First-Derivative Discontinuities of the Space-Time Metric Tensor

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First-derivative discontinuities of the space-time metric tensor which is the solution of the Einstein gravitational equations are investigated. Expressions are obtained for discontinuities on nonisotropic and isotropic hypersurfaces. It is shown that for a nonisotropic discontinuity hypersurface the expression in the general case contains four scalars. It is proven that in this case there exists a coordinate system in which the first derivatives of the metric tensor are continuous. For an isotropic discontinuity hypersurface the expression for the discontinuity contains six scalars in the general case. First derivative discontinuities of a spherically symmetric space-time metric tensor are considered in various coordinate systems.

It is known that the metric space-time tensor g_{ik} of general relativity theory, which is a solution of Einstein's gravitation equations, can have discontinuous first derivatives. The presence of these discontinuities depends in many respects on the choice of the coordinate system. Indeed, let the functions $f^i(x^1, \ldots, x^4)$, which define the transition from the coordinate system x^i to the coordinates $\tilde{x}^i$, be continuous together with their first derivatives. The second derivatives of the function f^i are at least piecewise smooth. Since the transformation law for $\partial g_{ik}/\partial x^j$ is given by

$$\frac{\partial \tilde{g}_{ik}}{\partial \tilde{x}^j}=\frac{\partial g_{lm}}{\partial x^n}\frac{\partial x^l}{\partial \tilde{x}^i}\frac{\partial x^m}{\partial \tilde{x}^k}\frac{\partial x^n}{\partial \tilde{x}^j}+g_{lm}\left(\frac{\partial^2 x^l}{\partial \tilde{x}^i\,\partial \tilde{x}^j}\frac{\partial x^m}{\partial \tilde{x}^k}+\frac{\partial^2 x^m}{\partial \tilde{x}^k\,\partial \tilde{x}^j}\frac{\partial x^l}{\partial \tilde{x}^i}\right),$$

it follows that continuity of the derivatives $\partial \tilde{g}_{ik}/\partial \tilde{x}^j$ is determined by the continuity of $\partial g_{lm}/\partial x^n$ and by the continuity of the second derivatives of the transformation functions. Therefore the derivatives $\partial \tilde{g}_{ik}/\partial \tilde{x}^j$ will be discontinuous in the coordinate system $\tilde{x}^i$ even if the derivatives $\partial g_{lm}/\partial x^n$ are continuous, if the second derivatives of the transformation functions are piecewise smooth. In other words, if the first derivatives of g_{ik} are discontinuous in a given system of coordinates and there exists another coordinate system in which they are continuous, then the functions that transform the first system of coordinates into the second have discontinuous second derivatives.

A simple example confirming the foregoing is the Schwarzschild exact solution. In terms of the curvature coordinates, this solution is[1]

$$ds_-^2=\left(\frac{3}{2}\sqrt{1-ga^2}-\frac{1}{2}\sqrt{1-gr^2}\right)^2 dt^2-\frac{dr^2}{1-gr^2}-r^2\,d\sigma^2, \quad (1)$$

$$ds_+^2=\left(1-\frac{2m}{r}\right)dt^2-\frac{dr^2}{1-2m/r}-r^2\,d\sigma^2, \quad (2)$$

where

$$d\sigma^2=d\theta^2+\sin^2\theta\,d\varphi^2, \quad ga^2=2m.$$

The metric ds_-^2 determines space-time in a sphere of radius a and filled with a homogeneous liquid, and ds_+^2 defines space-time outside the sphere. By calculating the derivaties of the metrics (1) and (2) and by comparing them at $r = a$, we easily verify that the derivatives with respect to r of the metric tensor component at dr^2 are different. Therefore the derivatives of the metrics (1) and (2) are not continuous functions. On the other hand, Rosen[2] has shown that the complete Schwarzschild solution has continuous first derivatives in a homogeneous coordinate system. This means that the functions that transform the curvature coordinates into homogeneous coordinates should have discontinuous second derivatives at $r = a$.

