Scattering and Transformation of Nonlinear Periodic Waves in an Inhomogeneous Medium

G. M. Zaslavskiĭ
Physics Institute, Siberian Division, USSR Academy of Sciences
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The problem of scattering of a nonlinear periodic wave by inhomogeneities in a medium is considered with the nonlinear wave equation describing the oscillations of a chain of anharmonic oscillators (the Fermi-Pasta-Ulam problem) as an example, for two limiting cases: localized (delta-function) inhomogeneity and smoothly-varying inhomogeneity. The method is based on the use of a two-stream approximate solution, which is obtained in the paper and describes a state with two coupled nonlinear waves propagating in opposite directions. For a localized inhomogeneity, under certain conditions, the scattering matrix is obtained and the phase shift and the change of the wavelength as a result of the scattering are calculated. In the case of a smooth inhomogeneity, the WKB method is used to find the reflected wave connected with the change of the adiabatic invariant of the system. It is shown that when a nonlinear wave is scattered by inhomogeneities there is a tendency for multiple-stream motion to be produced.

1. INTRODUCTION

THE present work is connected with an investigation of the influence of inhomogeneities on the propagation of nonlinear periodic waves in media with dispersion (henceforth called simply "nonlinear waves"). In the general theory of nonlinear waves, this problem is among the most important ones and is connected with a large number of different physical applications. The existing methods are based on the approximation that can be called adiabatic, in analogy with the linear case, and is the analog of the WKB method, namely, the solution of the corresponding problem is sought in the case of a weakly-inhomogeneous medium as the solution of the homogeneous problem, but in which the parameters are slowly varying functions (relative to the adiabaticity parameter). The most highly developed method in this direction is that of Whitham[1,2]. Different developments of the adiabatic method are contained also in[3,4]. An important circumstance in the construction of the theory in the adiabatic approximation is the use of the adiabatic invariants of nonlinear waves, analogous to the adiabatic invariants of dynamic systems. Moser proved[6] that the adiabatic invariant of a nonlinear wave is conserved in all orders of perturbation theory, showed that the change of the adiabatic invariant is exponentially small, and calculated this change.

A characteristic feature of the existing approximate methods is that the scattering of the nonlinear wave by the inhomogeneities is not completely described. If we use again the terminology of the linear problems, we can state that the existing methods do not describe the reflection of the nonlinear wave from the inhomogeneity. Even in the case of smooth inhomogeneities, in which the effect of the reflection is expected to be exponentially small, the appearance of a "reflected wave" is of fundamental significance. The main difficulty in the development of the theory in this direction is obvious: in view of the nonlinearity of the equations of motion, the sum of the solutions (of the incident and reflected waves) is not a solution. We can say more accurately that the appearance of a reflected wave would make it necessary to describe (at least) two-stream motions, whereas the existing methods describe only single-stream motions. If the inhomogeneity region scattering the wave is very small, then the question of joining the solutions on the discontinuity arises. A discussion of a number of difficulties encountered in this case is contained in[2,7,8].

The present paper is devoted to a study of the scattering of nonlinear waves in an inhomogeneous medium with allowance for the appearance of the reflected wave. Although certain limitations will be introduced, and the model of a nonlinear string will be considered for the sake of concreteness and clarity, the method presented has nevertheless a certain degree of generality.

Section 2 deals with the properties of the chosen model of a nonlinear medium with dispersion, describes a number of transformations, and introduces the variational principle and Hamilton's equations of motion. In Sec. 3 are considered approximate two-stream solutions in the case of a homogeneous medium; these solutions serve as the basis for the solution in the inhomogeneous case. This is followed by an investigation of the case of δ-like (localized) inhomogeneity (Sec. 4) and the case of a smooth (adiabatic) inhomogeneity (Sec. 5).

2. EQUATIONS OF MOTION AND VARIATIONAL PRINCIPLE

We consider as the initial system the model of a nonlinear string, describing long-wave motions in a one-dimensional chain of coupled nonlinear operators. This model was proposed by Fermi for the study of the properties of nonlinear systems. Leaving out the discussion of its properties (see the review[9]), we write down directly the equation of motion in the case of cubic nonlinearity:

$$c^{-2}y_{tt} = y_{xx}(1 + y_x^2) + y_{xxxx}, \tag{2.1}$$

where y is the displacement of the oscillators, c is the speed of sound, and the nonlinearity parameter preceding the y_x^2 term is set equal to unity, as is also the coefficient preceding the dispersion term y_{xxxx}. The inhomogeneous case corresponds to $c = c(x)$, meaning an uneven distribution of the oscillator masses along the lattice.

