomentum of the phonon system takes place principally in the bulk of the sample, and that the effect of the boundaries can be neglected. Such a situation arises for $d^2 \gg l_N l_U$, where d is the characteristic transverse dimension of the sample and l_N and l_U are the free path lengths of phonons relative to N and U processes, respectively.[5] This inequality is satisfied with a great deal to spare in experiments on second sound[8] and on the measurement of the thermal conductivity of solid He^4[9,10] at $T \gtrsim 0.8$°K. The measurement of the velocity of ordinary (first) sound in solid helium has also been carried out on sufficiently massive samples.[11] At the same time, the solution of the problem in the limit $d^2 \gg l_N l_U$ is greatly simplified, since all the physical quantities in this case depend only on a single spatial coordinate.

2. THERMO-ABSORPTION OF SOUND

We now find the dependence of Γ on V for an isotropic dielectric without dispersion in the case $\hbar\Omega \gg T$ that is of practical interest. The coefficients of absorption of transverse and longitudinal sound have the form:[1]

$$\Gamma_t = -\frac{\hbar A_t}{16\pi^2\rho^3 c_t^4 p}\int \frac{(\mathbf{kp})^2(\mathbf{ke})^2}{k^2}\frac{\partial N(\mathbf{k})}{\partial k}\delta\left(\frac{c_t}{c_l}-\frac{\mathbf{kp}}{kp}\right)d\mathbf{k}, \quad (1)$$

$$\Gamma_l = -\frac{\hbar A_l}{16\pi^2\rho^3 c_t^4 p}\int p^2k^2\frac{\partial N(\mathbf{k})}{\partial k}\delta\left(1-\frac{\mathbf{kp}}{kp}\right)d\mathbf{k}, \quad (2)$$

where p and k are the wave vectors of the sound wave and the thermal phonon, respectively, A_t and A_l are certain quadratic functions of the third-order elastic constants, e is the polarization vector of the sound

wave, c_l and c_t are the velocities of longitudinal and transverse sound, ρ is the density of the crystal, and N(k) is the phonon distribution function. In writing down (2), it was assumed that the longitudinal sound attenuates as the result of interaction with parallel longitudinal phonons, an assumption confirmed by experiment (see, for example,[12]). For simplicity, we assume that the heat flow is parallel to p. Then, substituting the Bose-Einstein distribution with drift in (1) and (2), we get

$$\Gamma_t(V,T)=\frac{\pi^3A_t\alpha^2(1-\alpha^2)}{60\hbar^3\rho^3c_t^8}\frac{pT^4}{(1-\alpha V/c_t)^4},\qquad \Gamma_l(V,T)=\frac{\pi^3A_l}{30\hbar^3\rho^3c_l^8}\frac{pT^4}{(1-V/c_l)^4} \tag{3}$$

where $\alpha = c_t/c_l$. These formulas are valid for $1 - V/c_l \gg T/\Theta$; however, this condition is always satisfied, since V cannot exceed c_l.

Experimentally, it is most convenient to observe thermo-absorption of sound in those crystals where hydrodynamic behavior of the phonon gas has been observed, i.e., in solid helium and in NaF crystals.[8,9,13] We shall give estimates for these materials. For $T \ll \Theta$, the temperature gradient is connected with the drift velocity of the phonons by the relation

$$\frac{dT}{dx}\cong-\frac{4\pi^2T^4}{45\lambda\hbar^3c_t^2}\frac{V}{c_t}\left[1-\left(\frac{V}{c_t}\right)^2\right]^{-3}, \tag{4}$$

where λ is the thermal conductivity of the crystal. We have taken into account here only the contribution of the transverse branches of the phonon spectrum, since the contribution of the longitudinal branch is small. In solid He^4, c_t = 330 m/sec, α = 0.55 and λ = 50 W/cm-deg for T = 0.9°K.[9,14] Then, at $V \ll c_t$, we get for the relative changes of Γ_t and Γ_l:

