PION-DEUTERON SCATTERING LENGTH

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By summation of an infinite series of nonrelativistic Feynman diagrams, a formula which is a generalization of the Brueckner formula is obtained for the π-meson—deuteron scattering length. It is then demonstrated that allowance for terms corresponding to the kinetic energy of nucleons in the intermediate state significantly changes the Brueckner result. The effects of p-wave πN interaction in diagrams with single or double scattering are estimated. The general relations obtained are applied to a numerical calculation of the πd scattering length. For this purpose the πN scattering lengths presented in survey[1] and the Hulthen wave function for the deuteron are employed. The result is $a_{\pi d} = -0.047$ F.

1. INTRODUCTION

IT is known that the s-wave lengths for scattering of pions by nucleons are small. At zero energy

$$f_{\pi N} = b_0 + b_1 \mathbf{I}\boldsymbol{\tau}, \tag{1}$$

where $b_0 = (-0.012 \pm 0.004)\mu^{-1} = (-0.017 \pm 0.006)$ F, $b_1 = (-0.097 \pm 0.007)\mu^{-1} = (-0.137 \pm 0.010)$ F, and $\mathbf{I}$ and $\boldsymbol{\tau}$ are the pion and nucleon isospin operators[1)]. Nonetheless, multiple-scattering effects must be taken into account even in the calculation of the length for scattering by a deuteron. The point is that the πd-scattering amplitude in the impulse approximation ($f_d^{imp} = 2b_0$) is small because b_0 is small, and is therefore comparable with the contribution of the multiple scattering, which is of the order of b_1^2/R (R is the deuteron radius).

There are two different approaches to the calculation of the πd-scattering length. The first, proposed by Brueckner[2], is based on the smallness of the pion mass in comparison with the nucleon mass ($\mu/m \approx 1/7$). It is assumed that the nucleons experience a small recoil after scattering the pion, and the πd scattering amplitude can be regarded as the amplitude for scattering by a system of two immobile centers[2,3], averaged over the wave function of the deuteron. Brueckner succeeded in obtaining an expression for the πd scattering amplitude, in which account is taken of all multiplicities of pion rescattering. A somewhat more accurate expression was obtained in[4] by summing nonrelativistic Feynman diagrams. It takes into account the deuteron binding energy and is rigorous in the limit as $\mu/m \to 0$. The second approach was developed by a number of workers[5-8] and reduces in essence to inclusion of the double scattering, besides the impulse approximation; in diagram language this corresponds to inclusion of diagrams a and b of Fig. 1. The calculation of the diagram b is carried out approximately, without allowance for the deuteron binding energy and neglecting the kinetic energy of the nucleons in the intermediate state. As will be shown below, this leads to a calculation error on the order of 50–70%. In addition, one cannot discard beforehand the terms corresponding to large scattering multiplicities. The expansion of the amplitude in powers of the scattering multiplicity is not, generally speaking, an expansion in terms of the parameter $b_{\pi N}/R$, as will be shown in Sec. 2.

In the present paper we take multiple pion scattering into account by using a nonrelativistic diagram technique[9,10]. In Sec. 2 we sum the diagram series that makes the main contribution to the πd-scattering length (the diagrams of Fig. 1), and average the amplitude over the isotopic variables. Unlike the earlier paper[4], we retain here kinematic corrections of order μ/m. The result of the summation is a modification of Brueckner's formula[2] if we leave out from the propagators, for convenience in summation, certain "small" terms of order μ/m. These terms are taken into account in Sec. 2. It is shown that they alter the result strongly. In the same section, the accuracy of the proposed method is compared with the accuracy of calculations by others. In Sec. 4 we take into account the corrections made to the amplitude by the contribution of the p-wave πN interaction and discuss other possible corrections. The results enable us, in principle, to calculate the nonabsorptive part of the πd-scattering length with an accuracy of several per cent. The real accuracy with which this quantity is calculated is now governed by the accuracy with which the πN-scattering lengths are known.

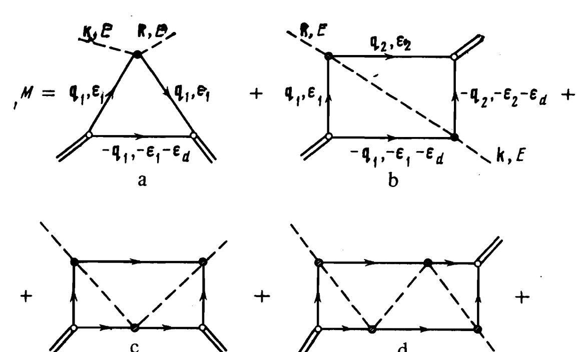

FIG. 1

[1)]The values of b_0 and b_1 were taken from Ericson's review [1]. Our notation also follows this review. We note that, in accord with the present practice in the literature on πN and πd scattering, we define the scattering length as the amplitude at zero energy (and not the amplitude with the sign reversed).

Preliminary results of the work were presented at the 4th International Conference on High Energy Physics and Nuclear Structure[11].