We seek the solution of (2.1) at c = const in the form of a nonlinear periodic wave

$$y = y(x - ut). \tag{2.2}$$

From (2.1) and (2.2) we obtain the following equation for the determination of y:

$$z'^2 + {}^1/_2 z^4 - (u^2/c^2 - 1)z^2 = C, \quad z \equiv y' \tag{2.3}$$

and its solution is best analyzed on the phase plane (see the figure). The case $C < 0$ corresponds to periodic waves of constant sign; $C = 0$ describes a single wave (soliton), and at $C > 0$ we have an alternating-sign periodic wave. In addition, we must have $u^2 > c^2$ everywhere. We consider henceforth, without loss of generality, the case $C < 0$, from which we can make a simple transition to the limit of the linear case, and $z > 0$. It follows from (2.3) that

$$\begin{gathered} z = \gamma_2 \operatorname{dn}[2^{-1/2}\gamma_2(x - ut); \varkappa], \\ \gamma_{1,2} = \{(u^2/c^2 - 1) \mp [(u^2/c^2 - 1)^2 - 2|C|]^{1/2}\}^{1/2}, \\ \varkappa = (\gamma_2^2 - \gamma_1^2)^{1/2}/\gamma_2. \end{gathered} \tag{2.4}$$

The spatial period of the solution (2.4) is defined by

$$\lambda = \frac{2\pi}{k} = \frac{2\sqrt{2}}{\gamma_2} F\left(\frac{\pi}{2}, \varkappa\right). \tag{2.5}$$

Let us simplify (2.1). On going from a chain of oscillators to the Eq. (2.1) for a continuous medium, it must be assumed that the dispersion term y_{xxxx} is small in comparison with y_{xx} (otherwise higher derivatives appear). We also assume that the nonlinearity is moderate, i.e.,

$$y_x^2 \ll 1, \tag{2.6}$$

although the term $y_x^2 y_{xx}$ may be large in comparison with the dispersion term. The inequality (2.6) means that the region of the solutions near the boundary where the nonlinear wave breaks is excluded from consideration. Under the foregoing limitations, the zeroth approximation for (2.1) takes the form

$$c^{-2}y_{tt} = y_{xx} \quad \text{or} \quad y_x = \pm c^{-1} y_t. \tag{2.7}$$

For solutions of the Riemann type, (2.2), the different signs in (2.7) correspond to different signs of u, i.e., to waves traveling in different directions. The next approximation is obtained by substituting (2.7) in (2.1)

$$c^{-2}y_{tt} = y_{xx} + c^{-4}y_t^2 y_{tt} + c^{-4}y_{tttt}. \tag{2.8}$$

It is easily seen from (2.4) and (2.6) that the approximation in question corresponds to the inequality

$$u^2/c^2 - 1 \ll 1 \quad \text{or} \quad \alpha \equiv |u|/c - 1 \ll 1. \tag{2.9}$$

In this case

$$|C|/16\alpha^2 \ll 1. \tag{2.10}$$

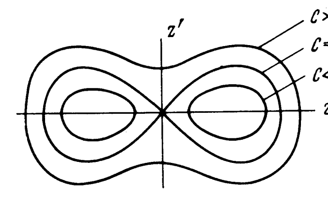
we introduce also the parameter N:

$$N = \sqrt{\alpha}/k, \tag{2.11}$$

which is equal (apart from a numerical factor of the order of unity) to the ratio of the period of the nonlinear wave to the width of its crest. In addition, in the Fourier expansion of z

$$z = \sum_{m=-\infty}^{\infty} z_m e^{imkx}$$

we have



$$\begin{gathered} z_m \approx 2\sqrt{\alpha}/N \approx 2k, \quad n \lesssim N, \\ z_m \sim e^{-n/N}, \quad n \gg N, \end{gathered} \tag{2.12}$$

i.e., the nonlinearity of the problem at $N \gg 1$, and at $N \sim 1$ we have a nearly-harmonic case[1)].

We proceed now to formulate the problem of the scattering of a nonlinear wave by an inhomogeneity. We follow the procedure in the linear case, namely: assume that there exists in the system a stationary motion that is periodic in time and has a frequency ω. We determine the form of the solution as a function of the spatial coordinate for specified boundary conditions. By virtue of the periodicity in t, we expand (2.8) in a Fourier series:

$$\begin{gathered} \frac{d^2 y_n}{dx^2} + n^2\frac{\omega^2}{c^2}\left(1 + n^2\frac{\omega^2}{c^2}\right) y_n \\ + \frac{\omega^4}{c^4} n \sum_{n_1,n_2,n_3} n_1 n_2 n_3 y_{n_1} y_{n_2} y_{n_3} \delta(n - n_1 - n_2 - n_3) = 0, \\ y = \sum_{n=-\infty} (y_n e^{in\omega t} + y_n^* e^{-in\omega t}), \end{gathered} \tag{2.13}$$

where, generally speaking, $c = c(x)$. In particular, for the nonlinear wave (2.2) at $c = \text{const}$ we have

$$y_n = e^{-inkx} a_n, \tag{2.14}$$

where a_n does not depend on t and x, and the wave number k is given by

$$k = \omega/u. \tag{2.15}$$

The system (2.13) is to be solved, and a feature of the nonlinear-wave scattering is that $\sim N$ of its harmonics y_n are strongly coupled and do not vary independently of one another.