$$\delta\Gamma_t/\Gamma_t\approx 4dT/dx,\quad \delta\Gamma_l/\Gamma_l\approx 6.7dT/dx, \tag{5}$$

where the temperature gradient is measured in deg/cm. It is seen from (5) that a 10% change in Γ_l is already reached for dT/dx = 0.15 deg/cm. In crystals of NaF, $c_t = 3.1\times10^3$ m/sec, α = 0.55 and λ = 150 W/cm-deg for T = 12°K.[13] For small V/c_t, we get

$$\delta\Gamma_t/\Gamma_t\approx 0.033dT/dx,\quad \delta\Gamma_l/\Gamma_l\approx 0.06dT/dx. \tag{6}$$

Thus the considered effect is not difficult to observe with currently available technology.

As is well known, a wave interacting with a beam of particles (or quasiparticles) is amplified if the drift velocity of the particles exceeds the phase velocity of the wave. In principle, this effect should be observed even in dielectrics. For example, for a sufficiently high phonon drift velocity, amplification of the surface waves should occur in the crystal, since their phase velocity is always less than c_t. In ferromagnets and antiferromagnets, sound amplification by a flux of magnons is evidently possible if their drift velocity is larger than the sound velocity. However, to assess the possibility of the experimental observation of these effects, a more detailed investigation is necessary, an investigation which goes beyond the bounds of this research.

3. SELF-CONSISTENT SOUND PROPAGATION AND HEAT FLOW. FUNDAMENTAL EQUATIONS

We now proceed to the consideration of the nonlinear damping of sound under the conditions of phonon hydrodynamics. The nonlinear hydrodynamic equations describing the motion of a phonon gas in a deformed dielectric crystal, valid for arbitrary (not small) velocities of phonon drift, were first obtained by Éfros.[15] Somewhat later, Nielsen and Shklovskiĭ[16] derived these equations for the case of the absence of elastic deformations, but with account of the dissipation of the quasimomentum of the phonon gas, which was completely ignored by Éfros.[15] We shall use the results of[16], generalizing them to the case of propagation of a high-frequency sound wave in the crystal. Thus, the phenomenological equations which describe the self-consistent sound propagation and heat flow in the stationary case will have the form

$$\frac{dq}{dx}=-\Gamma(V,T)q,\qquad \frac{d\Pi(V,T)}{dx}=\left(\frac{\partial P}{\partial t}\right)_{\text{coll}}+\left(\frac{\partial P}{\partial t}\right)_{\text{so}},\quad \frac{dQ(V,T)}{dx}=\left(\frac{\partial E}{\partial t}\right)_{\text{so}}. \tag{7}$$

Here q is the energy flux density of the sound wave, $\Pi(V, T)$ is the quasimomentum flux density of the phonon gas, and Q(V, T) is the thermal flux density. The collision term $(\partial P/\partial t)_{coll}$ is equal to the rate of dissipation of the quasimomentum per unit volume of the crystal P(V, T) due to U processes and impurity scattering. The terms $(\partial P/\partial t)_{so}$ and $(\partial E/\partial t)_{so}$ describe the drag and heating, respectively, of the phonons by the sound.[6,7] The one-dimensional character of the problem allows us to write out Eqs. (7) in scalar form, although in fact the quantities q, Q, P, and V are the x components of the corresponding vectors and Π represents the xx component of the tensor Π_{ij}. In writing down (7), we have neglected the viscous terms.[4,5] It can be shown that account of the viscosity of the phonon gas would have led in the given case to the appearance in the final formula of inconsequentially small corrections of the order of $l_N\Gamma$.

