The Influence of the Finite Width of the Conduction Band on the Heating of Electrons in an Electric Field

I. B. Levinson and Ya. Yasevichyute
L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences
Institute of Semiconductor Physics, Lithuanian Academy of Sciences
Submitted November 18, 1971
Zh. Eksp. Teor. Fiz. 62, 1902–1912 (May, 1972)

The influence of Stark quantization in a semiconductor on the heating of the electrons is investigated. A model is considered in which the allowed electron band has a finite width along the direction of the electric field and is infinite across the field, while the phonon energy and the lattice temperature are smaller than the width of the band. It is shown that the ultraquantum range of field intensities breaks up into two parts. In weaker fields the energy dissipation does not depend on the field strength E and therefore the current $j \sim E^{-1}$; on the other hand in stronger fields the energy dissipation decreases with increasing E, and therefore the current falls off faster than E^{-1}.

INTRODUCTION

In the majority of cases the physical phenomena associated with the heating of electrons in semiconductors by a strong electric field **E** are insensitive to the finite width $\Delta\mathscr{E}$ of the conduction band, since usually the field **E** is so weak, and the scattering is so large, that the average energy $\langle\mathscr{E}\rangle$ of the electrons turns out to be very much smaller than $\Delta\mathscr{E}$. However, if the field E becomes sufficiently large, then it may turn out that $\langle\mathscr{E}\rangle \approx \Delta\mathscr{E}$, and then it is necessary to take the finite width of the conduction band into consideration. It should be kept in mind here that two substantially different situations are possible. In weaker fields the electron increases its energy up to the top of the band by means of "diffusion," thereby undergoing many collisions. In stronger fields the electron increases its energy up to the top of the band by "dynamical" means, being accelerated by the field during the time interval between two collisions. In the first situation the finite width of the band can be taken into account within the framework of the classical kinetic equation,[1] and simply amounts to a restriction of the momentum space; on the other hand quantum effects are essential in the second situation, these effects being due to the quantization of the periodic motion of the electron reflected from the boundaries of the Brillouin zone (the Stark levels). Recent experimental data offer evidence that these levels play a role in galvanomagnetic phenomena.[2] A very general theory of the influence of Stark levels on electron heating has been developed by Bryksin and Firsov[3] on the basis of a diagram technique. However, if we confine our attention to the case of a weak electron-phonon coupling, then we can use the more easily understood quantum-kinetic equation of[4] and thereby obtain, in a simpler way, a whole series of results, including some which were not previously known.

To obtain results in a more understandable form, we employ the following model for the energy spectrum. In the first place it is assumed that the forbidden bands are much wider than the allowed bands, so that interband tunneling due to the influence of the field is unimportant. Second, we assume that the electrons are essentially free when they are moving transverse to the field, but are essentially bound when moving along the field, that is,

$$\mathscr{E}(\mathbf{p}) = \frac{p_\perp^2}{2m} - \Delta\mathscr{E}\cos\frac{p_\parallel a}{\hbar}. \tag{1}$$

Here $p_\parallel$ and $\mathbf{p}_\perp$ denote the components of **p** along and transverse to the field **E**, m is the effective mass in the transverse direction, $\Delta\mathscr{E}$ is the half-width of the band in the longitudinal direction, and a is the lattice period along **E**. Actually Eq. (1) means that the band is much wider across **E** than along **E**, and the energies $\mathscr{E}$ of interest to us are much lower than the transverse width of the band.

With regard to the electron-phonon interaction, we shall assume that it is weak, that it occurs with phonons which are in equilibrium at a low temperature $T \ll \Delta\mathscr{E}$, and that the energy of the important phonons is small: $\hbar\Omega \ll \Delta\mathscr{E}$. It is also assumed that the scattering by phonons has the same kind of axial symmetry about the direction of **E** as the spectrum has. For orientation, we note that the Stark levels appear when $eEa \gg \hbar/\tau$ (τ is the electronic relaxation time); this gives the condition $E > 10^4$ V/cm for typical values $a = 5 \times 10^{-8}$ cm and $\tau = 10^{-12}$ sec.

We point out that an energy spectrum which approximately corresponds to (1) may exist in crystals with a graphite structure when the field is oriented along the hexagonal axis, and it may also exist in "superlattices," which can be obtained by forming alloys with periodically varying concentrations.[8,9] In the latter case $a = 10^{-6}$ cm, and the Stark levels can be observed in weaker fields, $E > 10^3$ V/cm. Certain aspects of the kinetic effects in such "superstructures" have been considered in[8,10].