2. ZERO-ANGLE πd-SCATTERING AMPLITUDE AT LOW ENERGIES

In this section we sum the diagram series that makes the main contribution to the πd-scattering amplitude (Fig. 1). We calculate the zero-angle πd-scattering amplitude, assuming the πN-interaction to be s-wave and its amplitude to be constant. We disregard for the time being the fact that the πN-scattering amplitude is a matrix in the isotopic variables. We proceed to calculate each of the diagrams of Fig. 1. Diagram 1a corresponds to the following invariant amplitude[2)] $M_{\pi d}^{(1)}$:

$$M_{\pi d}^{(1)}=\frac{i^2(-2im)^3}{(2\pi)^4}$$
$$\times\int\frac{M_1^2F(\mathbf{q}_1)F(\mathbf{q}_1)A_1\,d\mathbf{q}_1\,d\varepsilon_1}{(\mathbf{q}_1^2-2m\varepsilon_1-i\eta)(\mathbf{q}_1^2+2m\varepsilon_1+2m\varepsilon_d-i\eta)(\mathbf{q}_1^2-2m\varepsilon_1-i\eta)}. \quad (2)$$

Here $M_1^2 = 8\pi\alpha/m^2$ ($\alpha^2 = m\epsilon_d$, ϵ_d is the deuteron binding energy and m is the nucleon mass); $A_1(A_2)$ is the invariant amplitude for pion scattering by the first (second) nucleon; $F(\mathbf{q})$ is the deuteron form factor, defined by the relation

$$\frac{F(\mathbf{q})}{q^2+\alpha^2}=\frac{1}{(8\pi\alpha)^{1/2}}\int e^{i\mathbf{q}\mathbf{r}}\psi_d(\mathbf{r})\,d\mathbf{r}. \quad (3)$$

Integrating with respect to energy in (2) and using (3), we obtain

$$M_{\pi d}^{(1)}=A_1\int\psi_d^2(\mathbf{r})\,d\mathbf{r}=A_1. \quad (4)$$

We proceed now to consider double scattering. The corresponding diagram is shown in Fig. 1b. Writing down the corresponding Feynman integral and integrating with respect to the energy variables, we obtain

$$M_{\pi d}^{(2)}=\frac{\mu\alpha A_1A_2}{4\pi^5}$$
$$\times\int\frac{F(\mathbf{q}_1)F(\mathbf{q}_2)\,d\mathbf{q}_1\,d\mathbf{q}_2}{(q_1^2+\alpha^2)[(\mathbf{k}+\mathbf{q}_1-\mathbf{q}_2)^2+2\mu(\varepsilon_d-E)+\mu q_1^2/m+\mu q_2^2/m](q_2^2+\alpha^2)} \quad (5)$$

Since $\mu/m \ll 1$, we can omit at first the terms $\mu q_1^2/m$ and $\mu q_2^2/m$ in the pion propagator. The error incurred thereby is evaluated in Sec. 3. It should be noted that at low energies we do not exaggerate the accuracy by omitting the terms $\mu q_1^2/m$ and $\mu q_2^2/m$ but retaining the term $2\mu\epsilon_d$ in the denominator of (5). In fact, by discarding terms of the type $\mu q^2/m$ we increase the value of the integral (5). $2\mu\epsilon_d$ is also positive, and therefore, by omitting this term, we increase the integral still more, i.e., we increase the error.

Assume, for concreteness, that $E < \epsilon_d$. By simple transformations[4] we easily arrive at a final expression for $M_{\pi d}^{(2)}$:

$$M_{\pi d}^{(2)}=A_1\left(\frac{\mu A_2}{2\pi}\right)\int\psi_d^2(\mathbf{r})\frac{e^{ikr-\varkappa r}}{r}\,d\mathbf{r}, \quad (6)$$

where $\kappa^2 = 2\mu(\epsilon_d - E)$, and E and k are the energy and momentum of the incoming pion. Proceeding analogously, i.e., omitting the terms $\mu q^2/m$ in all the pion propagators, we can calculate any term of the series of Fig. 1. For odd N we have

$$M_{\pi d}^{(N)}=A_1\left(\frac{\mu A_1}{2\pi}\right)^{(N-1)/2}\left(\frac{\mu A_2}{2\pi}\right)^{(N-1)/2}\int\psi_d^2(\mathbf{r})\frac{e^{-(N-1)\varkappa r}}{r^{N-1}}\,d\mathbf{r}. \quad (7)$$

For even N, analogously,

$$M_{\pi d}^{(N)}=A_1\left(\frac{\mu A_1}{2\pi}\right)^{N/2-1}\left(\frac{\mu A_2}{2\pi}\right)^{N/2}\int\psi_d^2(\mathbf{r})\frac{e^{ikr-(N-1)\varkappa r}}{r^{N-1}}\,d\mathbf{r}. \quad (8)$$

We shall find it convenient to change over from the invariant amplitudes $M_{\pi d}^{(N)}$ and A_i (i = 1, 2) to the ordinary amplitudes $f_{\pi d}^{(N)}$ and f_i (i = 1, 2)[3)]

$$f_{\pi d}^{(N)}=\frac{\mu m_d}{2\pi(\mu+m_d)}M_{\pi d}^{(N)},\quad f_i=\frac{\mu m}{2\pi(m+\mu)}A_i. \quad (9)$$

In terms of this notation, the amplitude

$$f_{\pi d}=\sum_{n=1}^{\infty}f_{\pi d}^{(n)}$$

is given by

$$f_{\pi d}=\frac{1}{1+\mu/m_d}\int\psi_d^2(\mathbf{r})\left\{\left(\tilde f_1+\tilde f_2+\frac{2\tilde f_1\tilde f_2}{r}e^{ikr-\varkappa r}\right)\right.$$
$$+\frac{\tilde f_1\tilde f_2}{r^2}e^{-2\varkappa r}\left(\tilde f_1+\tilde f_2+2\frac{\tilde f_1\tilde f_2}{r}e^{ikr-\varkappa r}\right)$$
$$\left.+\left(\frac{\tilde f_1\tilde f_2}{r^2}e^{-2\varkappa r}\right)^2\left(\tilde f_1+\tilde f_2+2\frac{\tilde f_1\tilde f_2}{r}e^{ikr-\varkappa r}\right)+\ldots\right\}d\mathbf{r}, \quad (10)$$

where $\tilde{f}_i = (m + \mu)f_i/m$. Transforming the expression in the curly brackets, we obtain ultimately

$$f_{\pi d}=\frac{1}{1+\mu/m_d}\int\psi_d^2(\mathbf{r})\frac{\tilde f_1+\tilde f_2+2\tilde f_1\tilde f_2r^{-1}e^{ikr-\varkappa r}}{1-\tilde f_1\tilde f_2r^{-2}e^{-2\varkappa r}}\,d\mathbf{r}. \quad (11)$$