We introduce the Hamiltonian for the system (2.13):

$$\begin{gathered} H = \frac{1}{2}\sum_n \frac{dy_n}{dx}\frac{dy_{-n}}{dx} + \frac{1}{2}\frac{\omega^2}{c^2}\sum_n n^2\left(1 + n^2\frac{\omega^2}{c^2}\right) y_n y_{-n} \\ + \frac{1}{4}\frac{\omega^4}{c^4}\sum_{n_1,n_2,n_3,n_4} n_1 n_2 n_3 n_4 y_{n_1} y_{n_2} y_{n_3} y_{n_4} \delta(n_1 + n_2 + n_3 + n_4) \end{gathered} \tag{2.16}$$

and Hamilton's equations of motion equivalent to (2.13):

$$\frac{dy_n}{dx} = \frac{\partial H}{\partial p_{-n}}, \quad \frac{dp_n}{dx} = -\frac{\partial H}{\partial y_{-n}}, \quad p_n \equiv \frac{dy_n}{dx}. \tag{2.17}$$

The Hamiltonian H has a form more general than (2.16), namely:

$$H = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt\left\{\frac{1}{2}y_x^2 - \frac{1}{2c^2}y_t^2 + \frac{1}{2c^4}y_{tt}^2 + \frac{1}{12c^4}y_t^4\right\}. \tag{2.18}$$

Accordingly we obtain in place of (2.17) the equations of motion

$$\frac{\partial y}{\partial x} = \frac{\delta H}{\delta y_x}, \quad \frac{\partial y_x}{\partial x} = -\frac{\delta H}{\delta y}, \tag{2.19}$$

1) The parameter N for the exact equation (2.1), according to (2.4) and (2.5), is equal to $F(\pi/2,\chi)$, and the character of the spectrum (2.12) remains the same.

which are equivalent to (2.8).

We point out the main feature of the variational principles (2.16), (2.17) and (2.18), (2.19): they describe problems in which the frequency characteristics of the temporal spectrum of the solution remain unchanged, as determined by the initial conditions of the problem. This is what makes the problem of wave scattering in an inhomogeneous medium unique[2)].

In the case of a homogeneous medium it follows from (2.18) and (2.19) that

$$dH/dx=0,\quad c=\text{const}. \tag{2.20}$$

3. TWO-STREAM SOLUTIONS

The initial equation (2.8) in the homogeneous case, which is the only one considered in the present section, has two types of solutions in the form of nonlinear waves corresponding to the different signs of the velocity u. We shall use the notation

$$y^{(+)}=y(x-ut),\quad y^{(-)}=y(x+ut). \tag{3.1}$$

A solution of the type $y = c_1y^{(+)} + c_2y^{(-)}$ exists, naturally, in the case of a very small nonlinearity. Nonetheless, even in the case of a sufficiently large nonlinearity we can construct a solution containing a superposition of waves traveling to the right and to the left. Such a solution will be called here "two-stream" (its analog for a related problem was constructed in[10])[3)].

We seek for (2.8) at c = const an approximate solution that is periodic in time with a period $2\pi/\omega$, in the form

$$y(x,t)\approx y^{(+)}(x-u^{(+)}t)+y^{(-)}(x+u^{(-)}t), \tag{3.2}$$

under the conditions

$$\alpha^{(\pm)}\ll 1,\quad N^{(\pm)}\gg 1. \tag{3.3}$$

We start from the fact that the expression (3.2) can be taken as the zeroth approximation, We shall determine below the condition under which that the part of the Hamiltonian which corresponds to the interaction of the waves $y^{(+)}$ and $y^{(-)}$ is small and can be regarded as a perturbation.

We represent the Hamiltonian (2.18) in the form

$$H=H^{(+)}+H^{(-)}+H_I, \tag{3.4}$$

$$H^{(\pm)}=\frac{1}{2}\sum_n\frac{dy_n^{(\pm)}}{dx}\frac{dy_{-n}^{(\pm)}}{dx}+\frac{1}{2}\frac{\omega^2}{c^2}\sum_n n^2\left(1+n^2\frac{\omega^2}{c^2}\right)y_n^{(\pm)}y_{-n}^{(\pm)}$$
$$+\frac{1}{4}\frac{\omega^4}{c^4}\sum_{n_1,n_2,n_3,n_4}n_1n_2n_3n_4y_{n_1}^{(\pm)}y_{n_2}^{(\pm)}y_{n_3}^{(\pm)}y_{n_4}^{(\pm)}\delta(n_1+n_2+n_3+n_4);$$

$$H_I=\frac{1}{4}\frac{\omega^4}{c^4}\sum_{n_1,n_2,n_3,n_4}n_1n_2n_3n_4(4y_{n_1}^{(+)}y_{n_2}^{(-)}y_{n_3}^{(-)}y_{n_4}^{(-)} \tag{3.5}$$
$$+6y_{n_1}^{(+)}y_{n_2}^{(+)}y_{n_3}^{(-)}y_{n_4}^{(-)}+4y_{n_1}^{(-)}y_{n_2}^{(+)}y_{n_3}^{(+)}y_{n_4}^{(+)})\delta(n_1+n_2+n_3+n_4),$$