For the solution of the set of Eqs. (7) we need to express all the quantities entering into (7) in terms of q, V, and T, which can be done only for some specific model of the crystal. The functions P(V, T), Q(V, T), and $\Pi(V, T)$ have been calculated previously[16] for the case of a spectrum consisting of a single acoustic branch without dispersion. We also restrict ourselves to the consideration of this simple model, claiming only qualitative results and correction estimates. Thus,

$$P=\frac{BT^4v}{c(1-v^2)^3},\quad Q=c^2P,\quad \Pi=\frac{BT^4}{4}\frac{1+3v^2}{(1-v^2)^3}\quad B=\frac{2\pi^2}{45(\hbar c)^3} \tag{8}$$

where c is the sound velocity and v = V/c. The collision term in the limits of the hydrodynamic region can depend explicitly only on V and T. Without loss of generality we can represent it in the form

$$\left(\frac{\partial P}{\partial t}\right)_{\text{coll}}=-\frac{P(V,T)}{\tau(V,T)}. \tag{9}$$

The expressions for $(\partial P/\partial t)_{so}$ and $(\partial E/\partial t)_{so}$ in the considered mode are easily found from the corresponding conservation laws:

$$\left(\frac{\partial P}{\partial t}\right)_{\text{so}}=-\frac{1}{c}\frac{dq}{dx},\quad \left(\frac{\partial E}{\partial t}\right)_{\text{so}}=-\frac{dq}{dx}, \tag{10}$$

The absorption coefficient $\Gamma(V, T)$ was calculated above for the Debye model. In the case of a spectrum consisting of a single branch, the expression for $\Gamma(V, T)$ can be easily shown to take the form

$$\Gamma(V,T)=\frac{\Gamma_0}{(1-v)^4}\left(\frac{T}{T_0}\right)^4, \tag{11}$$

where $\Gamma_0 = \Gamma(0, T_0)$.

We now turn to the set of equations (7). Substituting the second relation (10) in the third equation of (7) and then integrating it, we get

$$q = Q_0 - Q(V, T), \tag{12}$$

where Q_0 is the integration constant and is equal to the density of the total (acoustic and thermal) energy flux in the sample. Replacing V by v everywhere, and substituting (8)–(12) in the first two equations of (7), we find

$$\frac{1+5v^2}{1-v^2}\frac{dv}{dx}+\frac{4v}{T}\frac{dT}{dx}=\frac{\Gamma_0}{(1-v)^4}\left[v_0(1-v^2)^3-v\left(\frac{T}{T_0}\right)^4\right],$$
$$\frac{3v(1+v^2)}{1-v^2}\frac{dv}{dx}+\frac{1+3v^2}{T}\frac{dT}{dx}$$
$$=\frac{\Gamma_0}{(1-v)^4}\left[v_0(1-v^2)^3-v\left(\frac{T}{T_0}\right)^4\right]-\frac{v}{l(v,T)} \tag{13}$$

Here $l(v, T) = c\tau(v, T)$, $v_0 = 3Q_0/cT_0C_0$ and $C_0 = 3BT_0^3$ is the heat capacity per unit volume of the crystal at $v = 0$ and $T = T_0$. The constant v_0 has the meaning of the drift velocity (referred to the sound velocity) corresponding to the thermal flux density $Q = Q_0$ at the temperature $T = T_0$.

The function $l(v, T)$ remains undetermined in Eqs. (13). The specific form of $l(v, T)$ can be established only by using some approximations for the collision integral in the kinetic equation. We shall not compute $l(v, T)$ here, especially since the expressions obtained in this fashion would be rather difficult to estimate. Instead, we represent $l(v, T)$ in the form of a power series in v and $\theta = (T - T_0)/T_0$ and then find the energy flux density of the sound wave as a function of the coordinates in first nonlinear approximation, with the help of (8) and (12). This allows us to determine the character of the developing nonlinearity, and to estimate the value of the sound intensity for which the nonlinear effects become appreciable.