Similar calculations can be performed for $E > \epsilon_d$, and the amplitude $f_{\pi d}$ will take the form

$$f_{\pi d}=\frac{1}{1+\mu/m_d}\int\psi_d^2(\mathbf{r})\frac{\tilde f_1+\tilde f_2+2\tilde f_1\tilde f_2r^{-1}e^{ikr+ipr}}{1-\tilde f_1\tilde f_2r^{-2}e^{2ipr}}\,d\mathbf{r}, \quad (12)$$

where $p^2 = 2\mu(E - \epsilon_d)$. Formulas (11) and (12) take into account (in the approximation $\mu/m = 0$) the three-particle unitarity of the amplitude $f_{\pi d}$: $f_{\pi d}$ is real below the deuteron breakup threshold and complex above. These formulas are a generalization of the Brueckner formula[2,4] and take into account the additional deuteron binding energy in the relations for κ and p, and also the kinematic corrections μ/m_d and μ/m that arise on going from the laboratory frame of the c.m.s. of the pion and deuteron and of the pion and nucleon[4)].

In addition to the summed diagrams of the series in Fig. 1, contributions to the scattering amplitude should be made also by diagrams containing the rescattering of nucleons in the intermediate state. However, as shown earlier[4], this contribution is small. First, any diagram with nucleon rescattering, for example the diagram of Fig. 2, contains the factor $(\mu/m)^{1/2}$. Second, such a diagram has smallness of order $(b_0/b_1)^2$ in comparison with the double-scattering diagram (Fig. 1b). The point is that after the nucleon rescattering the pion can be scattered both by the deuteron and by the proton.

[2)]The invariant amplitudes M and A are connected with the S-matrix elements of the corresponding processes by relations of the type $S_{if} = \delta_{if} + i(2\pi)^4 M_{if}\delta^4(\tilde{P}_i - \tilde{P}_f)$, where $\tilde{P}_i$ and $\tilde{P}_f$ are the 4-momenta of the initial and final states.

[3)]The amplitudes f are defined in such a way that $|f|^2 = d\sigma/d\Omega$ in the c.m.s. of the corresponding reaction.

[4)]The formulas for scattering through a nonzero angle differ from (11) and (12) only in the presence of a factor exp $\{i\Delta r/z\}$ in the integrand, where Δ is the momentum transfer.

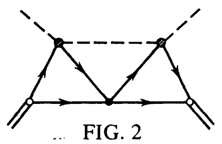

FIG. 2

This results in a sum of diagrams that gives the value of b_0. The same can be said also concerning the first scattering of the pion. All taken together make the contribution of the diagrams with nucleon rescattering not larger than 2–3% of the contribution of the diagrams of Fig. 1.

Let us return to formulas (11) and (12). We note that the integral contained in them converges, in principle, even with a wave function of zero effective radius of the forces, even though each of the terms of the series (see (7) and (8)) diverges in this case. Incidentally, examining (7) and (8) we can easily verify that the parameter of the expansion in the scattering multiplicities is not the quantity $b_{\pi N}/R$. The convergence of formulas (11) and (12) at zero is ensured by the term $\tilde{f}_1\tilde{f}_2/r^2$ in the denominator. This term, however, becomes significant only at very small distances of the order of $f_{\pi N} = (0.1-0.2)$ F. The convergence of the integrals is therefore actually ensured by the wave function of the deuteron at much larger distances $\sim 1/\mu$. For the same reason, the contributions of the terms of the triple, quadruple, and higher order scattering is small. It is sensitive to the behavior of the wave function at small distances, but does not exceed 8–10%.[5)]

The vertices of the diagrams of Fig. 1 can contain not only elastic scattering, but also charge exchange. This is easiest to take into account by recalling that the πN-scattering amplitude is an operator in isospin space (1). Accordingly, it is necessary to regard as an operator in each expression (6)–(8) the quantity

$$f_{\pi d}^{(1)}=\frac{1}{1+\mu/m_d}\int \psi_d^2(\mathbf{r})\hat{P}_1\,d\mathbf{r},\ f_{\pi d}^{(2)}=\frac{1}{1+\mu/m_d}\int \psi_d^2(\mathbf{r})\frac{e^{ikr-\varkappa r}}{r}\hat{P}_2\,d\mathbf{r} \quad (13)$$

etc., where

$$\hat{P_1}=\tilde{b}_0+\tilde{b}_1\mathbf{I}\boldsymbol{\tau}_{N_1}+\tilde{b}_0+\tilde{b}_1\mathbf{I}\boldsymbol{\tau}_{N_2}, \quad (14a)$$

$$\hat{P_2}=(\tilde{b}_0+\tilde{b}_1\mathbf{I}\boldsymbol{\tau}_{N_1})(\tilde{b}_0+\tilde{b}_1\mathbf{I}\boldsymbol{\tau}_{N_2})+(\tilde{b}_0+\tilde{b}_1\mathbf{I}\boldsymbol{\tau}_{N_2})(\tilde{b}_0+\tilde{b}_1\mathbf{I}\boldsymbol{\tau}_{N_1}) \quad (14b)$$

etc. Here

$$\tilde{b}_0=(m+\mu)b_0/m,\qquad \tilde{b}_1=(m+\mu)b_1/m. \quad (15)$$

Recognizing that the deuteron isospin is equal to zero, each of the terms of the series (10) can be easily averaged over the charge states of the pion[6)]:

$$\hat{P}_1=2\tilde{b}_0,\quad \hat{P}_2=2(\tilde{b}_0^2-2\tilde{b}_1^2),\quad \hat{P}_3=2\tilde{b}_0(\tilde{b}_0^2-2\tilde{b}_1^2)-4\tilde{b}_1^3,\quad \hat{P}_4=2(\tilde{b}_0^2-2\tilde{b}_1^2)^2-4\tilde{b}_1^4 \quad (16)$$

etc.