This paper deals with discontinuities of the first derivatives of the metric tensor g_{ik}. Section 1 gives the conditions for joining the discontinuity of the first derivatives on the hypersurface. In Sec. 2, a representation is obtained for the discontinuities on a nonisotropic hypersurface. It is shown that in this case the discontinuities are determined by four scalars specified on the discontinuity hypersurface. It is proved that in a semi-geodesic coordinate system the derivatives $\partial g_{ik}/\partial x^j$ are continuous if the discontinuity hypersurface is non-singular. A representation of the discontinuities on an isotropic hypersurface is obtained in Sec. 3. It is shown that in this case the discontinuities of the first derivatives of the metric tensor are determined by six scalars specified on the discontinuity hypersurface. The discontinuities of the derivatives of the metric tensor of spherically-symmetrical space-time in special coordinate systems are considered in Sec. 4.

1. CONDITIONS FOR JOINING THE FIRST DERIVATIVES OF THE METRIC TENSOR ON THE DISCONTINUITY HYPERSURFACE

Let g_{ik} be the space-time metric tensor of general relativity theory, which is a solution of Einstein's gravitational equation, and let S be a sufficiently smooth hypersurface, and let the first derivatives $\partial g_{ik}/\partial x^j$ becomes discontinuous on going through this surface. Such a hypersurface can be the boundary between regions of space filled with substances having different properties, or the boundary between a substance and vacuum. If the energy-momentum tensor of the substance has on the hypersurface S a singularity of the δ-function type, such a hypersurface is called singular with energy-momentum density δT_{ik}. The analog of a singular hypersurface in Newtonian gravitation theory is a simple layer.

Let the hypersurface S divide the region in which g_{ik} is defined into two subregions Ω_+ and Ω_-, in each of which g_{ik} are continuous together with their first and second derivatives. The tensor g_{ik} is continuous in the entire region of definition. We define the discontinuities of the first derivatives of g_{ik} in the following manner:

$$\left[\frac{\partial g_{ik}}{\partial x^j}\right]=\lim\left(\frac{\partial g_{ik}(M_l^+)}{\partial x^j}-\frac{\partial g_{ik}(M_l^-)}{\partial x^j}\right), \quad M_l^+\to M,\ M_l^-\to M, \quad (1.1)$$

where M, M_l^+, and M_l^- belong to S, Ω_+, and Ω_-, respectively. In the definition of $[\partial g_{ik}/\partial x^j]$ we assume that the limit of the right-hand side of the expression exists and does not depend on the sequence of the points M^l. From the definition (1.1) and from the law governing the transformation of the derivatives $\partial g_{ik}/\partial x^j$ it follows that their discontinuity is a tensor with respect to the coordinate transformations of class C^2.

The discontinuities (1.1) should be joined on the hypersurface S. The joining conditions take the form[3]

$$(\Gamma_{ik}{}^j - {}^1/_2 g_{ik}(g^{jm}\Gamma_{ml}{}^l - g^{ml}\Gamma_{ml}{}^j) - {}^1/_2(\delta_i{}^j\Gamma_{kl}{}^l + \delta_k{}^j\Gamma_{il}{}^l)]n_j = 0, \quad (1.2)$$

If S is a non-singular hypersurface. In the case of a singular hypersurface with energy-momentum density δT_{ik}, the joining conditions are

$$[\Gamma_{ik}{}^j - {}^1/_2 g_{ik}(g^{jm}\Gamma_{ml}{}^l - g^{ml}\Gamma_{ml}{}^j) - {}^1/_2(\delta_i{}^j\Gamma_{kl}{}^l + \delta_k{}^j\Gamma_{il}{}^l)]n_j = \delta T_{ik}. \quad (1.3)$$

In (1.2) and (1.3), n_j is the four-dimensional normal to S.

Let us transform the conditions (1.2) and (1.3). Contracting, for example, (1.3) with g_{ik}, we obtain

$$[g^{jm}\Gamma_{ml}{}^l - g^{ml}\Gamma_{ml}{}^j]n_j = \delta T_l{}^l, \quad (1.4)$$

and the conditions (1.3) become

$$[\Gamma_{ik}{}^j - {}^1/_2(\delta_i{}^j\Gamma_{kl}{}^l - \delta_k{}^j\Gamma_{il}{}^l)]n_j = \delta T_{ik} - {}^1/_2 g_{ik}\delta T_l{}^l. \quad (1.5)$$