4. CALCULATION OF THE SOUND-WAVE INTENSITY IN THE FIRST NONLINEAR APPROXIMATION

Assuming $v \ll 1$ and $\theta \ll 1$, we expand v and θ in terms of increasing smallness:

$$v = v^{(1)} + v^{(2)} + \ldots, \quad \theta = \theta^{(1)} + \theta^{(2)} + \ldots . \tag{14}$$

We further express q in terms of v and θ with accuracy to terms of second order:

$$q = q^{(1)} + q^{(2)}, \quad q^{(1)} = {}^1/_3 cT_0C_0(v_0 - v^{(1)}),$$
$$q^{(2)} = -{}^1/_3 cT_0C_0(v^{(2)} + 4v^{(1)}\theta^{(1)}). \tag{15}$$

It is clearly seen from (15) that it suffices to know only the drift velocity v in second approximation to find the nonlinear correction $q^{(2)}$. In order to obtain equations for $v^{(1)}$, $\theta^{(1)}$, and $v^{(2)}$, it is necessary to expand the coefficients before the derivatives and the right sides of Eq. (13) in two-dimensional Taylor series. We consider separately the expansion of the dissipative term $v/l(v, T)$. It is clear from general considerations that $l(v, T)$ should be an even function of v. We can therefore write

$$\frac{v}{l(v,T)}=\frac{v}{l_0}(1+a\theta+\ldots)+v^3f(\theta)+\ldots, \tag{16}$$

where $l_0 = l(0, T_0)$, a is a constant, and $f(\theta)$ is some function of temperature. Thus the dissipative term remains proportional to v in the first nonlinear (quadratic) approximation in v. Further, substituting (14) in (13), with account of the expansions given above, and equating terms of the same order of smallness, we obtain

$$\frac{dv^{(1)}}{dx}=\Gamma_0(v_0-v^{(1)}), \quad \frac{d\theta^{(1)}}{dx}=\Gamma_0 v_0-\left(\Gamma_0+\frac{1}{l_0}\right)v^{(1)}; \tag{17}$$
$$\frac{dv^{(2)}}{dx}+\Gamma_0 v^{(2)}=4\Gamma_0 v^{(1)}\left(v_0-v^{(1)}-\theta^{(1)}-\frac{1}{\Gamma_0}\frac{d\theta^{(1)}}{dx}\right). \tag{18}$$

The character of the solutions of Eqs. (17) and (18) is determined by the form of the boundary conditions for $v^{(1)}$, $\theta^{(1)}$, and $v^{(2)}$. We consider the case $Q_0 = 0$, which is not difficult to realize experimentally. Here $Q(x) = -q(x)$ in any transverse cross section of the sample; consequently,

$$v^{(1)}(0) = -3q_0 / cT_0C_0 = -v_*,$$

where $q_0 = q(0)$ is the sound intensity at the input to the crystal (for $x = 0$). This regime results if the output face of the sample is thermally isolated and the sound is practically entirely damped over the length of the crystal. For $\theta^{(1)}$, we assume the condition $\theta^{(1)} = 0$, i.e., we assume that the input face is maintained at a constant temperature T_0. The constant of integration in the expression for $v^{(2)}$ will be determined from the condition $q^{(2)}(0) = 0$, which reduces to the equality $v^{(2)}(0) = 0$, as is not difficult to see.

The solution of the set of equations (17) and (18) does not present any difficulty. However, the expressions thus obtained for $v^{(2)}$ and $q^{(2)}$ will contain secular terms. To eliminate the secular terms, it is necessary, as in the case of nonlinear mechanics,[17,18] to take into account the "renormalization" of the absorption coefficient Γ_0 due to nonlinear effects, i.e., to replace Γ_0 in (17) and (18) by $\gamma = \Gamma_0 + \Gamma^{(1)}$, where $\Gamma^{(1)}/\gamma \sim v^{(1)} \ll 1$. Here, an additional term appears in Eq. (18) proportional to the correction $\Gamma^{(1)}$, which allows us to eliminate the secular terms by suitable choice of $\Gamma^{(1)}$. Thus, recognizing that $v_0 = 0$ in the case under discussion, we get