We note that in diagram language the aforementioned averaging over the pion charge states, say in double scattering, corresponds to taking the sum of a diagram with elastic πp and πn scattering with a diagram where the order of the scattering is interchanged (i.e., first πn scattering and then πp scattering); this is followed by subtracting the diagram with charge exchange. The minus sign of the last diagram is due to the fact that the proton and neutron exchange places in one of the deuteron vertices, as a result of which this vertex reverses sign[12].

The complete formula with allowance for the virtual charge exchange in rescatterings of all multiplicities is given in the Appendix.

3. ALLOWANCE FOR TERMS CORRESPONDING TO THE NUCLEON KINETIC ENERGY IN THE INTERMEDIATE STATE

As already noted, by discarding the diagrams with nucleon rescattering, we incur an error on the order of several per cent. It is therefore meaningful to calculate the series of diagrams of Fig. 1 likewise with an accuracy of several percent. The errors in (11) and (12) are much larger, for two approximations were used in their derivations: a) we discarded the terms of the type $\mu q^2/m$ in the pion propagators (these terms correspond to the kinetic energies of the nucleons in the intermediate states), b) we neglected the p-wave pion-nucleon interaction. We shall now get rid of the first assumption, and will take the p-wave part of the πN scattering into account in the next section. We make use of the fact that the main contribution to the πd-scattering amplitude is made by diagrams with single and double scattering. Principal attention is therefore paid precisely to these diagrams.

Let k = 0. Introducing the symbol

$$\varphi(\mathbf{q})=F(\mathbf{q})\,/\,(q^2+\alpha^2).$$

for the quantity proportional to the wave function of the deuteron in the momentum representation, we rewrite formula (5) for the diagram of Fig. 1b:

$$M^{(2)}=C\int\varphi(\mathbf{q}_1)\varphi(\mathbf{q}_2)\frac{d\mathbf{q}_1d\mathbf{q}_2}{(\mathbf{q}_1-\mathbf{q}_2)^2+\mu(q_1^2+q_2^2)/m+2\mu\varepsilon_d}, \quad (17)$$

where $C=\mu\alpha A_1A_2/4\pi^5$. At first glance it may appear that neglect of the term $\mu(q_1^2+q_2^2)/m$ leads to an error of the order of μ/m. Actually this error is larger and is more readily of order $(\mu/m)^{1/2}$ (this statement would be perfectly exact at $\epsilon_d = 0$). To verify this, we write (17) in the form

$$M^{(2)}=C\int\frac{\varphi(\mathbf{Q}+\mathbf{q}/2)\varphi(\mathbf{Q}-\mathbf{q}/2)\,d\mathbf{q}\,d\mathbf{Q}}{(1+\mu/2m)q^2+2\mu Q^2/m+2\mu\varepsilon_d}. \quad (18)$$

We have introduced here the new variables

$$\mathbf{q}=\mathbf{q}_1-\mathbf{q}_2,\quad \mathbf{Q}={}^1/_2(\mathbf{q}_1+\mathbf{q}_2). \quad (19)$$

We regard μ/m as a parameter, which we denote by ξ, and calculate $\partial M^{(2)}/\partial\xi|_{\xi=0}$:

$$\left.\frac{\partial M^{(2)}}{\partial\xi}\right|_{\xi=0}=2C\int\varphi\left(\mathbf{Q}+\frac{\mathbf{q}}{2}\right)\varphi\left(\mathbf{Q}-\frac{\mathbf{q}}{2}\right)\frac{Q^2d\mathbf{q}\,d\mathbf{Q}}{(q^2+2\mu\varepsilon_d)^2}+\frac{C}{2}\int\varphi\left(\mathbf{Q}+\frac{\mathbf{q}}{2}\right)\varphi\left(\mathbf{Q}-\frac{\mathbf{q}}{2}\right)\frac{q^2d\mathbf{q}\,d\mathbf{Q}}{(q^2+2\mu\varepsilon_d)^2}. \quad (20)$$

The first integral is determined by small $q \sim (2\mu\epsilon_d)^{1/2}$. In the arguments of the functions φ we can therefore put approximately q = 0. In the second integral it is useful to integrate first with respect to d**Q**, introducing the quantity

$$S(\mathbf{q})=\int\varphi\left(\mathbf{Q}+\frac{\mathbf{q}}{2}\right)\varphi\left(\mathbf{Q}-\frac{\mathbf{q}}{2}\right)d\mathbf{Q}\sim\int\psi_d^2(\mathbf{r})\,e^{i\mathbf{q}\mathbf{r}}\,d\mathbf{r}.$$

[5)]If we discard $f_1f_2e^{-2\pi r}/r^2$ in the denominator of (11), then we are left with only single and double scattering. Thus, only a small region of spatial integration, with dimensions on the order of the pion-nucleon scattering length, contributes to the multiple scattering.

[6)]We note that the connection between P_{n+2} and P_n, obtained by Moyer and Koltun [7], is not quite correct. Formulas (38)–(40) of [4] are consequently likewise not quite correct. This does not affect, however, the numerical calculations, because they account correctly for the principal terms (single and double scattering).