In the case of a nonsingular hypersurface, we obtain

$$[\Gamma_{ik}{}^j - {}^1/_2(\delta_i{}^j\Gamma_{kl}{}^l + \delta_k{}^j\Gamma_{il}{}^l)]n_j = 0. \quad (1.6)$$

From the kinematic joining conditions[4,5] it follows that

$$[\partial g_{ik}/\partial x^j] = h_{ik}n_j, \quad (1.7)$$

where h_{ik} is a symmetrical tensor specified on S. Using (1.7), we can easily show that

$$[\Gamma_{ik}{}^j] = {}^1/_2 g^{jl}(h_{il}n_k + h_{kl}n_i - h_{ik}n_l), \quad (1.8)$$

$$[\Gamma_{il}{}^l] = \sigma n_i, \quad 2\sigma = h_{ik}g^{ik}. \quad (1.9)$$

Substituting (1.8) and (1.9) in (1.5), we obtain after simple transformations the joining conditions in the form

$$\begin{aligned} {}^1/_2 n_i(h_{kl}n^l - \sigma n_k) + {}^1/_2 n_k(h_{il}n^l - \sigma n_i) \\ - {}^1/_2 h_{ik}(n^l n_l) = \delta T_{ik} - {}^1/_2 g_{ik}\delta T_l{}^l. \end{aligned} \quad (1.10)$$

Contracting (1.10) with n^k and taking (1.4) into account, we can show that the singular density tensor δT_{ik} should satisfy the condition

$$\delta T_{ik}n^k = 0. \quad (1.11)$$

It follows from (1.7) that the discontinuities of the derivatives $\partial g_{ik}/\partial x^j$ are determined by the tensor h_{ik}. We obtain a representation for the discontinuities by determining h_{ik} from the joining conditions (1.10).

2. REPRESENTATION OF DISCONTINUITIES ON A NON-ISOTROPIC HYPERSURFACE

Let S be a non-isotropic hypersurface. Then $(n^l n_l) \neq 0$ and, contracting (1.10) with g_{ik}, we obtain

$$(h_{ik}n^k - 2\sigma n_i)n^i = -\delta T_l{}^l, \quad (2.1)$$

from which it follows that

$$h_{ik}n^k - 2\sigma n_i = -\frac{n_i}{(n^l n_l)}\delta T_l{}^l + \tau_i. \quad (2.2)$$

In (2.2), τ_i is a vector specified on S and orthogonal to n^i in the metric g_{ik}. Substituting (2.2) in (1.10), we obtain

$$(n^l n_l)h_{ik} = n_i a_k + n_k a_i - 2\left(\delta T_{ik} - \frac{1}{2}g_{ik}\delta T_l{}^l + \frac{n_i n_k}{(n^l n_l)}\delta T_l{}^l\right), \quad (2.3)$$

where $a_i = \sigma n_i + \tau_i$ is an arbitrary vector specified on S. It is easily seen that the vector a_i is determined by four scalars, namely the projections on n^i and $\tau^i_{(\alpha)}$ where $\tau^i_{(\alpha)}$ are linearly-independent vectors on S, orthogonal to n^i in the metric g_{ik}. Expression (2.3) gives a representation of the discontinuities of the first derivatives of the tensor g_{ik} on a singular hypersurface. Putting $\delta T_{ik} = 0$ in (2.3), we obtain a representation of h_{ik} on a non-singular hypersurface:

$$(n^l n_l)h_{ik} = n_i a_k + n_k a_i. \quad (2.4)$$

The presence of nontrivial solutions (2.4) of the joining conditions (1.10) with zero right-hand sides indicates that the space-time metric is not smooth in the general coordinate system, i.e., the first derivatives are discontinuous.

It follows from (2.4) that the discontinuities of the first derivatives of g_{ik} on a non-isotropic hypersurface with $\delta T_{ik} = 0$ are determined by the vector a_i.

The obtained representation allows us to prove the following statement: in a coordinate system x^i, one of the coordinate hypersurface of which $(x^1 = \text{const})$ coincides with a non-singular hypersurface S and is orthogonal to the remaining coordinate hypersurfaces, the first derivatives $\partial g_{ik}/\partial x^j$ are continuous, with the possible exception of the derivative $\partial g_{11}/\partial x^1$.