1. THE QUANTUM KINETIC EQUATION

Let us start from the general equation (1.16) given in[4]; for the case we are interested in it is necessary to write this equation down in the Bloch-function representation instead of the plane-wave representation. For the assumed model of the spectrum, the equation (without any magnetic field) retains the form of (3.8) in[4], the only difference being that the meaning of the electron-phonon interaction constant $|c_q|^2$ is changed somewhat; we now denote this constant by B(q) (where q is the phonon momentum). We also modify the collision term by making the following substitution in Eq. (1.20) of[4]

$$\int_{-\infty}^{t} dt_0 \ldots \to \frac{1}{2}\int_{-\infty}^{+\infty} dt_0 \ldots$$

It is assumed here, as it were, that the collision act is completed by the instant t under consideration.[5] In other words, this substitution means

$$P/\mathscr{E} - i\pi\delta(\mathscr{E}) \to -i\pi\delta(\mathscr{E}),$$

i.e., it corresponds to the fact that the electron transitions under the influence of the electron-phonon interaction remain, but the renormalization of the electron spectrum under the influence of this interaction is not taken into consideration.

After this has been done the equation for the Wigner density f(p), which in the present case coincides with the diagonal element of the electron density matrix, will have the following form:

$$eE\frac{\partial}{\partial p_\parallel} f(\mathbf{p}) = I(f|\mathbf{p}), \tag{1.1}$$

where the collision term is formally analogous to the classical term

$$I(f|\mathbf{p}) = \int d\mathbf{p}'[f(\mathbf{p}')W(\mathbf{p}',\mathbf{p}) - f(\mathbf{p})W(\mathbf{p},\mathbf{p}')]. \tag{1.2}$$

The "transition probability" W consists of two parts which correspond to the emission and absorption of phonons:

$$W(\mathbf{p},\mathbf{p}') = W^+(\mathbf{p},\mathbf{p}') + W^-(\mathbf{p},\mathbf{p}') \tag{1.3}$$

$$W^\pm(\mathbf{p},\mathbf{p}') = \int d\mathbf{q}\frac{2\pi}{\hbar}B(q)\left[N(q)+\frac{1}{2}\pm\frac{1}{2}\right]\delta(\mathbf{p}-\mathbf{p}'\mp\mathbf{q}) \times D^\pm(\mathbf{p},\mathbf{p}',\Omega(q)), \tag{1.4}$$

where N(q) denotes the equilibrium number of phonons and

$$D^\pm(\mathbf{p},\mathbf{p}',\Omega) = \mathrm{Re}\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dt\,\exp\left\{\frac{i}{\hbar}\int_0^t dt' \times[\mathscr{E}(\mathbf{p}-e\mathbf{E}t') - \mathscr{E}(\mathbf{p}'-e\mathbf{E}t')\mp\hbar\Omega]\right\}. \tag{1.5}$$

The "probability" W depends on the field E through the functions $D^\pm$, and the quantum nature of Eq. (1.1) is contained in this fact. The transition to the classical equation is evident from the equation

$$\lim_{E\to 0} D^\pm(\mathbf{p},\mathbf{p}',\Omega) = \delta(\mathscr{E}(\mathbf{p}) - \mathscr{E}(\mathbf{p}')\mp\hbar\Omega).$$

It is obvious that the Stark "levels" will appear only if they are not washed out by the collisions, i.e., if the distance eEa between these levels is very large compared to $\hbar/\tau$, where τ is the effective time for the scattering of an electron by phonons. But, on the other hand, this condition means that $\omega_E\tau \gg 1$, that is, the collisions are a slow process in comparison with the oscillations along the field, which have a frequency $\omega_E = eEa/\hbar$. Therefore Eq. (1.1) can be iterated with respect to the collision term. Proceeding in analogy to what was done in[6], let us set

$$f(\mathbf{p}) = f^0(\mathbf{p}) + f'(\mathbf{p}), \tag{1.6}$$

where f′ is small in comparison with f^0, and then by iterating with respect to I we have

$$eE\frac{\partial}{\partial p_\parallel} f^0(\mathbf{p}) = 0, \tag{1.7}$$

$$eE\frac{\partial}{\partial p_\parallel} f'(\mathbf{p}) = I(f^0|\mathbf{p}). \tag{1.8}$$

It follows from Eq. (1.7) that f^0 depends only on $\mathbf{p}_\perp$ and furthermore

$$f'(\mathbf{p}_\perp, p_\parallel) = \frac{1}{eE}\int_{-\pi/a\hbar b}^{p_\parallel} dp_\parallel' I(f'|\mathbf{p}_\perp, p_\parallel') + C(\mathbf{p}_\perp), \tag{1.9}$$

where $b = 2\pi/a$ is the reciprocal lattice vector along E and C is an arbitrary function. From the periodicity of f with respect to $p_\parallel$ (the period being equal to $\hbar b$) it follows that

$$\oint dp_\parallel I(f^0|\mathbf{p}_\perp, p_\parallel) = 0, \tag{1.10}$$

which is the equation for the determination of f^0.

