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Effect of the Dimensions and Shape of a Metal Specimen on the Value of the Torque in a Rotating Magnetic Field

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The value of the magnetic moment induced by an alternating magnetic field is calculated under the condition that the length of the free path is comparable with the specimen dimensions. The dependence of the moment on the specimen dimensions and on the nature of the reflection of the electrons by the surface is found for plane, cylindrical, and spherical specimens. This dependence may differ significantly from that for the static conductivity. This fact can be used for experimental determination of the relaxation time and of the coefficient of specularity of the reflection of the electrons by the boundary, and also for observation of the hydrodynamic mechanism of electronic conduction.

THE method of determining the conductivity of metal specimens from the value of the induced magnetic moment in an alternating magnetic field[1] has obtained quite broad dissemination. It is used, for example, for determining the galvanomagnetic properties of metals and for detecting open electron trajectories[2,3]. In the experiment it is usual to determine the torque acting on a freely suspended specimen placed in a rotating magnetic field H perpendicular to the axis of rotation. As is well known, the value of the torque K for bodies of revolution can be expressed in terms of the imaginary part α of the magnetic polarizability of the specimen in an alternating magnetic field:

$$K = \alpha V H^2,$$

here V is the volume of the specimen.

For small specimens of pure metals at low temperatures, when the length of the free path of the electrons is comparable with the dimensions of the specimen, scattering of the electrons by the surface may prove important[1]. Under these conditions, the value of the magnetic moment may depend appreciably on the dimensions and shape of the specimen, and also on the character of the reflection of the electrons from the boundary. Because of the nonuniformity of the electric field, the dependence of the magnetic moment on the free-path length l and on the coefficient of specularity q of reflection the electrons by the surface will be different from that of the static conductivity $\sigma(q, l)$. Independent measurements of σ and α can be used to determine the values of l and q. In the low-frequency range, the imaginary part of the magnetic polarizability is proportional to the frequency ω[4]. By investigating the dependence of α on frequency, it is possible to determine a correction proportional to ω^3, and thus to obtain still another quantity dependent on q and l.

In the case of diffuse scattering of the electrons by the surface (q = 0), a characteristic dimension of the specimen will play the role of effective free-path length. As a result, the effective conductivity and, consequently, the magnetic moment should decrease with decrease of the dimensions of the specimen. The case of specular scattering (q = 1) requires more careful treatment. It turns out that in this case the result will depend on the geometry of the specimen and the direction of the magnetic field. Formally, this can be understood from the following arguments.

We first consider a spherical specimen: the alternating magnetic field induces circular currents in it; that is, only the component E_φ differs from zero. From Maxwell's equations in the first approximation with respect to ω, it can easily be shown that in a spherical coordinate system $E_\varphi \sim r \sin\theta$. The inhomogeneous part of the kinetic equation, $E_\varphi v_\varphi$, will in this case be a conserved quantity, proportional to the projection of the angular momentum along the z axis. Therefore for specular reflection of the electrons by the boundary, the dependence of the distribution function on the coordinates will be the same as for $l \ll R$, where R is the radius radius of the specimen. As a result, the magnetic moment for q = 1 will be determined by the expression obtained for $l \ll R$. (A similar result occurs for the static conductivity of a plate.)

A different situation occurs for a metal plate placed in an alternating magnetic field parallel to the surface. The induced currents flow in opposite directions near the top and bottom surfaces of the specimen. As a result, an electron that undergoes an addition to its speed because of the electric field at one surface, and that reaches the opposite surface without collisions, will decrease the electric current. Therefore the value of the magnetic moment should differ from the value corresponding to $l \ll d$ (2d = plate thickness), even for specular reflection of the electrons by the boundary. An immediate consequence of this result is the occurrence of Sondheimer oscillations of the induced magnetic moment of a plate in a constant magnetic field perpendicular to the surface, for arbitrary conditions of reflection of the electrons by the surface. Of interest also is a result obtained for a hydrodynamic mechanism of electrical conductivity: since the current density vanishes on the boundary for an arbitrary value of q, a maximum on the curve of magnetic moment vs temperature should be observed for an arbitrary type of reflection of the electrons by the surface.