Let us prove this. In the indicated coordinate system, the normal to S is given by $n_i = 0$, $i \neq 1$, $n_1 = 1$. From the representation (2.4) it follows that $h_{ik} = 0$ if $i, k \neq 1$. Further, since $g_i = 0$ when $i \neq 1$, it follows that $h_{1i} = 0$, $i \neq 1$. On the other hand, we have from (2.4)

$$(n^l n_l)h_{1i} = a_i, \quad i \neq 1,$$

from which it follows that $a_i = 0$ when $i \neq 1$. Consequently, the tensor h_{ik} has only one component

$$(n^l n_l)h_{11} = 2a_1, \quad (2.5)$$

which can possibly differ from zero. Therefore only the derivative $\partial g_{11}/\partial x^1$ can be discontinuous.

Using the foregoing statement, we can readily prove that the first derivatives of the tensor g_{ik} are continuous in a semi-geodesic coordinate system constructed on the basis of S. Indeed, in the semi-geodesic coordinate system constructed on the basis of S, the equation of the hypersurface S takes the form $x^1 = 0$ where x^1 is the canonical parameter of the geodesics drawn through each point of S in the direction of n^1. In such a coordinate system $g^{1i} = 0$ if $i \neq 1$ and $g_{11} = \pm 1$[1]. By virtue of the previous proof where we have $a_i = 0$, $i \neq 1$ and from (2.5) we have $a_1 = 0$. It follows then from the representation (2.4) that $h_{ik} = 0$, and therefore the first derivatives of g_{ik} are continuous, i.e., the semi-geodesic system of coordinates is admissible according to Lichnerowicz[1].

In the case of a singular hypersurface, it is easy to show that, in the coordinate system indicated above, h_{ik} is determined by T_{ik} and is given by

$$(n^l n_l) h_{ik} = -2(\delta T_{ik} - {}^1/_2 g_{ik}\delta T_l^l + {}^1/_2 g_{11} n_i n_k \delta T_l^l).$$

We note that the joining conditions (1.10) on a non-isotropic hypersurface are equivalent to the conditions

$$(n^l n_l) h_{ik}\tau^i_{(\alpha)}\tau^k_{(\beta)} = -2(\delta T_{ik} - {}^1/_2 g_{ik}\delta T_l^l)\tau^i_{(\alpha)}\tau^k_{(\beta)}, \qquad (2.6)$$

if S is singular and

$$h_{ik}\tau^i_{(\alpha)}\tau^k_{(\beta)} = 0 \qquad (2.7)$$

in the case of a non-singular hypersurface. The equivalence follows from the representations (2.3) and (2.4).

3. REPRESENTATIONS OF DISCONTINUITIES ON AN ISOTROPIC HYPERSURFACE

Let S be an isotropic discontinuity hypersurface, i.e., $(n^l n_l) = 0$. Then the joining conditions (1.10) on S take the form

$${}^1/_2 n_i(h_{kl}n^l - \sigma n_k) + {}^1/_2 n_k(h_{il}n^l - \sigma n_i) = \delta T_{ik} - {}^1/_2 g_{ik}\delta T_l^l. \quad (3.1)$$

Let $\xi^i_{(\alpha)}$ be vectors on S, different from n^i and linearly-independent at each point of S. The symbol (α) will henceforth stand for the indices pertaining to the numbering of the vectors. The tensor h_{ik} is defined uniquely by contractions of the type $h_{ik}n^i\xi^k_{(\alpha)}$, $h_{ik}\xi^i_{(\alpha)}\xi^k_{(\beta)}$, $h_{ik}n^i n^k$, where $\alpha, \beta = 1, 2, 3$. Since the tensor h_{ik} is contained in the joining condition (3.1) in the form $h_{ik}n^k$, it is impossible to determine contractions of the type $h_{ik}\xi^i_{(\alpha)}\xi^k_{(\beta)}$ from (3.1). Therefore the tensor h_{ik} on an isotropic hypersurface is defined in the general case to within six arbitrary scalars specified on S.

Let us make the choice of the vectors $\xi^i_{(\alpha)}$ more precise. We separate the linearly-independent vectors $\tau^i_{(\alpha)}$ which do not contain n^i and are orthogonal to the vector n^i in the metric g_{ik}. We shall show that at each point of the hypersurface S the set of vectors $\tau^i_{(\alpha)}$ is two-dimensional. Indeed, let M_0 be a point on S, and let n^i be the normal to S at M_0. Then there exists a coordinate system in which the space-time metric is

$$dS_{M_0}^2 = (dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2,$$

and the vector n_i takes the form (1, 1, 0, 0).