Then

$$\left.\frac{\partial M^{(2)}}{\partial \xi}\right|_{\xi=0} \approx \frac{16\pi^3 C}{(2\mu\varepsilon_d)^{1/2}} \int_0^\infty \varphi^2(Q) Q^4 dQ + 2\pi C \int_0^\infty \frac{S(q) q^4 dq}{(q^2 + 2\mu\varepsilon_d)^2}. \quad (21)$$

The first term is a large quantity because of the smallness of $(2\mu\epsilon_d)^{1/2}$ and becomes infinite as $\epsilon_d \to 0$. We see therefore that the correction can in any case not be linear in ξ.

The structure of the correction that appears when the term $\mu(q_1^2 + q_2^2)/m$ in (17) is taken into account is easiest to see by using the deuteron wave function in Gaussian form

$$\psi_d(r) = (\pi\gamma)^{-3/4} \exp(-r^2/2\gamma), \quad (22a)$$

where $\gamma = 1/(86\ \mathrm{MeV/c})^2$[13]. Accordingly,

$$\varphi(p) \sim \exp(-\gamma p^2/2). \quad (22b)$$

With $\varphi(p)$ in this form, we can integrate with respect to q in (18):

$$M^{(2)} \sim \frac{1}{(1+\mu/2m)\gamma^{1/2}} \int_0^\infty \exp(-\gamma Q^2) \times Q^2 \left\{1 - \frac{(\pi\gamma)^{1/2} B}{2} \exp\left(\frac{\gamma B^2}{4}\right)\left[1 - \Phi\left(\frac{\gamma^{1/2} B}{2}\right)\right]\right\} dQ. \quad (23)$$

Here

$$B = (2\mu Q^2/m + 2\mu\varepsilon_d)^{1/2}(1 + \mu/m)^{-1/2}, \quad (24)$$

$\Phi(x)$ is the probability integral. The correction is given here in practice by the second term proportional to B under the integral sign. We see that at small ϵ_d it is actually proportional to $(\mu/m)^{1/2}$ together with B.

Thus, the correction due to the terms of the type $\mu q^2/m$, which take into account the motion of the nucleons in the intermediate state, should be quite large. To calculate it exactly we have performed a numerical calculation in accordance with formula (18) with two deuteron wave functions: Gaussian (22) and Hulthen:

$$\psi_d(r) = \left[\frac{\alpha\beta(\alpha+\beta)}{2\pi(\beta-\alpha)^2}\right]^{1/2} \frac{e^{-\alpha r} - e^{-\beta r}}{r}, \quad (25a)$$

$$\varphi(p) \sim (p^2 + \alpha^2)^{-1}(p^2 + \beta^2)^{-1} \quad (25b)$$

with parameter $\beta = 240$ MeV/c. We note that $\langle r^{-1}\rangle \approx 0.5\ \mathrm{F}^{-1}$ for both wave functions. We retain the symbol $M^{(2)}$ for the exact value of the integral (17) and take $\widetilde{M}^{(2)}$ to mean the same integral, but calculated without the term $\mu(q_1^2 + q_2^2)/m$, i.e., actually the expression (6), and $\widetilde{M}^{(2)}_{\mathrm{Br}}$ to mean the Brueckner value of the double-scattering amplitude, i.e., the quantity obtained from (17) by neglecting both the term $\mu(q_1^2 + q_2^2)/m$ and the term $2\mu\epsilon_d$, which is proportional to the deuteron binding energy. We introduce the quantities

$$\alpha_1 = \frac{M^{(2)}_{\mathrm{Br}} - M^{(2)}}{M^{(2)}}, \quad \alpha_2 = \frac{\widetilde{M}^2 - M^{(2)}}{M^{(2)}}, \quad \alpha_3 = \frac{M^{(2)}_{\mathrm{Br}} - \widetilde{M}^{(2)}}{\widetilde{M}^{(2)}}. \quad (26)$$

The results are as follows: with the Hulthen wave function

$$\alpha_1 = 0.71, \quad \alpha_2 = 0.37, \quad \alpha_3 = 0.25; \quad (27)$$

with the Gaussian wave function

$$\alpha_1 = 0.82, \quad \alpha_2 = 0.42, \quad \alpha_3 = 0.29. \quad (28)$$

Thus, the result of formula (6) is approximately 40% higher. This is not a correction to the entire amplitude, only to the double scattering, but in a certain sense this is precisely the quantity of interest. The single-scattering term (impulse approximation) is always present and is very easy to calculate. It is therefore of interest to compare the calculation with the difference between the experimental πd-scattering length and the length calculated in the single-scattering approximation. We recall that the main emphasis in Brueckner's was precisely on the calculation of this difference.

The term of the form $\mu(q_1^2 + q_2^2)/m$ in (17) can be calculated approximately by replacing q_1 and q_2 by a certain constant value q_{eff}. To obtain the correct result with the Hulthen wave function, q_{eff} must be equal to 110 MeV/c. For a Gaussian function this quantity is somewhat smaller. When q_{eff} is used, all the terms of the series of Fig. 1 are integrated in the same manner as in Sec. 2, and we obtain formula (10)—(12), in which, however, ϵ_d is replaced by $\epsilon_d + q_{\mathrm{eff}}^2/m$. It is just in this manner that we shall estimate henceforth the influence of terms corresponding to the kinetic energy of the nucleons in the intermediate state on the expressions for triple, quadruple scattering, etc.