Hereafter we shall restrict ourselves to calculation of the imaginary part of the magnetic polarizability in the approximation in which the skin depth appreciably exceeds the dimensions of the specimen. (We assume weak, quasistatic magnetic fields: $\Omega\tau \ll 1$, $\omega\tau \ll 1$, where $\Omega = eH/mc$ is the cyclotron frequency.) The real part of the polarizability is different from zero only in the next (as compared with α) approximation with respect to L/δ and is proportional to ω^2. The classical kinetic equation is used, in the relaxation-time approximation. The scattering of the electrons by the surface is described by introduction of the coefficient of specularity of the reflection, q. The energy spectrum of the electrons is assumed to be isotropic and quadratic. Results are presented for plane, cylindrical, and spherical specimens.

1. We consider a plane metal specimen in an alternating magnetic field **H** parallel to the surface. All quantities vary with time as $e^{-i\omega t}$. Assuming the frequency ω so small that the skin depth δ appreciably exceeds the specimen thickness 2d, we shall suppose that in Maxwell's equation

$$\mathrm{rot}\,\mathbf{E} = i\omega \mathbf{H}/c \tag{1}$$

the amplitude of the magnetic field is independent of the coordinates. Then in the first approximation with respect to δ/d

$$E = i\omega Hz/c. \tag{2}$$

To calculate the electric current, we shall use the kinetic equation linearized with respect to the electric field,

$$\pm v_z df^{\pm}/dz + f^{\pm}/\tau = eEv_x, \tag{3}$$

where the functions $f^{\pm}$ correspond to $v_z \gtrless 0$. On substituting the electric field (2) in the right member, we find

$$j(z) = e\langle v_x(f^+ + f^-)\rangle_+,$$

where the angular brackets with the plus sign denote an average over the part of the Fermi surface where $v_z > 0$. Once the electric current is known, it is easy to determine the value of the magnetic moment and of the imaginary part of the polarizability:

$$\mathfrak{M} = \frac{1}{c}\int_{-d}^{d} zj(z)\,dz, \qquad \mathfrak{M} = 2idH\alpha. \tag{4}$$

The solution of the kinetic equation for a plate is well known; therefore we present only the results of the calculations:

$$\mathfrak{M} = \frac{3}{2}\mathfrak{M}_0\int_0^1 \frac{dx(1-x^2)}{1-q^2e^{-2t/x}}\Big\{1 - q^2e^{-2t/x} - \frac{3x}{t}\Big[1 - q + \frac{2x}{t}(1+q)\Big]$$
$$\times\Big[1 - \frac{2x}{t} + \Big(1 - q + (1+q)\frac{2x}{t}\Big)e^{-t/x} - q\Big(1 + \frac{2x}{t}\Big)e^{-2t/x}\Big]\Big\}, \tag{5}$$

where

$$\mathfrak{M}_0 = \frac{k_0^2 d^3 H}{6\pi}, \quad k_0^2 = \frac{4\pi i\omega\sigma_0}{c^2}, \quad \sigma_0 = \frac{ne^2\tau}{m};$$

$\mathfrak{M}_0$ and σ_0 are the values of the magnetic moment and of the conductivity in the case of small free-path length, $t = 2d/l$, and $l = v_F\tau$.

For $t \gg 1$ we easily derive from (5)

$$\mathfrak{M} = \mathfrak{M}_0\Big[1 - \frac{9}{8}\frac{1-q}{t} - \frac{12}{5}\frac{q}{t^2}\Big]. \tag{6}$$

For $t \ll 1$ it is necessary to distinguish two limiting cases: for $t \ll 1 - q$,

$$\mathfrak{M} = \mathfrak{M}_0\frac{t}{1-q^2}\Big[\frac{9 - 30q + 65q^2}{16} + 3q(1-2q)\ln 2\Big] \tag{7}$$

and for $1 - q \ll t \ll 1$,

$$\mathfrak{M} = \mathfrak{M}_0\frac{t}{4q^2}\Big[\frac{16 + 195q - 221q^2}{100} + \frac{6}{5}q(3q-2)\ln 2 - \frac{3}{10}(1-q^2)(\gamma + \ln t)\Big], \tag{8}$$

where $\gamma = 0.577$ is Euler's constant.

In order to derive formulas (7) and (8), it is necessary to replace the denominator in (5) by $1 - q^2$ or by $q^2 2tx^{-1}$, respectively, and to use the expansions of the functions

$$E_n(t) = \int_1^\infty \frac{e^{-tx}}{x^n}dx$$

in powers of t.