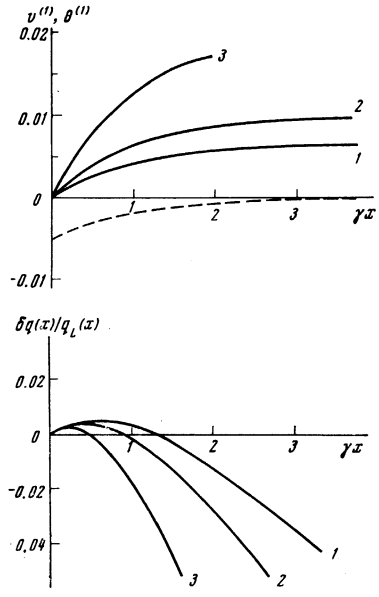
$$\frac{dv^{(1)}}{dx}+\gamma v^{(1)}=0, \qquad \frac{d\theta^{(1)}}{dx}+\gamma(1+\Lambda)v^{(1)}=0; \tag{19}$$

$$\frac{dv^{(2)}}{dx}+\gamma v^{(2)}=-4\gamma v^{(1)}\left(v^{(1)}+\theta^{(1)}+\frac{1}{\gamma}\frac{d\theta^{(1)}}{dx}\right)+\Gamma^{(1)}v^{(1)}, \tag{20}$$

where $\Lambda = (l_0\gamma)^{-1}$. Integrating the set of equations (19) and (20) with account of the boundary conditions, and substituting the result in (15), we find:

$$v^{(1)}=-v_*e^{-\gamma x}, \quad \theta^{(1)}=v_*(1+\Lambda)(1-e^{-\gamma x}), \quad \Gamma^{(1)}=4v_*\gamma(1+\Lambda), \tag{21}$$
$$q^{(1)}=q_0e^{-\gamma x}, \quad q^{(2)}=(12q_0^2/cT_0C_0)e^{-\gamma x}(2+\Lambda)(1-e^{-\gamma x}). \tag{22}$$

Equations (21) and (22) are evidently valid for any values of x if the inequality $4v_S(1 + \Lambda) \ll 1$ is satisfied.



Plots of the x dependence of the drift velocity $v^{(1)}$, of the temperature of the heating $\theta^{(1)}$ (in first approximation) and of the ratio $\delta_{q/qL}$ for three values of Λ: 1–0.3, 2–1.0 and 3–3.0. The dashed curve on the upper drawing is the plot of $v^{(1)}(x)$.

The investigated nonlinear effect is characterized by the difference $q(x) - q_L(x) = \delta q(x)$, which represents the nonlinear correction to the sound intensity $q_L(x) = q_0 \exp(-\Gamma_0 x)$, calculated without account of the drag and heating of the thermal phonons by the sound wave. By means of (15), (21) and (22), we get

$$\delta q(x) = q_L(x)\{[1 + 4v_s(2 + \Lambda)(1 - e^{-\gamma x})]\exp[-4v_s(1 + \Lambda)\gamma] - 1\}. \quad (23)$$

The figure shows plots of $v^{(1)}$, $\theta^{(1)}$, and the ratio $\delta q/q_L$ against x for the three values of Λ: 0.3, 1.0, and 3.0. These quantities are obtained if we take the three values of l_0, equal to 1, 3, and 10 cm, which correspond to solid He^4 for $T_0 = 0.8$, 0.7, 0.6°K, and $\gamma = 0.33\ cm^{-1}$. Such a value of γ should be observed in solid He^4 for the same temperatures and for sound frequencies of the order of 100 MHz.[6] The parameter v_s is taken to be equal to 5×10^{-3}.