where H_I describes the interaction of two opposing waves (3.1). Using (2.17), we write down the equations of motion obtained from H by varying the equations with respect to $y_n^{(\pm)}$:

$$\frac{d^2y_n^{(\pm)}}{dx^2}+n^2\frac{\omega^2}{c^2}\left(1+n^2\frac{\omega^2}{c^2}\right)y_n^{(\pm)}$$
$$+\frac{\omega^4}{c^4}n\sum_{n_1,n_2,n_3}n_1n_2n_3y_{n_1}^{(\pm)}y_{n_2}^{(\pm)}y_{n_3}^{(\pm)}\delta(n-n_1-n_2-n_3) \tag{3.6}$$
$$=-3\frac{\omega^4}{c^4}n\sum_{n_1,n_2,n_3}n_1n_2n_3[y_{n_1}^{(+)}y_{n_2}^{(-)}y_{n_3}^{(-)}+y_{n_1}^{(+)}y_{n_2}^{(+)}y_{n_3}^{(-)}]\delta(n-n_1-n_2-n_3).$$

Since the right-hand side of (3.6), which takes the wave interaction into account, is small, it is convenient, according to (2.3) and (2.14), to seek the solution in the form

$$y_n^{(\pm)}=[\mp A_n^{(\pm)}(x)/ink^{(\pm)}+b_n^{(\pm)}(x)]\exp\{\mp ink^{(\pm)}x\}, \tag{3.7}$$

where $A_n^{(\pm)}(x)$ are functions that vary slowly in comparison with the exponential and are in fact the Fourier amplitudes $z_n^{(\pm)}$, while $b_n^{(\pm)}$ are rapidly-varying small corrections. Substitution of (3.7) in (3.6) and averaging of the rapidly oscillating functions yield

$$\frac{dA_n^{(\pm)}}{dx}\approx\mp\frac{3}{2}ink^{(\pm)}\sum_{n_1,n_2}\{A_{n_1}^{(\mp)}A_{n_2}^{(\mp)}A_n^{(\pm)}\delta(n_1+n_2)$$
$$+A_{n_1}^{(\mp)}A_{n_2}^{(\pm)}A_0^{(\pm)}\delta(n-n_1-n_2)\}. \tag{3.8}$$

It is necessary to leave under the summation sign in (3.8) the zeroth approximation, and consequently the expression in the braces does not depend on x. From this we get a solution of (2.8), which is conveniently represented in the form

$$A_n^{(\pm)}(x)=a_n^{(\pm)}\exp\{\mp in\Delta k^{(\pm)}x\}, \tag{3.9}$$

where $a_n^{(\pm)}$ are unperturbed values of the amplitudes, which do not depend on x, and the shift of the wave number $\Delta k^{(\pm)}$ is equal to

$$\Delta k^{(\pm)}=\frac{3}{2}k^{(\pm)}\left\{\sum_{n_1}|a_{n_1}^{(\mp)}|^2+\frac{a_0^{(\pm)}}{a_n^{(\pm)}}\sum_{n_1,n_2}a_{n_1}^{(\mp)}a_{n_2}^{(\pm)}\delta(n-n_1-n_2)\right\}. \tag{3.10}$$

It should be pointed out that, generally speaking, the second term in (3.10) can depend on n. This would make a solution in the form (3.9) with a shift of the wave number meaningless. However, by virtue of the conditions (3.3), expressions (2.12) are valid for $a_n^{(\pm)}$. This yields $a_0^{(\pm)}/a_n^{(\pm)}\approx 1$, and the expression in the second sum of (3.10) is likewise independent of n.

With the aid of (2.12), by performing the simple summation in (3.10), we obtain, accurate to a constant factor of the order of unity,

$$\Delta k^{(+)}/k\approx\alpha^{(-)}/N^{(-)},\quad \Delta k^{(-)}/k\approx\alpha^{(+)}/N^{(+)}, \tag{3.11}$$

where account is taken of the fact that $k^{(+)}\sim k^{(-)}\sim k$ by virtue of the smallness of the corrections Δk. At $\alpha^{(+)}\sim\alpha^{(-)}\sim\alpha$, the latter is obvious, since $N^{(\pm)}\gg 1$. If the values of $\alpha^{(\pm)}$ differ appreciably, the expressions (3.11) remain in force, and their region of applicability will be defined shortly.

According to (2.15), the change of the wave numbers (3.11) leads to a shift in the wave velocity, given by

$$\Delta u^{(\pm)}=-|u^{(\pm)}|\Delta k^{(\pm)}/k\approx-\alpha^{(\mp)}/N^{(\mp)}. \tag{3.12}$$

This is indeed the main result of the present section.

We present the necessary estimates. Let, for example, $\alpha^{(+)}>\alpha^{(-)}$. In order for perturbation theory to be

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[2)]It should also be noted that in the nonlinear case the Hamiltonians may greatly differ from one another, while retaining the frequency characteristics of the temporal or spatial spectrum.