We note that the terms under discussion become more significant when $k \neq 0$. Numerical calculation shows, for example, that at k = 25 MeV/c we have for the Hulthen function

$$\alpha_1 = 0.50 + 0.36\,i, \quad \alpha_2 = 0.59, \quad (29)$$

and for the Gaussian function

$$\alpha_1 = 0.64 + 0.42\,i, \quad \alpha_2 = 0.70 \quad (30)$$

(Brueckner's result becomes already complex in this case).

In earlier studies of the πd-scattering length (see, e.g.,[7,8]), the term with double scattering was actually expressed in the "Brueckner" form, i.e., both the motion of the nucleons in the deuteron and the deuteron binding energy were neglected in its calculation. As seen from the values of the ratio α_1 in (27) and (28), the absolute value of such an expression is 50—70% higher than the correct result.

4. ALLOWANCE FOR p-WAVE πN SCATTERING

We proceed to take into account the p-wave part of the πN interaction in the diagrams of Figs. 1a and 1b. Even at zero incident-pion momentum, the pion momentum relative to the deuteron nucleons differs from zero, owing to the intranuclear motion of the latter. This leads, as will be shown below, to noticeable but not very large corrections (on the order of 20%). In the calculation of such corrections we shall therefore neglect the terms $\sim \mu/m$ in a number of cases. The πN-scattering amplitude at low energies is given by[1]

$$f_{\pi N} = b_0 + b_1 \mathbf{I}\boldsymbol{\tau} + (c_0 + c_1 \mathbf{I}\boldsymbol{\tau})\mathbf{k}\mathbf{k}' + i(d_0 + d_1 \mathbf{I}\boldsymbol{\tau})\boldsymbol{\sigma}[\mathbf{k}\mathbf{k}'], \quad (31)^*$$

where $\mathbf{k}$ and $\mathbf{k}'$ are the pion momenta in the c.m.s. before and after scattering. Expressions for b_0 and b_1 were given in the Introduction,

$$c_0 = (0.208 \pm 0.008)\mu^{-3}, \quad c_1 = (0.180 \pm 0.005)\mu^{-3},$$
$$d_0 = (-0.193 \pm 0.005)\mu^{-3}, \quad d_1 = (-0.060 \pm 0.004)\mu^{-3}.$$

We consider the diagram 1a. In forward scattering, the third term of (31), which contains the factor $\boldsymbol{\sigma}\cdot\mathbf{k} \times \mathbf{k}'$, makes not contribution (there is no preferred pseudovector along which the mean value of the opera-

*$[\mathbf{k}\mathbf{k}'] \equiv \mathbf{k} \times \mathbf{k}'$.

tor σ can be directed). If the incident pion has a momentum **k** in the lab, then in the meson + nucleon c.m.s. the momentum is

$$\tilde{\mathbf{k}} = \frac{m}{m+\mu}\mathbf{k} - \frac{\mu}{m+\mu}\mathbf{q}_1. \tag{32}$$

The s-wave term

$$M_s^{(1)} = 2b_0N^2\int|\varphi(\mathbf{q}_1)|^2 d\mathbf{q}_1 = 2b_0 \tag{33}$$

(N is a normalization factor) should be compared with the p-wave term

$$M_p^{(1)} = 2c_0N^2\int|\varphi(\mathbf{q}_1)|^2\left(\frac{m}{m+\mu}\mathbf{k} - \frac{\mu}{m+\mu}\mathbf{q}_1\right)^2 d\mathbf{q}_1 =$$
$$= 2c_0N^2\left\{k^2\left(\frac{m}{m+\mu}\right)^2\int|\varphi(\mathbf{q}_1)|^2 d\mathbf{q}_1 + \left(\frac{\mu}{m+\mu}\right)^2\int|\varphi(\mathbf{q}_1)|^2 q_1^2 d\mathbf{q}\right\} =$$
$$= 2c_0\left\{k^2\left(\frac{m}{m+\mu}\right)^2 + \left(\frac{\mu}{m+\mu}\right)^2\overline{q^2}\right\}. \tag{34}$$

At k = 0 the ratio of the p-wave term to the s-wave term is

$$\frac{M_p^{(1)}}{M_s^{(1)}} = \frac{c_0}{b_0}\left(\frac{\mu}{m+\mu}\right)^2\overline{q^2} = -0{,}17. \tag{35}$$

We took into account here the fact that $\overline{q^2} = (105\ \text{MeV/c})^2$ for both the Hulthen and the Gaussian wave function. If $k \neq 0$, the term with k^2 in (34) makes a contribution. Its ratio to $M_s^{(1)}$ is

$$\frac{c_0}{b_0}\left(\frac{m}{m+\mu}\right)^2 k^2.$$

Its value is −0.27 and −0.62 at k = 20 and 30 MeV/c.

We proceed to the diagram 1b. If we recognize that the pion momentum in the intermediate state is equal to $\mathbf{k} + \mathbf{q}_1 - \mathbf{q}_2$, we can readily see that instead of $\mathbf{k}\cdot\mathbf{k}'$ we should write in (31) for the left-hand upper vertex

$$(\mathbf{kk}')_1 = \left(\frac{m}{m+\mu}\mathbf{k} - \frac{\mu}{m+\mu}\mathbf{q}_1\right)\left(\mathbf{k} + \mathbf{q}_1 - \mathbf{q}_2 - \frac{\mu}{m+\mu}(\mathbf{k}+\mathbf{q}_1)\right),$$

and for the right-hand lower vertex

$$(\mathbf{kk}')_2 = \left(\frac{m}{m+\mu}\mathbf{k} + \frac{\mu}{m+\mu}\mathbf{q}_2\right)\left(\mathbf{k} + \mathbf{q}_1 - \mathbf{q}_2 - \frac{\mu}{m+\mu}(\mathbf{k}-\mathbf{q}_2)\right).$$