The results have a clear physical meaning. In the case considered, the phonon drift is directed counter to the direction of propagation of sound and the region of developed drift extends to the input face of the same (see the figure). Inasmuch as the heating of the crystal near this face is small, the principal effect on the local sound absorption coefficient here is exerted precisely the drift of the thermal phonons. This leads to local acoustic "clearing" of the crystal (the correction $\delta q(x)$ is positive). As the coordinate x increases, the drift velocity decreases and the heating increases, which brings about a change in the sign of $\delta q(x)$, i.e., the crystal "darkens." We estimate the value of q_0 for which the nonlinear sound damping becomes appreciable in crystalline He^4. For definiteness, we set $\gamma x = 2$ and $\Lambda = 1.0$. It is natural to take c to be equal to the velocity of transverse sound c_t, since the transverse branches of the spectrum in the real crystal give the principal contribution to the thermal flux and the quasimomentum per unit volume. Then, calculating the heat capacity C_0, it is not difficult to find, by use of (23), that δq amounts to 10% of q_L for $T_0 = 0.6°K$ if $q_0 = 0.03\ W/cm^2$. For $T_0 = 0.8°K$, δq is equal to $0.1\ q_L$, for $q_0 = 0.1\ W/cm^2$. Thus the nonlinear absorption of sound can be found without any difficulty with existing experimental technology. Further, inasmuch as the values of q_0, for which nonlinear damping begins to appear, are small and decrease with decrease in the temperature, this effect should be kept in mind in the measurement of ordinary linear damping of high frequency sound in pure crystals of solid helium.

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[1]L. D. Landau and Yu. B. Rumer, Phys. Z. Sowjetunion **11,** 18 (1937).
[2]B. L. Timan, Fiz. Tverd. Tela **6,** 950 (1964) [Sov. Phys.-Solid State **6,** 732 (1964)].
[3]Lattice Dynamics. W. P. Mason, Ed. [Russian translation, Mir P., 1968].
[4]R. N. Gurzhi, Usp. Fiz. Nauk **94,** 689 (1968) [Sov. Phys.-Usp. **11,** 255 (1968)].
[5]R. A. Guyer and J. A. Krumhansl, Phys. Rev. **148,** 766 (1966); Phys. Rev. **148,** 778 (1966).
[6]A. I. Sokolov, Zh. Eksp. Teor. Fiz. Pis'ma Red. **11,** 138 (1970) [JETP Lett. **11,** 85 (1970)].
[7]A. I. Sokolov, Fiz. Tverd. Tela **13,** 1021 (1971) [Sov. Phys.-Solid State **13,** 850 (1971)].
[8]C. C. Ackerman and R. A. Guyer, Ann. Phys. (N.Y.) **50,** 128 (1968).
[9]L. P. Mezhov-Deglin, Zh. Eksp. Teor. Fiz. **49,** 66 (1965) [Sov. Phys.-JETP **22,** 47 (1966)]; Zh. Eksp. Teor. Fiz. **52,** 866 (1967) [Sov. Phys.-JETP **25,** 568 (1967)].
[10]R. M. Hogan, R. A. Guyer, and H. A. Fairbank, Phys. Rev. **185,** 356 (1969).
[11]R. Wanner and J. P. Franck, Phys. Rev. Lett. **24,** 365 (1970).
[12]J. de Klerk and P. G. Klemens, Phys. Rev. **147,** 585 (1966).
[13]H. E. Jackson, C. T. Walker, and T. F. McNelly, Phys. Rev. Lett. **25,** 26 (1970).
[14]J. H. Vignos and H. A. Fairbank, Phys. Rev. **147,** 185 (1966).
[15]A. L. Éfros, Zh. Eksp. Teor. Fiz. **54,** 1764 (1968) [Sov. Phys.-JETP **27,** 948 (1968)].
[16]H. Nielsen and B. I. Shklovskiĭ, Zh. Eksp. Teor. Fiz. **56,** 709 (1969) [Sov. Phys.-JETP **29,** 386 (1969)].
[17]L. D. Landau and E. M. Lifshitz, Mekhanika (Mechanics), Nauka, 1965 [Addison-Wesley, 1969].
[18]A. Blaquiere, System Analysis, Academic, 1966.

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