[3)]The method described below is used to construct two-stream solutions in the general case (2.1).

valid it is obviously necessary that the corrections to the velocity be sufficiently small, namely,

$$\Delta u^{(+)} \ll \alpha^{(+)}, \quad \Delta u^{(-)} \ll \alpha^{(-)}. \tag{3.13}$$

Taking (2.11) into account, we get from (3.13) the following system of inequalities:

$$(N^{(-)})^2 \gg N^{(+)} > N^{(-)} \gg 1. \tag{3.14}$$

A nontrivial fact here is the first inequality. When $\alpha^{(+)} \sim \alpha^{(-)}$ the conditions (3.13), according to the second inequality in (3.3), are automatically satisfied. We can verify in exactly the same manner that the corrections connected with $b_n^{(\pm)}$ can be neglected in first order of perturbation theory, owing to the inequalities (3.3).

It is also of interest to discuss the result (3.12) from another point of view. Since $N^{(\pm)} \gg 1$ and $k^{(+)} \sim k^{(-)} \sim k$, the wave interaction reduces to an almost periodic passage of crests (solitons) of different waves through one another. Let us estimate the change of the phase of the solution in one period, i.e., as a result of a single passage of the crests. Obviously, according to (2.10), it is equal to

Substituting (3.12) and (3.16) in (3.15) we obtain

$$\Delta\psi^{(+)} = \Delta\psi^{(-)} \sim \sqrt{\alpha^{(+)}\alpha^{(-)}} \ll 1, \tag{3.17}$$

i.e., the change of phase is always small in view of the first inequality of (3.3). The result is typical for two opposing waves, since $u^{(+)} > c$, $u^{(-)} < -c$ and $u^{(+)} - u^{(-)} > 2c$. In the case of two waves propagating in the same direction ($u_{1,2} > c$), the following singularity arises. Expressions (3.12) remain in force[10], but the relative velocity can be very small and

$$t_{1,2} \sim 1/k|u_1 - u_2| = 1/k|\alpha_1 - \alpha_2| \sim 1/k\alpha_1 \tag{3.18}$$

when

$$1 \gg \alpha_1 \gg \alpha_2. \tag{3.19}$$

The quantity c in (3.16) is replaced in (3.18) by the small quantity α_1, and we get

$$\Delta\psi_1 = \Delta\psi_2 \sim \sqrt{\alpha_2/\alpha_1} \ll 1. \tag{3.20}$$

This last inequality, naturally, is valid only because of the condition (3.19).[4)]

4. LOCAL INHOMOGENEITY

In the preceding section we constructed a two-stream solution in the form of nonlinear waves traveling opposite to each other with corresponding renormalized velocities. This solution allows us to approach the problem of scattering of a nonlinear wave by inhomogeneities. We consider first the so-called localized inhomogeneity, which can be produced, for example. by an impurity atom with a mass that differs from the masses of the remaining atoms of the chain.

[4)]We note that the statement made in[11], that the interaction of two solitons or two periodic waves cannot be considered within the framework of perturbation theory, is incorrect and even contradicts the results of[11]. Indeed, from formulas (13) of[11] there follows in the case $\alpha_1 \gg \alpha_2$ an expression that coincides with that obtained from perturbation theory. This comparison can be easily made, and we omit it here. A similar agreement of the results is obtained also for the shift of the velocities of nonlinear periodic waves.

In this case we can write

$$\frac{1}{c^2(x)} = \frac{1}{c_0^2} + \frac{\varepsilon}{c_0^2}\delta(x - x_0), \tag{4.1}$$

where c_0 is the constant speed of sound at $\epsilon = 0$, and the perturbation connected with the defect is assumed small, i.e.,

$$\varepsilon/\lambda \ll 1, \tag{4.2}$$

where λ is the characteristic wavelength. Substituting (3.1) in (2.8) we obtain in place of (2.13)

$$\frac{d^2 y_n}{dx^2} + n^2\frac{\omega^2}{c_0^2}\left(1 + n^2\frac{\omega^2}{c_0^2}\right)y_n + \frac{\omega^4}{c_0^4} n \sum_{n_1,n_2,n_3} n_1 n_2 n_3 y_{n_1} y_{n_2} y_{n_3} \delta(n - n_1 - n_2 - n_3) = \varepsilon n^2 \frac{\omega^2}{c_0^2}\delta(x) y_n, \tag{4.3}$$

where we put for convenience $x_0 = 0$. If y_n is the solution of (4.3) then it should obviously satisfy the following (exact boundary conditions:

$$y_n(+0) = y_n(-0) \equiv y_n(0), \quad \frac{dy_n(+0)}{dx} = \frac{dy(-0)}{dx} + \varepsilon\frac{n^2\omega^2}{c_0^2} y_n(0), \tag{4.4}$$

which follow directly from (4.3).