At k = 0 we have, accurate to terms of order $(\mu/m)^2$,

$$(\mathbf{kk}')_1 = -\frac{\mu}{m+\mu}\mathbf{q}_1(\mathbf{q}_1-\mathbf{q}_2),\quad (\mathbf{kk}')_2 = \frac{\mu}{m+\mu}\mathbf{q}_2(\mathbf{q}_1-\mathbf{q}_2). \tag{36}$$

These quantities enter in the form of the sum

$$(\mathbf{kk}')_1 + (\mathbf{kk}')_2 = -\frac{\mu}{m+\mu}(\mathbf{q}_1-\mathbf{q}_2)^2. \tag{37}$$

If we use the Gaussian wave function of the deuteron, we can easily see that allowance for the p-wave scattering in each vertex introduces a smallness of order $(\overline{q^2}/\mu^2)\mu/m$. We need therefore take into account only combinations with s-wave scattering in one of the vertices and p-wave scattering in the other. (We denote the corresponding expression by $M_{sp}^{(2)}$.) For the same reason, we can disregard the term with the coefficients d_0 and d_1 in (31). Indeed, because of the absence of a preferred pseudovector direction, this term cannot interfere with the s-wave terms in the other vertex. If we write down expressions of the type (31) for both vertices of the diagram 1b and add a diagram with the nucleons in reverse order, we find that the product of the vertices yields

$$\{(b_0 + b_1\mathbf{I}\boldsymbol{\tau}_1)(c_0 + c_1\mathbf{I}\boldsymbol{\tau}_2) + (b_0 + b_1\mathbf{I}\boldsymbol{\tau}_2)(c_0 + c_1\mathbf{I}\boldsymbol{\tau}_1)\}$$
$$\times[(\mathbf{kk}')_1 + (\mathbf{kk}')_2].$$

After averaging over the charge states of the pion, we obtain, with allowance for (37)

$$2(b_0c_0 - 2b_1c_1)(-\mu\overline{q^2}/m) \tag{38}$$

as against $2(b_0^2 - 2b_1^2)$ for the case when only the s-wave πN interaction is taken into account (such an amplitude is designated $M_{ss}^{(2)}$). Thus, an expression of type (18) for $M_{ss}^{(2)}$ with $C = 2(b_0^2 - 2b_1^2)$ should be compared with the following formula for $M_{sp}^{(2)}$:

$$-2(b_0c_0 - 2b_1c_1)\frac{\mu}{m+\mu}\int\varphi\left(\mathbf{Q}+\frac{\mathbf{q}}{2}\right)\varphi\left(\mathbf{Q}-\frac{\mathbf{q}}{2}\right)$$
$$\times\frac{q^2 d\mathbf{q}\, d\mathbf{Q}}{q^2 + \mu(2Q^2 + q^2/2)/m + 2\mu\varepsilon_d}. \tag{39}$$

If we use a Gaussian wave function, their ratio is

$$\frac{M_{sp}^{(2)}}{M_{ss}^{(2)}} = -\frac{\mu}{m+\mu}\frac{b_0c_0 - 2b_1c_1}{b_0^2 - 2b_1^2}$$
$$\times\int\frac{\exp(-\gamma Q^2 - \gamma q^2/4)\, q^4Q^2\, dQ\, dq}{q^2 + \mu(2Q^2 + q^2/2)/m + 2\mu\varepsilon_d}$$
$$\times\left[\int\frac{\exp(-\gamma Q^2 - \gamma q^2/4)\, q^2Q^2\, dq\, dQ}{q^2 + \mu(2Q^2 + q^2/2)/m + 2\mu\varepsilon_d}\right]^{-1} = -0{,}29. \tag{40}$$

We see that allowance for the p-wave part of the pion-nucleon interaction has a strong influence on the value of the double-scattering amplitude. We note that whereas the p-wave πN scattering was taken into account by Moyer and Koltun[7] for diagrams of the type 1a, nothing was done with respect to the diagram 1b in the preceding investigations. When p-wave terms are taken into account in both vertices of diagram 1b, we obtain in place of (38)

$$2\left(\frac{\mu}{m+\mu}\right)^2\left\{\frac{1}{3}(d_0^2 - 2d_1^2)[q^2Q^2 - (\mathbf{qQ})^2] - (c_0^2 - 2c_1^2)\left[(\mathbf{qQ})^2 - \frac{q^4}{4}\right]\right\} \tag{41}$$

The corresponding correction to $M_{ss}^{(2)}$ amounts to only a fraction of one per cent.

We have neglected in the calculations the D-wave part of the deuteron function. It can alter the result very little. At k = 0, as can be easily seen, the S- and D-wave parts of the deuteron wave function can not interfere with each other, and the integral contains simply the sum $\psi_S^2(r) + \psi_D^2(r)$. Thus, the result is altered only because the radial dependence is changed in a small part (6—7%) of the integrand. This is certainly a small effect.

5. ESTIMATE OF NUMERICAL VALUE OF πd-SCATTERING LENGTH

We have thus examined the series of diagrams of Fig. 1 and estimated the following two corrections to the Brueckner formula[2]; the correction due to allowance for the kinetic energy of the nucleons in the intermediate state (the recoil of the nucleons in the virtual-scattering process) in double scattering, and the correction for the p-wave πN-interaction in single and double scattering. We now estimate the numerical value of the πd-scattering length. Although the proposed method has an accuracy of several per cent, the real accuracy is low because of the uncertainty in the information on the s-wave lengths of the πN scattering. We have used the set of lengths from[1] and the Hulthen wave function of the deuteron. (In the estimate of (4) we used the Gaussian function.)