We introduce the total number of electrons having a given transverse momentum, i.e., the number of electrons on the trajectory of the rapid motion:

$$F(\mathbf{p}_\perp) = \int dp_\parallel f^0(\mathbf{p}) = \hbar b f^0(\mathbf{p}_\perp), \tag{1.11}$$

and the transition probability averaged over the initial trajectory of the rapid motion and summed over the final trajectory of this motion is given by

$$W(\mathbf{p}_\perp, \mathbf{p}_\perp') = \oint\frac{dp_\parallel}{\hbar b}\oint dp_\parallel' W(\mathbf{p},\mathbf{p}'). \tag{1.12}$$

Then Eq. (1.10) takes the form

$$\int d\mathbf{p}_\perp[F(\mathbf{p}_\perp')W(\mathbf{p}_\perp',\mathbf{p}_\perp) - F(\mathbf{p}_\perp)W(\mathbf{p}_\perp,\mathbf{p}_\perp')] = 0. \tag{1.13}$$

This is the kinetic equation from which the fast process, namely the dynamic motion under the influence of the field, has been eliminated.

The description employed here is very similar in form to the classical description. Let us indicate how this description goes over into the quantum description containing the Stark levels. In order to do this, we define

$$\varepsilon(\mathbf{p}_\perp) = \frac{1}{\hbar b}\oint dp_\parallel \mathscr{E}(\mathbf{p}_\perp, p_\parallel), \tag{1.14}$$

which is catually the energy of the "zero" Stark level, and we then write down

$$\mathscr{E}(\mathbf{p}_\perp, p_\parallel) = \varepsilon(\mathbf{p}_\perp) + \Delta\mathscr{E}(\mathbf{p}_\perp, p_\parallel). \tag{1.15}$$

Then Eq. (1.5) can be transformed into

$$D^\pm(\mathbf{p},\mathbf{p}',\Omega) = \sum_{s=-\infty}^{+\infty}\delta(\varepsilon(\mathbf{p}_\perp) - \varepsilon(\mathbf{p}_\perp')\mp\hbar\Omega + seEa)D_s(\mathbf{p},\mathbf{p}'), \tag{1.16}$$

$$D_s(\mathbf{p},\mathbf{p}') = \mathrm{Re}\frac{\omega_E}{2\pi}\oint dt e^{-is\omega_E t}\exp\left\{\frac{i}{\hbar}\int_0^t dt'[\Delta\mathscr{E}(\mathbf{p} - e\mathbf{E}t') - \Delta\mathscr{E}(\mathbf{p}'-e\mathbf{E}t')]\right\} = D_s(-\mathbf{p},-\mathbf{p}') = D_{-s}(\mathbf{p}',\mathbf{p}). \tag{1.17}$$

By substituting (1.16) into (1.4) we find

$$W^\pm(\mathbf{p}_\perp,\mathbf{p}_\perp') = \sum_{s=-\infty}^{+\infty} W_s^\pm(\mathbf{p}_\perp,\mathbf{p}_\perp'), \tag{1.18}$$

$$W_s^\pm(\mathbf{p}_\perp,\mathbf{p}_\perp') = \int d\mathbf{q}\frac{2\pi}{\hbar}B(q)\left[N(q)+\frac{1}{2}\pm\frac{1}{2}\right] \times\oint\frac{dp_\parallel}{\hbar b}\oint dp_\parallel'\delta(\mathbf{p}-\mathbf{p}'\mp\mathbf{q})D_s(\mathbf{p},\mathbf{p}')\delta(\varepsilon(\mathbf{p}_\perp) - \varepsilon(\mathbf{p}_\perp')\mp\hbar\Omega(q) + seEa). \tag{1.19}$$

The delta-function in Eq. (1.19) corresponds to the con-

servation law associated with a transition between two Stark levels: $np_\perp \to n'p'_\perp$ with $n' = n - s$. Such a transition is simply a displacement of the state through a distance as toward the side of decreasing potential energy (for $s > 0$), and D_s represents the extent of the overlap of the wave functions of the initial and the shifted states. This interpretation is confirmed by an investigation of the quantity D_s. The expression inside the square brackets in Eq. (1.5) is a periodic function of t with frequency ω_E, an amplitude of the order of $\Delta\mathscr{E}$, and a mean value equal to zero. Therefore its integral is a periodic function of the same frequency with an amplitude of the order of $\Delta\mathscr{E}/\omega$. It is clear therefore that we have $D_s \to \delta_{s,0}$ in the ultraquantum case $\Delta\mathscr{E}/\hbar\omega_E \to 0$. This corresponds to the localization of the Stark functions within the limits of a single cell. Hence it is also clear that the quantity D_s is of the order of unity in the quasiclassical case $\Delta\mathscr{E}/\hbar\omega_E \to \infty$ for $s \lesssim \Delta\mathscr{E}/\hbar\omega_E$; this corresponds to an effective overlap distance $sa \approx \Delta\mathscr{E}/eE$, i.e., it corresponds to the size of the region in which the Stark functions are localized.