Assume that at M_0 the vector τ^i is orthogonal to n^i and does not contain it, and then $\tau^4_{(\alpha)} = \tau^1_{(\alpha)} = 0$, from which it follows that the set of vectors $\tau^i_{(\alpha)}$ is two-dimensional at each point of the hypersurface S. We denote these vectors by $\tau^i_{(1)}$ and $\tau^i_{(2)}$. For $\xi^i_{(1)}$ and $\xi^i_{(2)}$ we choose the vector fields $\tau^i_{(1)}$ and $\tau^i_{(2)}$. The vector field $\xi^i_{(3)}$ cannot be orthgonal to n_i, and therefore $(\xi^i_{(3)}n_i) \neq 0$. We make its choice more precise by means of the following conditions: $(\xi_{(3)}n) = 1$, $(\xi_{(3)}\tau_{(\alpha)}) = 0$, $(\xi_{(3)}\xi_{(3)}) = 0$. These conditions define the vector field $\xi^i_{(3)}$ uniquely, We shall henceforth omit the index 3 of the vector $\xi^i_{(3)}$.

Let us fine the contractions $h_{ik}n^i n^k$, $h_{ik}n^i\tau^k_{(\alpha)}$, $h_{ik}n^i\xi^k$. Contracting (2.1) with g_{ik}, we obtain

$$h_{ik}n^i n^k = -\delta T_l^l. \qquad (3.2)$$

Then, contracting (3.1) with $\tau^i_{(\alpha)}\xi^k$ and $\xi^i\xi^k$ we obtain respectively

$$h_{ik}n^i\tau^k_{(\alpha)} = 2\delta T_{ik}\xi^i\tau^k_{(\alpha)}, \qquad (3.3)$$

$$h_{ik}n^i\xi^k = \sigma + \delta T_{ik}\xi^i\xi^k. \qquad (3.4)$$

Any symmetrical tensor can be represented in the form of a bilinear symmetrical combination of four linearly-independent vector fields, therefore

$$h_{ik} = An_i n_k + B\xi_i\xi_k + \bar{B}^{\alpha\beta}\tau_{(\alpha)i}\tau_{(\beta)k} + 2A^\alpha n_{(i}\tau_{(\alpha)k)} + 2B^\alpha\tau_{(\alpha)(i}\xi_{k)} + 2Dn_{(i}\xi_{k)}, \qquad (3.5)$$

where A, B, A^α, B^α, $\bar{B}^{\alpha\beta}$, D are scalars on S, $\bar{B}^{\alpha\beta} = \bar{B}^{\beta\alpha}$, the sign (...) denotes symmetrization over the tensor indices contained in parentheses. Taking into account (3.2), (3.3), (3.4), and the normalization $h_{ik}g^{ik} = 2\sigma$, we obtain expressions for the scalars:

$$B = -\delta T_l^l, \quad B^\alpha = 2\delta T_{ik}\xi^i\tau^k_{(\beta)}\sigma^{\alpha\beta}, \qquad D = \sigma + \delta T_{ik}\xi^i\xi^k, \quad \sigma_{\alpha\beta}\bar{B}^{\alpha\beta} = -2\delta T_{ik}\xi^i\xi^k, \qquad (3.6)$$

where

$$\sigma_{\alpha\beta} = (\tau_{(\alpha)}\tau_{(\beta)}), \quad \sigma_{\alpha\rho}\sigma^{\rho\beta} = \delta_\alpha{}^\beta \quad (\alpha, \beta, \rho = 1, 2).$$