Assume now that a two-stream solution $\{y^{(+)}, y^{(-)}\}$ is specified at $x < 0$. Does there exist a two-stream solution $\{\bar{y}^{(+)}, \bar{y}^{(-)}\}$ at $x > 0$, such that the conditions (4.4) are satisfied at $x = 0$? The affirmative answer to this question, which is given below, means the existence of a solution of the equation

$$\{\bar{y}^{(+)}, \bar{y}^{(-)}\} = \hat{M}\{y^{(+)}, y^{(-)}\} \tag{4.5}$$

for the transition (scattering) matrix $\hat{M}$. By virtue of the nonlinearity of the problem, the matrix $\hat{M}$ is nonlinear, i.e., it depends on the parameters of the solution. We emphasize that (4.5) is not a trivial equation, since the nonlinear transformation $\hat{M}$ should transform the two-stream solution $\{y^{(+)}, y^{(-)}\}$ into a solution of the same class.

According to (3.2) we have

$$y_n = y_n^{(+)} + y_n^{(-)}, \quad x \leq 0, \qquad \bar{y}_n = \bar{y}_n^{(+)} + \bar{y}_n^{(-)}, \quad x \geq 0. \tag{4.6}$$

Differentiation of (4.6) with respect to x, with allowance for (3.7) and (3.9), yields

$$dy_n/dx = -in(k^{(+)}y_n^{(+)} - k^{(-)}y_n^{(-)}), \quad x \leq 0, \qquad d\bar{y}_n/dx = -in(\bar{k}^{(+)}\bar{y}_n^{(+)} - \bar{k}^{(-)}\bar{y}_n^{(-)}), \quad x \geq 0, \tag{4.7}$$

where $k^{(\pm)}$ and $\bar{k}^{(\pm)}$ are renormalized wave numbers. We put

$$B_n^{(\pm)} = y_n^{(\pm)}(0), \qquad \bar{B}_n^{(\pm)} = \bar{y}_n^{(\pm)}(0). \tag{4.8}$$

Substitution of (4.6)—(4.8) in (4.4) yields the system

$$\bar{B}_n^{(+)} + \bar{B}_n^{(-)} = B_n^{(+)} + B_n^{(-)},$$

$$\bar{k}^{(+)}\bar{B}_n^{(+)} - \bar{k}^{(-)}\bar{B}_n^{(-)} = k^{(+)}B_n^{(+)} - k^{(-)}B_n^{(-)} + in\varepsilon\frac{\omega^2}{c_0^2}(B_n^{(+)} + B_n^{(-)}),$$

which is conveniently rewritten in the form

$$\begin{bmatrix}\bar{B}_n^{(+)}\\ \bar{B}_n^{(-)}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\frac{\bar{u}^{(+)}}{\bar{u}^{(-)}} + \frac{\bar{u}^{(+)}}{u^{(+)}} + in\varepsilon\frac{\omega\bar{u}^{(+)}}{c_0^2} & \frac{\bar{u}^{(+)}}{\bar{u}^{(-)}} - \frac{\bar{u}^{(+)}}{u^{(-)}} + in\varepsilon\frac{\omega\bar{u}^{(+)}}{c_0^2}\\ \frac{\bar{u}^{(-)}}{\bar{u}^{(+)}} - \frac{\bar{u}^{(-)}}{u^{(+)}} - in\varepsilon\frac{\omega\bar{u}^{(-)}}{c_0^2} & \frac{\bar{u}^{(-)}}{\bar{u}^{(+)}} + \frac{\bar{u}^{(-)}}{u^{(-)}} - in\varepsilon\frac{\omega\bar{u}^{(-)}}{c_0^2}\end{bmatrix}\begin{bmatrix}B_n^{(+)}\\ B_n^{(-)}\end{bmatrix}$$

Here

$$u^{(\pm)} = \omega/c_0 k^{(\pm)}, \qquad \bar{u}^{(\pm)} = \omega/c_0\bar{k}^{(\pm)}.$$

With the aid of (3.11) and (3.12) we obtain the functions

$$u^{(\pm)}=u^{(\pm)}(B_n^{(+)},B_n^{(-)}),\qquad \bar{u}^{(\pm)}=\bar{u}^{(\pm)}(\bar{B}_n^{(+)},\bar{B}_n^{(-)}),\quad (4.10)$$

which do not depend on n. We consider the system (4.19) for $n \lesssim N^{(\pm)}$. Since $B_n^{(\pm)}$ and $\bar{B}_n^{(\pm)}$ depend in this case only on their number n, the equations of the system (4.9) do not become coupled, and the problem of determining the transition matrix $\hat{M}$ becomes algebraic. Since ϵ is small, the components of the matrix $\hat{M}$ can be obtained by perturbation theory.