If we start from the very beginning with a distribution $f_n(p_\perp)$ over the Stark levels, then it is obvious that the homogeneity of the system this distribution should not depend on n, since states with different values of n differ only by a spatial displacement along the field. After this remark, it now becomes clear that Eq. (1.13) is the equation of balance between the Stark levels.

2. THE SOLUTION OF THE KINETIC EQUATION

In the model which we are using, where the spectrum and the scattering are axially symmetric about the direction of **E**, the function F does not depend on the direction of $\mathbf{p}_\perp$, but depends only on $\epsilon = p_\perp^2/2m$; therefore one can write

$$F(\mathbf{p}_\perp) = \frac{1}{2\pi m} n(\varepsilon), \quad \int_0^\infty d\varepsilon n(\varepsilon) = n, \tag{2.1}$$

where n is the total electron concentration and $2\pi m$ is the density of states in the two-dimensional problem. Subsequently Eq. (1.13) can be integrated over the angle χ between $\mathbf{p}_\perp$ and $\mathbf{p}'_\perp$; as a result one obtains the following equation on the transverse-energy axis:

$$\int_0^\infty d\varepsilon' [n(\varepsilon')W(\varepsilon', \varepsilon) - n(\varepsilon)W(\varepsilon, \varepsilon')] = 0 \tag{2.2}$$

where it is convenient to normalize the probability in the following manner:

$$W(\varepsilon, \varepsilon') = 2\pi m \int \frac{d\chi}{2\pi} W(\mathbf{p}_\perp, \mathbf{p}_\perp'). \tag{2.3}$$

We now notice that we have $W(\epsilon, \epsilon') = W(\epsilon', \epsilon)$ in the limit as $\hbar\Omega \to 0$, and therefore the solution will be $n(\epsilon) = \text{const}$. One might think that the solution will be a slowly varying function for small values of $\hbar\Omega$, and therefore the integral equation can be transformed into the Fokker-Planck equation. On the other hand, in the limit as $E \to \infty$ the probabilities W_s with $s \neq 0$ are insignificant. In this case one can verify by direct substitution that $n(\epsilon) \sim e^{-\epsilon/T}$. As will be demonstrated below, the first limiting case is valid for $\hbar\Omega \ll (\Delta\mathscr{E})^2(2N+1)/eEa$, and the second limiting case is valid for $eEa \gg (\Delta\mathscr{E})^2(2N+1)/\hbar\Omega$. If these two inequalities are regarded as a limitation on the field strength, then we obtain the following two regions:

$$\begin{aligned} &\text{I)} \quad eEa \ll (\Delta\mathscr{E})^2(2N+1)/\hbar\Omega, \\ &\text{II)} \quad eEa \gg (\Delta\mathscr{E})^2(2N+1)/\hbar\Omega. \end{aligned} \tag{2.4}$$

It is seen that the two indicated limiting cases span the entire range of electric field strengths, where region I includes both the quasiclassical region $eEa \ll \Delta\mathscr{E}$ and the ultraquantum region $eEa \gg \Delta\mathscr{E}$.

To estimate the validity of either approximation, we use the scattering model in which

$$B(q) = B = \text{const}, \qquad \Omega(q) = \Omega = \text{const}.$$

One can easily calculate the probability $W(\epsilon, \epsilon')$ for this model.

First we introduce the reciprocal lifetime of the classical kinetic equation into this model

$$\frac{1}{\tau} = \frac{1}{\tau^+} + \frac{1}{\tau^-}, \quad \frac{1}{\tau^\pm} = \int d\mathbf{p}' W^\pm(\mathbf{p}, \mathbf{p}')\,|_{\varepsilon=0}. \tag{2.5}$$

As will be clear from what follows, we will be interested in those values of **p** such that $\epsilon \gg \Delta\mathscr{E}, \hbar\Omega$. If this is taken into account in the integration over $\mathbf{p}'$, then we obtain

$$\begin{aligned} &1/\tau^\pm = 2\pi mb \cdot 2\pi B(N + {}^1/_2 \pm {}^1/_2), \\ &1/\tau = 2\pi mb \cdot 2\pi B(2N+1) \equiv (2N+1)/\tau_0. \end{aligned} \tag{2.6}$$

From Eq. (1.17) it is seen that in our model the energy spectrum D_s does not depend on $\mathbf{p}_\perp$, and the integral over $p_\parallel$ and $p'_\parallel$ appearing in Eq. (1.20) can be evaluated:

$$\oint \frac{dp_\parallel}{\hbar b} \oint dp_\parallel' \delta(p_\parallel - p_\parallel' \mp q_\parallel) D_s(p_\parallel, p_\parallel') = J_s^2\left(2\frac{\Delta\mathscr{E}}{eEa}\sin\frac{aq_\parallel}{2\hbar}\right) \tag{2.7}$$

where J_s is a Bessel function. Subsequently, one can easily find from Eq. (1.20)

$$W_s^\pm(\varepsilon, \varepsilon') = \frac{1}{\tau^\pm} A_s\left(\frac{\Delta\mathscr{E}}{eEa}\right)\delta(\varepsilon - \varepsilon' \mp \hbar\Omega + seEa), \tag{2.8}$$

where

$$A_s(\xi) = \oint \frac{d\varphi}{2\pi} J_s\left(2\xi \sin\frac{\varphi}{2}\right). \tag{2.9}$$