It is significant that in the case of specular reflection (q = 1), the value of the magnetic moment is not equal to $\mathfrak{M}_0$, as is the case for the conductivity. In the case $l \gg d$ the magnetic moment is independent of the relaxation time (the term $(1-q)\ln t$ may be neglected), both for diffuse and for specular reflection of the electrons by the boundary.

We now consider the question of Sondheimer oscillations of the magnetic moment of a plate and of their difference from the oscillations of the static conductivity[5]. From (5) one can determine the function

$$\Phi(t) = \mathfrak{M}/\mathfrak{M}_0 t.$$

Then in a constant magnetic field $\mathbf{H}_0$ perpendicular to the surface of the plate,

$$\mathfrak{M} = \mathfrak{M}_0 t\,\mathrm{Re}\,\Phi(s),$$

where $s = 2d/l + i2d/r_H$; r_H is the Larmor radius of an electron in the magnetic field H_0. There is also a magnetic moment in the perpendicular direction, but it does not produce a torque about the z axis. Thus Sondheimer oscillations of the magnetic moment should occur even when q = 1. Determination of their explicit form requires numerical calculations.

It is also interesting to consider a hydrodynamic mechanism of electrical conduction in the plate. Following[6], we write the equation for the drift velocity u in the x direction:

$$-\nu d^2u/dz^2 + u/\tau = eE/m,$$

where $\nu = \frac{1}{15}v_F l_{ep}(T)$; l_{ep} ~ T^{-5} is the electron-phonon free-path length; the expression (2) must be substituted for E. The boundary conditions in this case can be written $u(\pm d) = 0$ for arbitrary conditions of scattering of the electrons by the boundary. As a result, we get for the magnetic moment

$$\mathfrak{M} = \mathfrak{M}_0\Big(1 + 3\frac{1 - \varkappa\,\mathrm{cth}\,\varkappa}{\varkappa^2}\Big),$$

where $\kappa = d(\nu\tau)^{-1/2}$. If $\kappa \ll 1$, then $\mathfrak{M} \sim \mathfrak{M}_0\kappa^2/15$. Thus an increase of the magnetic moment with rise of temperature should be observed for an arbitrary mechanism of scattering of the electrons by the surface.

2. Let the magnetic field be perpendicular to the axis of a cylindrical specimen of radius R. We introduce cylindrical coordinates with the z axis along the axis of the cylinder; then we find from (1) that in the first approximation, the electric field is

$$E_z = i\omega c^{-1}Hr\sin\varphi. \tag{9}$$

The magnetic moment is defined by the expression

$$\mathfrak{M} = \frac{1}{c}\int_0^R r^2\,dr\int_0^{2\pi} d\varphi\, j(r,\varphi)\sin\varphi,$$

and the imaginary part of the polarizability is determined from the condition

$$\mathfrak{M} = i\pi R^2 H\alpha.$$

In the cylindrical coordinate system, the kinetic equation takes the form[7,8]

$$v_r\frac{\partial f}{\partial r}+\frac{v_\varphi}{r}\frac{\partial f}{\partial\varphi}+\frac{v_\varphi^2}{r}\frac{\partial f}{\partial v_r}-\frac{v_\varphi v_r}{r}\frac{\partial f}{\partial v_\varphi}+\frac{f}{\tau}=eE_zv_z. \qquad (10)$$

The homogeneous equation corresponding to (10) has the following integrals:

$$\varepsilon=v_r^2+v_\varphi^2,\quad M=rv_\varphi,\quad \psi=\varphi-\operatorname{arctg}(v_r/v_\varphi), \qquad (11)$$

the first two of these correspond to conservation of energy and of angular momentum about the z axis. In the variables ϵ, M, ψ, and r we get an ordinary differential equation with the boundary conditions $f^-=qf^+$ for $r=R$ and $f^+=f^-$ for $r=M\epsilon^{-1/2}$, where the signs ± correspond to $v \gtrless 0$.