To obtain a result that can be easily interpreted, we consider the simplest case, when the difference between $\bar{u}$ and u in the matrix components in (4.9) can be neglected. In this case we can put

$$\bar{B}_n^{(+)}\approx e^{ink^{(+)}\delta^{(+)}}B_n^{(+)},\qquad \bar{B}_n^{(-)}\approx e^{-ink^{(-)}\delta^{(-)}}B_n^{(-)};\qquad(4.11)$$

$$\delta^{(+)}=\frac{1}{2}\varepsilon\left(\frac{u^{(+)}}{c_0}\right)^2\left(1+\frac{B_n^{(-)}}{B_n^{(+)}}\right)\approx\frac{1}{2}\varepsilon\left(1+\sqrt{\frac{\alpha^{(-)}}{\alpha^{(+)}}}\right),$$
$$\delta^{(-)}=\frac{1}{2}\varepsilon\left(\frac{u^{(-)}}{c_0}\right)^2\left(1+\frac{B_n^{(+)}}{B_n^{(-)}}\right)\approx\frac{1}{2}\varepsilon\left(1+\sqrt{\frac{\alpha^{(+)}}{\alpha^{(-)}}}\right).\qquad(4.12)$$

Since the parameters $k^{(\pm)}$ and $\alpha^{(\pm)}$ are specified at $x<0$, the solution at $x<0$ is determined uniquely by expressions (4.11) and (4.12). Since $\delta^{(\pm)}$ do not depend on n, according to (4.12), this proves the existence of the solution of (4.5). Indeed, it follows from (4.11) that if a certain two-stream solution is specified for $x<0$, then we have at $x>0$

$$\bar{y}=\bar{y}^{(+)}(x-\bar{u}^{(+)}t-\delta^{(+)})+\bar{y}^{(-)}(x+\bar{u}^{(-)}t+\delta^{(-)}),\qquad(4.13)$$

i.e., the solution for $x>0$ is also of the two-stream type.

Expression (4.13) shows that in the case of scattering of nonlinear waves by inhomogeneities there is produced a coordinate phase shift $\delta^{(\pm)}$, defined by expressions (4.12). The phase shift was first observed experimentally by Howe[7], but there was essentially no theory for it. We note also that the phase shifts $\delta^{(\pm)}$ enter in the second term of expression (3.10) for Δk, and this leads to a corresponding change of the wavelengths $\bar{y}^{(\pm)}$ after scattering by the inhomogeneity (a phenomenon likewise observed by Howe experimentally[7]).

So far we have disregarded the possible existence of different phase shifts in the quantities $B_n^{(+)}$ and $B_n^{(-)}$. Assume now that

$$B_n^{(+)}/B_n^{(-)}=|B_n^{(+)}/B_n^{(-)}|e^{-i\varphi_n}.$$

Since the solution has a structure of the type (4.13), we get $\varphi_n=n\varphi_0$. From this it follows immediately that in the general case the system (4.9) has in general no solution. More accurately speaking, a two-stream solution does not go over in general into a two-stream solution. The physical cause of this difficulty is that the character of the interaction of the crests of two opposing waves with an inhomogeneity can depend strongly on the mutual locations of the crests relative to the point x = 0, (i.e. on φ_0). This difficulty can be resolved when $\varphi_0 \ll 1/N$, i.e., when the distance between the maxima of the crests located in the vicinity of the point x = 0 is much smaller than the width of the crests. In this case we can expand $\exp(-in\varphi_0)$ in a series and the problem has a solution similar to (4.11)–(4.13).

The question of the scattering of a nonlinear wave by a localized inhomogeneity remains in general open. From the foregoing results it follows that the dephasing of the individual harmonics of the nonlinear wave (i.e., the weakening of their correlation) leads to serious difficulties. There are, however, two possible situations typical of "strong" scattering: 1) the solution exists in the class of multiple-stream motions, 2) there is no stationary solution.

5. SLOWLY VARYING INHOMOGENEITY

We write the condition for the slowness of the variation of c(x) in the usual form:

$$kL\gg 1,\qquad L\sim c(dc/dx)^{-1}.\qquad(5.1)$$

It is natural to seek the solution for y, without allowance for the reflection effects (i.e., in the WKB approximation) in the form

$$y^{(\pm)}=\sum_n y_n^{(\pm)}e^{in\omega t},\qquad y_n^{(\pm)}=B_n^{(\pm)}\exp\left\{\mp in\int^x k^{(\pm)}(x')dx'\right\},\quad(5.2)$$

where the pre-exponential factor is already included in $B_n^{(\pm)}$.

It is easy to see that after substituting (5.2) in (2.13), that in the first approximation y_n satisfies the equation

$$d^2y_n/dx^2+n^2k^2(x)y_n=0,\qquad(5.3)$$

where k(x) is given by the same expression as in the case c = const, but now depends parametrically on x via c(x).

We assume now for the time being that we solve Eqs. (5.3) for each of the y_n separately, disregarding the strong coupling between the y_n. In this case we know the following transition matrix[12]:

$$\begin{bmatrix}\bar{B}_n^{(+)}\\ \bar{B}_n^{(-)}\end{bmatrix}=\begin{bmatrix}e^{in\varphi}\sqrt{1+e^{-2n\delta}} & ie^{n\delta}\\ -ie^{-n\delta} & e^{-in\varphi}\sqrt{1+e^{-2n\delta}}\end{bmatrix}\begin{bmatrix}B_n^{(+)}\\ B_n^{(-)}\end{bmatrix},\qquad(5.4)$$

where $B_n^{(\pm)}$ corresponds to the solution for $x\to-\infty$, $\bar{B}_n^{(\pm)}$ is the same for $x\to\infty$, φ is a phase shift on the order of 1/kL,

$$\delta=-i\oint k(\xi)d\xi\sim kL>0,\qquad(5.5)$$

and the integral in (5.5) is taken along a closed contour in the complex ξ plane, enclosing the points ξ and ξ^* at which $k^2(\xi)$ has a simple zero.