Now let us examine region I of field strengths. By transforming the integral collision term into a differential term,[7] instead of Eq. (2.2) we have the equation

$$-\frac{\partial}{\partial\varepsilon} J(\varepsilon) = 0. \tag{2.10}$$

Here J denotes the flux on the energy axis

$$J(\varepsilon) = -\left[\frac{\partial}{\partial\varepsilon}(M_2(\varepsilon)n(\varepsilon)) + M_1(\varepsilon)n(\varepsilon)\right] \tag{2.11}$$

which can be expressed in terms of the first and second moments:

$$M_1(\varepsilon) = \int_0^\infty d\varepsilon' W(\varepsilon, \varepsilon')(\varepsilon - \varepsilon'), \tag{2.12}$$

$$M_2(\varepsilon) = {}^1/_2 \int_0^\infty d\varepsilon' W(\varepsilon, \varepsilon')(\varepsilon - \varepsilon')^2.$$

Since one must have $J(\epsilon) \to 0$ in the limit $\epsilon \to \infty$, then $J(\epsilon) \equiv 0$, and from Eq. (2.10) we obtain the distribution

$$n(\varepsilon) \sim \frac{1}{M_2(\varepsilon)} \exp\left\{-\int_0^\varepsilon d\varepsilon' \frac{M_1(\varepsilon')}{M_2(\varepsilon')}\right\}. \tag{2.13}$$

The conditions for the validity of the Fokker-Planck approximation involve the smallness of the neglected terms

$$\frac{\partial^2}{\partial\varepsilon^2}(M_3 n),\ \frac{\partial^3}{\partial\varepsilon^3}(M_4 n).$$

Assuming that the moments depend on ϵ according to a power law, one can differentiate only the exponential dependence of n on ϵ and thus obtain the following estimate:

$$\left(\frac{\partial}{\partial\varepsilon}\right)^{k-1}(M_k n)\approx M_k\left(\frac{\partial}{\partial\varepsilon}\right)^{k-1} n\approx M_k\left(\frac{M_1}{M_2}\right)^{k-1} n.$$

Then the conditions for the validity of the Fokker-Planck approximation become

$$M_3/M_2\ll M_2/M_1,\quad M_4/M_2\ll(M_2/M_1)^2. \tag{2.14}$$

In order to estimate the validity conditions, and also the average energy ϵ, we turn to our scattering model, where one can easily calculate the moments. We have the following results:

$$M_1=\frac{1}{\tau_0}\hbar\Omega,\quad M_2=\frac{1}{2}\frac{1}{\tau_0}(2N+1)(\Delta\mathscr{E})^2,$$
$$M_3=\frac{1}{2}\frac{1}{\tau_0}\hbar\Omega(\Delta\mathscr{E})^2, \tag{2.15}$$
$$M_4=\begin{cases}{}^3/_{32}\,\tau_0^{-1}(2N+1)(\Delta\mathscr{E})^4 & \text{for } eEa\ll\Delta\mathscr{E},\\ {}^1/_{24}\,\tau_0^{-1}(2N+1)(\Delta\mathscr{E})^2(eEa)^2 & \text{for } eEa\gg\Delta\mathscr{E}.\end{cases}$$

The first three moments turn out to be identical for the quasiclassical and ultraquantum cases.

In this model the energy distribution turns out to be given by

$$n(\varepsilon)\sim\exp\left(-\frac{\varepsilon}{T^*}\right),\quad T^*=\frac{(\Delta\mathscr{E})^2}{2\hbar\Omega}(2N+1), \tag{2.16}$$

so that the average transverse energy, T*, is very large compared to $\Delta\mathscr{E}$ or $\hbar\Omega$, as was assumed. The condition on the moment M_3 gives (in both the quasiclassical case and the ultraquantum case) $\hbar\Omega\ll\Delta\mathscr{E}(2N+1)$, which is automatically satisfied. In the quasiclassical case the condition on M_4 reduces to the same inequality, but in the ultraquantum case it coincides with condition I from (2.4), which also proves the validity of the Fokker-Planck expansion in the field-strength region I.