After some tedious calculations we find

$$\mathfrak{M}=\mathfrak{M}_0\frac{24}{\pi}\int_0^{\pi/2}d\psi\cos\psi\int_0^{\pi/2}\frac{\cos^2\theta\sin^4\theta\,d\theta}{1-q\mathscr{E}_1}\left\{\frac{\cos\psi(1+2\sin^2\psi)}{3\sin^3\theta}(1-q\mathscr{E}_1)\right.$$
$$+\frac{4}{k^3}[3+q-4(1+q)\mathscr{E}_{1/2}+(1+3q)\mathscr{E}_1]-$$
$$\left.-\frac{4}{k^2}\frac{\cos\psi}{\sin\theta}[q+2(1-q)\mathscr{E}_{1/2}-\mathscr{E}_1]-\frac{1-q}{k\sin^2\theta}(1-\mathscr{E}_1)\right\}. \qquad (12)$$

Here

$$\mathscr{E}_n\equiv\exp\left(-nk\frac{\cos\psi}{\sin\theta}\right),\quad k=\frac{2R}{l},\quad \mathfrak{M}_0=\frac{k_0^2R^4}{16}H.$$

For $k \gg 1$ we get a formula analogous to (6):

$$\mathfrak{M}=\mathfrak{M}_0\left[1-\frac{3}{2}\frac{1-q}{k}-\frac{16}{5}\frac{q}{k^2}\right]. \qquad (13)$$

For $k \ll 1$ we proceed as in the derivation of formulas (7) and (8), using the expansion as a series in powers of k of the functions $S_n(k)$, introduced in[7]:

$$S_n(k)=\int_0^{\pi/2}e^{-k/\sin\theta}\cos^2\theta\sin^{n-3}\theta\,d\theta.$$

The first terms of the expansion for $k \ll 1$ can also be obtained more simply by setting $\tau=\infty$ in the kinetic equation. For $1-q \gg k$, we have

$$\mathfrak{M}=\mathfrak{M}_0\frac{1+q}{1-q}\frac{4k}{5}, \qquad (14)$$

for $1-q \ll k \ll 1$,

$$\mathfrak{M}={}^7/_8\mathfrak{M}_0. \qquad (15)$$

For specular reflection of the electrons by the boundary, the value of the magnetic moment differs from $\mathfrak{M}_0$, although the ratio $\mathfrak{M}/\mathfrak{M}_0$ does not approach zero with increase of the free-path length, as is the case for the plane specimen (8).

If the alternating magnetic field is directed along the axis of the cylinder, experimental determination of the value of the magnetic moment is difficult; we shall nevertheless give the results for this case also, in order to be able to make a comparison with the results for the spherical specimen:

$$\mathfrak{M}=\mathfrak{M}_0\frac{24}{\pi}\int_0^{\pi/2}d\theta\sin^3\theta\int_0^{\pi/2}\frac{\sin^2\psi\cos\psi\,d\psi}{1-q\mathscr{E}_1}$$
$$\times\left[(1-q\mathscr{E}_1)\cos\psi-\frac{1-q}{k}\sin\theta(1-\mathscr{E}_1)\right], \qquad (16)$$
$$\mathfrak{M}_0=\frac{k_0^2R^4}{32}H.$$

From (16) it is clear that for specular scattering by the boundary, in this case, $\mathfrak{M}=\mathfrak{M}_0$.

For $k \gg 1$,

$$\mathfrak{M}=\mathfrak{M}_0\left[1-\frac{3}{2}\frac{1-q}{k}\right], \qquad (17)$$

for $k \ll 1$ and $1-q \gg k$,

$$\mathfrak{M}=\mathfrak{M}_0\frac{1+q}{1-q}\frac{2k}{5}, \qquad (18)$$

finally, for $1-q \ll k \ll 1$

$$\mathfrak{M}=\mathfrak{M}_0\text{——} \qquad (19)$$

3. Let the magnetic field be directed along the z axis. We introduce a spherical coordinate system; then the electric field, in first approximation, is

$$E_\varphi=\frac{i\omega}{2c}Hr\sin\theta. \qquad (20)$$

For the magnetic moment we have the formula

$$\mathfrak{M}=\frac{\pi}{c}\int_0^R r^3\,dr\int_0^\pi j(r,\theta)\sin^2\theta\,d\theta,$$

the imaginary part of the polarizability is determined by the relation

$$\mathfrak{M}={}^4/_3\pi iR^3H\alpha.$$

The kinetic equation for a spherical coordinate system has the following form:

$$v_r\frac{\partial f}{\partial r}+\frac{v_\theta}{r}\frac{\partial f}{\partial\theta}+\frac{v_\varphi}{r\sin\theta}\frac{\partial f}{\partial\varphi}+\frac{v_\theta^2+v_\varphi^2}{r}\frac{\partial f}{\partial v_r} \qquad (21)$$
$$+\frac{v_\varphi^2\cos\theta-v_rv_\theta\sin\theta}{r\sin\theta}\frac{\partial f}{\partial v_\theta}-\frac{v_\varphi(v_r\sin\theta+v_\theta\cos\theta)}{r\sin\theta}\frac{\partial f}{\partial v_\varphi}+\frac{f}{\tau}=eE_\varphi v_\varphi.$$