Since $\delta\gg 1$, it is necessary to consider in the first approximation only the reflection effect, which is of the order of $e^{-\delta}$, i.e., the reflection of the first harmonics; the reflections of the remaining harmonics can be neglected.

We now find on the right and left sides of the reflection region a solution for which the first harmonic satisfies relations (5.4) with n = 1, and for all the remaining n we have

$$\bar{B}_n^{(\pm)}\approx B_n^{(\pm)}e^{\pm in\varphi},\qquad n\neq 1.\qquad(5.6)$$

So far we have considered the transition matrix (5.4) for Eq. (5.3) without paying attention to the fact that the Fourier component y_n is a harmonic of the nonlinear solution. To construct the solution for $x\to-\infty$ we recognize that

$$k(x)=\omega/u(x)=\omega/c(x)(1+\alpha),\qquad(5.7)$$

where the dependence of α on x can be neglected, since $\alpha \ll 1$, and consider only the c(x) dependence. As already noted, we are considering the simplest case of the singularity of the asymptotic solution (5.2), in which $k^2(x)$ has a simple zero in the complex plane, and consequently $1/c(x)$ has a branch point. Since the solutions in question are nonlinear, the quantity $k^{(\pm)}(x, \alpha^{(\pm)})$ depends on the coefficients $B_n^{(\pm)}$, but it is important in what follows that the analytic properties of $k^{(\pm)}$ do not depend on the amplitude by virtue of (5.7). This enables us to go around the singularities in the complex case in the same manner as in the linear case, with the following modification: if the coefficients $B_n^{(\pm)}$ change jumpwise on going from one level line to another in the solutions $y_n^{(\pm)}$ the values of $k^{(\pm)}$ also change jumpwise as a result of the change of $\alpha^{(\pm)}$. The corresponding calculations are similar to those in[12], with allowance for the change of $k^{(\pm)}$, and cause the expression (5.5) for δ in the matrix (5.4) to be replaced by a more complicated one. However, since α is small and $\delta \gg 1$, it suffices to use the approximate expression:

$$\delta = -i\omega \oint \frac{d\xi}{c(\xi)}.$$

Now the matrix (5.4) for n = 1 and the condition (5.6) for $n \neq 1$ play the role of boundary conditions similar to (4.9) in the preceding section.

Assume that $x \to \infty$ there is only the transmitted wave, i.e., $B_n^{(-)} = 0$ for all n and

$$y = \sum_n B_n^{(+)} \exp\left\{in\left[\omega t - \int^x k(x')\,dx'\right]\right\}, \qquad x \to \infty. \tag{5.8}$$

Then we have, according to (5.4) and (5.6),

$$\bar{y} \approx \bar{y}^{(+)} + \bar{y}^{(-)} + \Delta\bar{y}_1^{(+)}, \qquad x \to -\infty,$$
$$\bar{y}^{(+)} = \sum_n B_n^{(+)} \exp\left\{in\left[\omega t - \int^x k(x')\,dx' + \varphi\right]\right\},$$
$$\bar{y}^{(-)} = |B_1^{(+)}|\, e^{-\delta} \sin\left[\omega t + \int^x k^{(-)}(x')\,dx'\right],$$
$$\Delta\bar{y}_1^{(+)} = \frac{1}{2} B_1^{(+)} e^{-2\delta} \cos\left[\omega t - \int^x k(x')\,dx' + \varphi\right]. \tag{5.9}$$

The term $\Delta\bar{y}_1^{(+)}$ is immaterial in the first approximation, and the interaction between $\bar{y}^{(+)}$ and $\bar{y}^{(-)}$ can be considered in the same manner as in Sec. 2. The corrections to $\bar{y}^{(+)}$ are then of the order of $e^{-\delta}$, and the corrections to $\bar{y}^{(-)}$ are of the order of $\alpha^{(+)}e^{-\delta} \ll e^{-\delta}$. Thus, accurate to smaller terms, we have

$$\bar{y} \approx \bar{y}^{(+)} + \bar{y}^{(-)}, \qquad x \to -\infty, \tag{5.10}$$

where $\bar{y}^{(-)}$, defined in (5.8), is indeed the sought reflected wave.

Just as in the preceding section, there as a jump in phase between the incident wave $\bar{y}^{(+)}$ and the transmitted wave y ($\varphi \sim 1/kL$), and its more accurate value can be found in[12].

We note that when account is taken of the terms of next higher order in $e^{-\delta}$ in the solution y as $x \to -\infty$, the number of different "streams" will increase (there are already three of them in (5.9)), in agreement with the remark made in Sec. 4.

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