In the quasiclassical case the probability does not depend on the field strength, and therefore the distribution $n(\epsilon)$ also does not depend on the field strength (this result is independent of the model). In calculating the moments in the ultraquantum case it is seen that W_0 gives the major contribution to M_1, while the probabilities $W_{\pm 1}$ give the major contribution to M_2. By expanding the Bessel functions which appear there, one can see that M_1 and M_2 will turn out to be independent of E. Thus, in both regions the distribution turns out to be independent of the field in the lowest-order approximation. It is also clear from our estimates of the moments that, in both regions the ratio M_1/M_2, and therefore the average energy also, are of the same order of magnitude even though they may differ by a numerical factor of the order of unity. Therefore there is a weak dependence of the distribution and of the average energy on E in the transition region, $eEa\approx\Delta E$. We also note that the distribution $n(\epsilon)$ does not depend on the absolute magnitude of the scattering (in the scattering model, $n(\epsilon)$ does not depend on the value of the constant τ).

Now let us proceed to an investigation of field-strength region II, and let us write down

$$n(\varepsilon)=n_0(\varepsilon)+n_1(\varepsilon), \tag{2.17}$$

$$n_0(\varepsilon)=\frac{n}{T}e^{-\varepsilon/T},\qquad n_1\ll n_0. \tag{2.18}$$

One can easily verify that $n_0(\epsilon)$ is the solution of Eq. (2.2) provided we only keep the terms W_s with s = 0 in the expression for W. Assuming that the terms W_s with $s\neq 0$ are small, and using the method of successive approximations, we obtain the following equation for the determination of n_1:

$$\int d\varepsilon'[n_1(\varepsilon')W_0(\varepsilon',\varepsilon)-n_1(\varepsilon)W_0(\varepsilon,\varepsilon')] \tag{2.19}$$
$$=-\int d\varepsilon'[n_0(\varepsilon')W_1(\varepsilon',\varepsilon)-n_0(\varepsilon)W_1(\varepsilon,\varepsilon')].$$

Here we have neglected those terms W_s with $|s|\geq 2$ because they are small and of higher order in $\Delta E/eEa$. The terms with s = −1, which are generally of the same order of magnitude as the terms with s = +1, do not give any contribution because the arguments of the corresponding delta functions cannot vanish for $\epsilon > 0$.

Let us consider the right-hand side of the last equation in more detail. By using Eqs. (2.3) and (1.19) and neglecting the quantity $\hbar\Omega$ in W_1 in comparison with eEa, we find that the first term is a source, acting near $\epsilon = eEa$ in a region whose width is of the order of T, and the second term is a sink of the same strength near $\epsilon = 0$, in a region of the same width. Now it becomes clear that if we write the general solution of the inhomogeneous Eq. (2.19) in the form

$$n_1(\varepsilon)=ce^{-\varepsilon/T}+\bar{n}_1(\varepsilon),$$

where the first term is the general solution of the homogeneous equation corresponding to (2.19) and the second term is a particular solution of the inhomogeneous Eq. (2.19), then $\bar{n}_1$ describes the motion of the electrons from the source to the sink over the distance eEa. The characteristic length of variation of such a quantity $\bar{n}_1$ is eEa, which is larger than T*; therefore the integral operator on the left-hand side of Eq. (2.19), which operates on $\bar{n}_1$, can be transformed into the Fokker-Planck form, and the sources can be regarded as delta-functions. Taking all of this into consideration, the equation for $\bar{n}_1$ now takes the form

$$-\frac{\partial}{\partial\varepsilon}\bar{J}_1(\varepsilon)=n\frac{A_1}{\tau(E)}[\delta(\varepsilon)-\delta(\varepsilon-eEa)], \tag{2.20}$$

where $\bar{J}_1$ is obtained from J by replacing n by $\bar{n}_1$. Here $\tau(E)$ is determined by formula (2.6) for any arbitrary scattering mechanism; in (2.6) the quantity B(2N + 1), which depends on q, must be averaged over $q_\parallel$ using the weight function (2.7) and calculated for values of $q_\perp$ such that $q_\perp^2/2m = eEa$. For the scattering model $\tau(E) = \tau$. By integrating expression (2.20) we find

$$\bar{J}_1(\varepsilon)=\begin{cases}-nA_1\dfrac{1}{\tau(E)}, & \varepsilon<eEa\\ 0, & \varepsilon>eEa\end{cases}. \tag{2.21}$$

From this it follows that the distribution $\bar{n}_1(\epsilon)$ has the form (2.13) in the region $\epsilon > eEa$, and it falls to zero over a distance of the order of T*; in the range of field

strengths under consideration this distance is much smaller than eEa, i.e., one essentially has $\bar{n}_1 = 0$ in this region. For $\epsilon < eEa$ it is necessary to find the slowly-varying solution of Eq. (2.20), which can be obtained if we discard in $\bar{J}_1$ the term containing the derivative. As a result we have

$$\bar{n}_1(\varepsilon) = \begin{cases} nA_1/\tau(E)M_1(\varepsilon), & \varepsilon < eEa \\ 0 & \varepsilon > eEa \end{cases}. \tag{2.22}$$