The integrals of the homogeneous equation corresponding to (21) are the following: the energy $\epsilon=v_r^2+v_\theta^2+v_\varphi^2$; the components of the angular momentum

$$M_x=-rv_\theta\sin\varphi-rv_\varphi\cos\theta\cos\varphi,\quad M_y=rv_\theta\cos\varphi-rv_\varphi\cos\theta\sin\varphi,$$
$$M_z=rv_\varphi\sin\theta$$

($M_\perp^2=M_x^2+M_y^2$, $M^2=M_z^2+M_\perp^2$); and an additional integral ψ such that

$$\operatorname{tg}\psi=(v_\theta^2+v_\varphi^2)^{1/2}\frac{v_\theta\sin\theta-v_r\cos\theta}{v_rv_\theta\sin\theta+(v_\theta^2+v_\varphi^2)\cos\theta}.$$

Transformation to the variables ϵ, M, ψ, and r in the right member of the kinetic equation (21) in the general case can lead to difficulties. They can be overcome as follows: we go over to a system of coordinates so chosen that the angular momentum is directed along the z axis. In the new system, the relation between the variables and the integrals of the motion will be the same as in the cylindrical coordinate system (11). On transforming back to the original system, we get, for example,

$$\sin^2\theta=\frac{M_z^2}{M^2}+\frac{M_\perp^2}{M^2\varepsilon}\left[\frac{M}{r}\cos\psi-\left(\varepsilon-\frac{M^2}{r^2}\right)^{1/2}\sin\psi\right]^2.$$

After this, it is easy to express the remaining variables also in terms of the integrals of the motion.

In the approximation of interest to us, however, no difficulties arise, since

$$E_\varphi v_\varphi\sim rv_\varphi\sin\theta=M_z,$$

and the kinetic equation (21) reduces to the equation

$$\pm\left(\varepsilon-\frac{M^2}{r^2}\right)^{1/2}\frac{df\pm}{dr}+\frac{f\pm}{\tau}=\frac{ie\omega}{2c}HM_z \qquad (22)$$

with the boundary conditions $f^- = qf^+$ for $r = R$ and $f^+ = f^-$ for $r = M\epsilon^{-1/2}$.

The result for the magnetic moment looks like this:

$$\mathfrak{M} = \frac{15}{2}\mathfrak{M}_0 \int_0^1 \frac{dx}{1-qe^{-kx}} \left\{ x^2(1-x^2)(1-qe^{-kx}) - \frac{1-q}{k} x(1-x^2)(1-e^{-kx}) \right\}, \tag{23}$$

where $k = 2R/l$ and $\mathfrak{M}_0 = k_0^2 R^5 H/30$.

If $k \gg 1$,

$$\mathfrak{M} = \mathfrak{M}_0 \left[1 - \frac{15}{8}\frac{1-q}{k} \right]. \tag{24}$$

For $k \ll 1$ and $1 - q \gg k$,

$$\mathfrak{M} = \mathfrak{M}_0 \frac{1+q}{1-q}\frac{5k}{16}, \tag{25}$$

for $1 - q \ll k \ll 1$,

$$\mathfrak{M} = \mathfrak{M}_0 \tag{26}$$

In the case of spherical and cylindrical specimens in a longitudinal magnetic field, the transformed kinetic equation has the form (22), a characteristic feature of which is a right member independent of r. The same property is possessed by the kinetic equation for a plate or wire in the determination of the static conductivity. Precisely because of this property, as follows from (16) and (23), the magnetic moment is equal to $\mathfrak{M}_0$ for $q = 1$; this corresponds to the results for the static conductivity. But in other cases the dependence of $\mathfrak{M}$ on q and l differs from the corresponding dependence for the static conductivity; this may prove useful for measurement of such electronic parameters as the relaxation time and the nature of the reflection by the surface, and also for observation of the hydrodynamic mechanism of electrical conduction.

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