We represent the scalars $\bar{B}^{\alpha\beta}$ in the form

$$\bar{B}^{\alpha\beta} = -\sigma^{\alpha\beta}\delta T_{ik}\xi^i\xi^k + B^{\alpha\beta}, \qquad (3.7)$$

where $B^{\alpha\beta} = B^{\beta\alpha}$ and $B^{\alpha\beta}\sigma_{\alpha\beta} = 0$. We write the tensor h_{ik} in the form

$$h_{ik} = \overset{0}{h}_{ik} + \overset{\delta}{h}_{ik},$$

where $\overset{\delta}{h}_{ik}$ is determined by the density δT_{ik}, and $\overset{0}{h}_{ik}$ does not depend on δT_{ik}. Substituting (3.6) and (3.7) in (3.5), we obtain

$$\overset{0}{h}_{ik} = (n_i a_k + n_k a_i) + B^{\alpha\beta}\tau_{(\alpha)i}\tau_{(\beta)k}, \qquad (3.8)$$

$$\overset{\delta}{h}_{ik} = (\xi_i b_k + \xi_k b_i) - \tau_{(\alpha)i}\tau_{(\beta)k}\sigma^{\alpha\beta}\delta T_{ln}\xi^l\xi^n, \qquad (3.9)$$

where

$$a_i = An_i + \sigma\xi_i + A^\alpha\tau_{(\alpha)i},$$

$$b_i = n_i\delta T_{ln}\xi^l\xi^n + 2\tau_{(\alpha)i}\sigma^{\alpha\rho}\delta T_{ln}\xi^l\tau^n_{(\rho)} - {}^1/_2\xi_i\delta T_l^l.$$

Expressions (3.8) and (3.9) give a representation of the discontinuities on the isotropic hypersurface.

Introducing a special coordinate system, we can subject the metric tensor g_{ik} to four coordinate conditions in[1]. Therefore, in such a coordinate system four out of the six scalars in (3.8), which define the discontinuity on the non-singular hypersurface, can be set equal to zero. We introduce an isotropic semi-geodesic system of coordinates constructed on the basis of S[1]. In such a coordinate system, the space-time metric takes the form

$$ds^2 = g_{4\alpha}dx^4 dx^\alpha + g_{\alpha\beta}dx^\alpha dx^\beta \quad (\alpha, \beta = 1, 2, 3), \qquad (3.10)$$

where x^4 is a canonical parameter of the isotropic geodesics and $g_{4\alpha}$ does not depend on x^4. The equation of the hypersurface S in such a coordinate system is $x^4 = 0$. Then the vector $n_i = (1, 0, 0, 0)$ and the vectors $\tau_{(\alpha)i}$ have a zero component $\tau_{(\alpha)4}$. The coordinate conditions for the metric (3.10) yield $h_{4i} = 0$. On the other hand, from the representation (3.8) we have

$$h_{4i} = a_i + n_i a_4 = 0,$$

from which it follows that $a_i = 0$. Consequently, in this coordinate system the tensor h_{ik} is determined by two scalars, and the discontinuity of the first derivatives of g_{ik} on the non-singular isotropic hypersurface cannot be eliminated in the general case.

The representations (2.4) and (3.8) enable us to establish the qualitative difference between the discontinuities on an isotropic and a non-isotropic hypersurface. This difference can be determined in the following manner. Let $\tau^i_{(\alpha)}$ be linearly-independent vectors on S, which do not contain n_i and are orthogonal to n^i in the metric g_{ik}. We continue $\tau^i_{(\alpha)}$ in a smooth manner into the vicinity of S. We define the transverse components $g_{(\alpha\beta)}$ of the metric g_{ik} by the expression

$$g_{(\alpha,\beta)} = g_{ik}\tau^i_{(\alpha)}\tau^k_{(\beta)}.$$

The discontinuity of the first derivatives of $g_{(\alpha\beta)}$ on S is given by

$$[\partial g_{(\alpha\beta)}/\partial x^j] = h_{ik}\tau^i_{(\alpha)}\tau^k_{(\beta)}n_j. \quad (3.11)$$

Substituting (2.4) in (3.11) we readily see that the first derivatives of the transverse components of the metric are continuous on a non-isotropic hypersurface with $\delta T_{ik} = 0$. In the case of an isotropic hypersurface, the first derivatives of $g_{(\alpha\beta)}$ can have a discontinuity, as can be readily verified by substituting (3.8) in (3.11). Thus, the behavior of the derivatives $\partial g_{(\alpha\beta)}/\partial x^j$ on a non-isotropic hypersurface differs qualitatively from the behavior of the first derivatives of $g_{(\alpha\beta)}$ on an isotropic hypersurface--they are continuous in the former case and may have a discontinuity in the latter.