The constant c is determined from the condition that the normalization remains the same,

$$\int_0^\infty d\varepsilon \bar{n}_1(\varepsilon) = 0,$$

which gives

$$c = -\frac{n}{T} A_1 \frac{1}{\tau(E)} \int_0^{eEa} \frac{d\varepsilon}{M_1(\varepsilon)}. \tag{2.23}$$

The condition for the validity of the approximation under consideration is that the number of electrons in the distribution $\bar{n}_1$ and in the distribution $ce^{-\epsilon/T}$ must be small. By using our scattering model we find

$$\bar{n}_1(\varepsilon) = \frac{n}{\hbar\Omega} A_1(2N+1), \quad c = -\frac{n}{T}\frac{eEa}{\hbar\Omega} A_1(2N+1). \tag{2.24}$$

Using these results for estimates, and noting that $A_1 \approx (\Delta\mathscr{E}/eEa)^2$, one can see that the validity condition coincides with condition II in (2.4).

Just as in the field-strength region I, the distribution does not depend on the absolute magnitude of the scattering.

3. CALCULATION OF THE CURRENT

The current is calculated in analogy to the classical case

$$j = \int d\mathbf{p} e v_\parallel(\mathbf{p}) f(\mathbf{p}) = Q/E, \tag{3.1}$$

where

$$Q = -\int d\mathbf{p}\mathscr{E}(\mathbf{p}) I(f_0|\mathbf{p})$$
$$= \int d\mathbf{p} f(\mathbf{p}) \int d\mathbf{p}' [\Delta\mathscr{E}(\mathbf{p}) - \Delta\mathscr{E}(\mathbf{p}')] W(\mathbf{p}, \mathbf{p}'), \tag{3.2}$$

Q may be interpreted as the loss of kinetic energy during collisions, i.e., as the Joule heating. The expression for j in terms of Q is obtained after substitution of (1.9), $v_\parallel = \partial\mathscr{E}/\partial p_\parallel$, and integration by parts. The second expression for Q is obtained after substituting the representation (1.15) into (3.2), using (1.10) and then (1.2), and finally redesignating the integration variables in one term.

A formula for the current, which can be interpreted from the viewpoint of Stark levels, is obtained after substituting relations (1.14), (1.4), and (1.16) into (3.2) and taking (1.10) into consideration. This gives

$$Q = Q^+ + Q^-,$$
$$Q^\pm = \sum_{s=-\infty}^{+\infty} \int d\mathbf{q} \frac{2\pi}{\hbar} B(q) \left[N(q) + \frac{1}{2} \pm \frac{1}{2} \right] \int d\mathbf{p}_\perp \int d\mathbf{p}_\perp' f(\mathbf{p}_\perp) \tag{3.3}$$
$$\times \delta(\mathbf{p}_\perp - \mathbf{p}_\perp' \mp \mathbf{q}_\perp) \delta(\varepsilon(\mathbf{p}_\perp') - \varepsilon(\mathbf{p}_\perp) \mp \hbar\Omega(q) + seEa) R_s(\mathbf{p}_\perp, \mathbf{p}_\perp'; q_\parallel),$$

$$R_s(\mathbf{p}_\perp, \mathbf{p}_\perp'; q_\parallel) = \int dp_\parallel \int dp_\parallel' [\Delta\mathscr{E}(\mathbf{p}) - \Delta\mathscr{E}(\mathbf{p}')] D_s(\mathbf{p}, \mathbf{p}') \delta(p_\parallel - p_\parallel' - q_\parallel)$$
$$= R_s(\mathbf{p}_\perp, \mathbf{p}_\perp'; -q_\parallel) = -R_{-s}(\mathbf{p}_\perp', \mathbf{p}_\perp; q_\parallel). \tag{3.4}$$

The symmetry property can be verified with the aid of Eq. (1.17). Formula (3.3) shows that the current arises during $n\mathbf{p}_\perp \to n'\mathbf{p}_\perp'$ transitions (where $n' = n - s$) between different Stark states, accompanied by the emission or absorption of a phonon.

If the energy $\mathscr{E}(\mathbf{p})$ is separated into the sum of the longitudinal and transverse parts, then R does not depend on the transverse momenta, and then it follows from the symmetry properties that $R_0 = 0$. In this case the transitions with s = 0 do not give any contribution to the current, that is, there is no contribution unless the longitudinal quantum number changes. This is quite understandable because if the energy is additive then the wave function of the coordinates along **E** does not depend on $\mathbf{p}_\perp$, and therefore a change of only $\mathbf{p}_\perp$ does not displace the electron along **E**. If the energy is nonadditive, then the transitions involving only a change of $\mathbf{p}_\perp$ also contribute to the current.

In the model we have been using, the spectrum is given by

$$R_s(q_\parallel) = 2\pi\hbar eEsJ_s^2\left(2\frac{\Delta\mathscr{E}}{eEa}\sin\frac{aq_\parallel}{2\hbar}\right). \tag{3.5}$$

Now let us go on to the explicit calculation of Q, first considering the field-strength region I. In the quasiclassical part of this region we must use the classical transition probabilities in Eq. (3.2). The energy delta functions in these transition probabilities contain the longitudinal energies of the electron and the phonon energy, these energies being small in comparison with the electron's transverse energy. However, it is impossible to neglect these energies since Q would then be equal to zero after integrating over $p_\parallel$ and $p_\parallel'$. Therefore one should expand the delta functions, keeping the longitudinal energies and neglecting the energy of the phonon. The situation is analogous in the ultraquantum part of the field-strength region I; here in calculating Q one can keep only the terms with $s = \pm 1$ in the summation over s and neglect the phonon energy in comparison with the transverse energy. However, it is impossible to neglect $\pm eEa$ because then the terms with $s = +1$ and $s = -1$ in the summation will exactly cancel each other since R_s is odd in s. In both cases the integration over the angles of $\mathbf{p}_\perp$ and $\mathbf{p}_\perp'$ can be carried out by using the formula

$$\oint d\varphi \oint d\varphi' \delta(\mathbf{p}_\perp - \mathbf{p}_\perp' + \mathbf{q}_\perp) = \frac{4}{\sqrt{x}}\theta(x), \tag{3.6}$$

where θ is the Heaviside step function and

$$x = (2m)^2[2\varepsilon\bar{\varepsilon} + 2\varepsilon'\bar{\varepsilon} + 2\varepsilon\varepsilon' - \varepsilon^2 - \varepsilon'^2 - \bar{\varepsilon}^2]; \quad \bar{\varepsilon} = q_\perp^2/2m.$$

As a result we obtain a single formula for the entire region I of field strengths:

$$Q = \frac{1}{2\pi}(2\Delta\mathscr{E})^2 \int d\mathbf{q} \frac{2\pi}{\hbar} B(q)(2N(q)+1)\sin^2\frac{aq_\parallel}{2\hbar} \tag{3.7}$$
$$\times \int d\varepsilon \left(-\frac{\partial n(\varepsilon)}{\partial\varepsilon}\right)\theta(4\varepsilon - \bar{\varepsilon})[(4\varepsilon - \bar{\varepsilon})\bar{\varepsilon}]^{-1/2}.$$

We obtain a very simple answer for our scattering model:

$$Q = 2\frac{\hbar\Omega}{\tau_0} n = 2\frac{\hbar\Omega}{\tau(2N+1)} n, \tag{3.8}$$

this result is valid in order of magnitude for the general case. Since $n(\epsilon)$, and therefore Q as well, do not depend on E, it follows that $j \sim E^{-1}$.

In the field-strength region II, in calculating Q one can substitute the distribution $n_0(\epsilon)$ into Eq. (3.2), replacing the exponential by a delta function; then one finds

$$Q = \frac{(\Delta\mathscr{E})^2}{2\tau(E)eEa} n. \qquad (3.9)$$

This gives $j \sim E^{-2}$ for our scattering model.

The authors express their gratitude to É. I. Rashba for a discussion of this work.

[1]Yu. A. Bychkov and A. M. Dykhne, Zh. Eksp. Teor. Fiz. **48,** 1168 (1965) [Sov. Phys.-JETP **21,** 779 (1965)].
[2]S. Maekawa, Phys. Rev. Lett. **24,** 1175 (1970).
[3]V. V. Bryksin and Yu. A. Firsov, in Proceedings of the Tenth International Conference on the Physics of Semiconductors, Cambridge, Mass., 1970, p. 767.
[4]I. B. Levinson, Zh. Eksp. Teor. Fiz. **57,** 660 (1969) [Sov. Phys.-JETP **30,** 362 (1970)].
[5]U. Fano, Phys. Rev. **96,** 869 (1954).
[6]I. I. Vosilyus and I. B. Levinson, Zh. Eksp. Teor. Fiz. **52,** 1013 (1967) [Sov. Phys.-JETP **25,** 672 (1967)].
[7]P. A. Kazlauskas and I. B. Levinson, Litov. Fiz. Sb. **6,** 233 (1966).
[8]L. Esaki and R. Tsu, IBM J. Res. Dev. **14,** 61 (1970).
[9]Zh. I. Alferov, Yu. V. Zhilyaev, and Yu. V. Shmartsev, Fiz. Tekh. Poluprovodn. **5,** 196 (1971) [Sov. Phys.-Semicond. **5,** 174 (1971)].
[10]R. F. Kazarinov and R. A. Suris, Fiz. Tekh. Poluprovodn. **5,** 797 (1971) [Sov. Phys.-Semicond. **5,** 707 (1971)].

Translated by H. H. Nickle
220