We note that the joining conditions (3.1) can be written in the form (3.2), (3.3), (3.4). The equivalence of these conditions follows from the representation (3.8), (3.9).

In those cases when space-time admits of a motion group, the scalars in the representations (2.4) and (3.8) should satisfy a system of differential equations and their number can decrease.

4. DISCONTINUITIES OF THE FIRST DERIVATIVES OF g_{ik} OF SPHERICALLY-SYMMETRICAL SPACE-TIME

Let S be a hypersurface with $\delta T_{ik} = 0$, on passing through which the first derivatives of the metric tensor g_{ik} of spherically-symmetrical space-time can have a discontinuity. We assume that in the vicinity of S the tensor g_{ik} is continuous. Then, if S is non-isotropic, we can prove by using the representation (2.4) that in a polar Gaussian coordinate system, in a homogeneous coordinate system, and in an isothermal coordinate system the first derivatives of g_{ik} are continuous.

Let us prove this. We write the metric of spherically-symmetrical space-time in the form[1]

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - e^\mu(d\theta^2 + \sin^2\theta d\varphi^2), \quad (4.1)$$

where ν, λ, and μ are functions of r and t.

We number the coordinates in the following manner:

$$(t, r, \theta, \varphi) \to (x^4, x^1, x^2, x^3).$$

If $\lambda = 0$ in (4.1), then the coordinate system is polar Gaussian; if $\nu = \lambda$ we have an isothermal coordinate system; if $\lambda = \mu$, then the coordinate system is called homogeneous. If $e^\mu = r^2$, we have curvature coordinates. The continuity of the first derivatives of the metric tensor (4.1) in the indicated coordinate systems is proved if we demonstrate that the vector a_i in the representation

$$(n^l n_l)h_{ik} = a_i n_k + a_k n_i \quad (4.2)$$

is a zero vector.

Since the hypersurface S is spherically-symmetrical, it can be specified by a level $r = r(t)$ and then $n_i = (\dot r, -1, 0, 0)$. The coordinate conditions for the metric (4.1) are $h_{ik} = 0$, $i \neq k$. On the other hand, the representation (4.2) yields

$$h_{41} = \dot r a_1 - a_4, \quad h_{42} = \dot r a_2, \quad h_{43} = \dot r a_3,$$
$$h_{12} = -a_2, \; h_{13} = -a_3, \quad h_{23} = 0,$$

from which it follows that $a_2 = a_3 = 0$, $a_4 = \dot r a_1$. Since the vector n_i has zero components n_2 and n_3, it follows from (4.2) that $h_{22} = h_{33} = 0$, i.e., the function μ in (4.1) has continuous first derivatives. The nonzero components of the tensor h_{ik} are

$$h_{44} = 2\dot r a_4, \quad h_{11} = -2a_1, \quad (4.3)$$

where $a_4 = \dot r a_1$. We assume that the coordinate system is polar Gaussian. Then $\lambda = 0$, therefore $h_{11} = 0$ and (4.3) yields $a_4 = a_1 = 0$, so that $a_i = 0$ and the derivatives of the metric tensor (4.1) are continuous in the indicated coordinate system. The continuity of the first derivatives of g_{ik} in the isothermal and homogeneous coordinate systems is proved in similar fashion. In curvature coordinates, the first derivatives of the metric tensor have the nontrivial discontinuity

$$h_{44} = 2\dot r^2 a_1, \quad h_{11} = -2a_1$$

and if S is static, only the derivative $\partial g_{11}/\partial r$ can be discontinuous[3,6].

In the case of an isotropic hypersurface S, using the representation (3.8), we can prove that the function μ in (4.1) has continuous first derivatives, and the metric tensor itself has noncontinuous first derivatives in the polar Gaussian and homogeneous coordinate systems. In the isothermal coordinate system and in the curvature coordinates, the discontinuities of $\partial g_{ik}/\partial x^j$ are nontrivial and are determined by a single scalar.

We note that the representations (2.4) and (3.8) can be written in a different form, this being connected with the possibility of representing g_{ik} in the form of a bilinear symmetrical expression of four linearly-independent vectors. We can obtain a representation for the singular density tensor δT_{ik} by starting from the condition (1.11). It is easy to verify that this representation contains six scalars in the general case.

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