At k = 0, calculation in accord with formula (11), with allowance for (16), yields $f_{\pi d} = -0.070$ F, and the single- and double-scattering contributions are -0.036 and $+0.003$ F, respectively. In the case of single scattering, we then took into account the correction (35) for the p-wave, and in the case of double scattering we took into account the correction for terms of the type $\mu(q_1^2 + q_2^2)/m$ [Eq. (27)] and for the p-wave [Eq. (36)]. In the multiple scatterings, the correction for nucleon motion was taken approximately into account by introducing q_{eff} (see Sec. 3). The result is the following value for the πd-scattering length:

$$f_{\pi d} = -0.047 \quad \text{F}. \tag{42}$$

Practically the same result is obtained with the Gaussian wave function. If we take the values of the s-wave πN-scattering lengths $b_0 = -0.008\ \mu^{-1}$ and $b_1 = -0.0953\ \mu^{-1}$, which were used in[7], we obtain $f_{\pi d} \approx -0.037$ F.

The value (42) was obtained by taking into account only diagrams that do not contain pion capture processes. From the experimental data on the reaction $\pi^+ + d \to 2p$[14] it is known that when absorptive diagrams are taken into account the πd scattering acquires an imaginary part Im $f_{\pi d} = 0.006$ F. The contribution of these diagrams to Re $f_{\pi d}$ is apparently of the same order[8,15]. (See, however, the paper by Beder[16], where a much larger value is given for Re f_{abs}. The validity of the use in that reference of dispersion relations without subtractions is, however, subject to question.)

In conclusion, we wish to point out that exact measurement of the πN- and πd-scattering lengths is at present an important experimental problem.

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APPENDIX

We generalize formula (11) by taking into account the possibility of virtual charge exchange in rescatterings of all multiplicities. For concreteness, we shall consider scattering of π^+ mesons by deuterons. It is convenient to introduce a system of basis functions corresponding to the isotriplet and isosinglet states of two nucleons:

$$\psi_1 = \omega^1\chi_0^0, \quad \psi_2 = 2^{-1/2}(\omega^1\chi_1^0 - \omega^0\chi_1^1) \tag{A.1}$$

(ω^{-1}, ω^0, and ω^1 are different charge states of the pions, χ_0^0 is the isosinglet function and χ_1^{-1}, χ_1^0, and χ_1^1 the isotriplet functions of the system of two nucleons).

In the space of the functions ψ_1 and ψ_2, the operators $\tilde{b}_0 + \tilde{b}_1 \mathbf{t}\cdot\boldsymbol{\tau}_{N_1}$ and $\tilde{b}_0 + \tilde{b}_1 \mathbf{t}\cdot\boldsymbol{\tau}_{N_2}$ are two-row matrices

$$\tilde{b}_0 + \tilde{b}_1\mathbf{t}\boldsymbol{\tau}_{N_1} = \begin{pmatrix} \tilde{b}_0 & 2^{1/2}\tilde{b}_1 \\ 2^{1/2}\tilde{b}_1 & \tilde{b}_0 - \tilde{b}_1 \end{pmatrix} \cdot \tilde{b}_0 + \tilde{b}_1\mathbf{t}\boldsymbol{\tau}_{N_2} = \begin{pmatrix} \tilde{b}_0 & -2^{1/2}\tilde{b}_1 \\ -2^{1/2}\tilde{b}_1 & \tilde{b}_0 - \tilde{b}_1 \end{pmatrix} \tag{A.2}$$

We denote by S_1 the sum of terms in the integrand of a formula similar to (10), in which scattering begins with the first particle. It is easy to see that

$$S_1 = [(\tilde{b}_0 + \tilde{b}_1\mathbf{t}\boldsymbol{\tau}_{N_1}) + (\tilde{b}_0 + \tilde{b}_1\mathbf{t}\boldsymbol{\tau}_{N_2})(\tilde{b}_0 + \tilde{b}_1\mathbf{t}\boldsymbol{\tau}_{N_1}) r^{-1} e^{i\mathbf{kr} - \varkappa r}]\Sigma_1, \tag{A.3}$$

$$\Sigma_1 = 1 + r^{-2}e^{-2\varkappa r}(\tilde{b}_0 + \tilde{b}_1\mathbf{t}\boldsymbol{\tau}_{N_2})(\tilde{b}_0 + \tilde{b}_1\mathbf{t}\boldsymbol{\tau}_{N_1}) + r^{-4}e^{-4\varkappa r}[(\tilde{b}_0 + \tilde{b}_1\mathbf{t}\boldsymbol{\tau}_{N_2})(\tilde{b}_0 + \tilde{b}_1\mathbf{t}\boldsymbol{\tau}_{N_1})]^2 + \ldots \tag{A.4}$$

Σ_1 is easily obtained by solving the matrix equation

$$\Sigma_1 = 1 + r^{-2}e^{-2\varkappa r}(\tilde{b}_0 + \tilde{b}_1\mathbf{t}\boldsymbol{\tau}_{N_2})(\tilde{b}_0 + \tilde{b}_1\mathbf{t}\boldsymbol{\tau}_{N_1})\Sigma_1. \tag{A.5}$$

Introducing the notation

$$\rho = e^{-2\varkappa r}/r^2,$$
$$\delta = 1 - \rho(2\tilde{b}_0^2 - 2\tilde{b}_0\tilde{b}_1 - 3\tilde{b}_1^2) + \rho^2[(\tilde{b}_0^2 - 2\tilde{b}_1^2)(\tilde{b}_0^2 - 2\tilde{b}_0\tilde{b}_1 - \tilde{b}_1^2) + 2\tilde{b}_1^4], \tag{A.6}$$

we get

$$\Sigma_1 = \frac{1}{\delta}\begin{pmatrix} 1 - \rho(\tilde{b}_0^2 - 2\tilde{b}_0\tilde{b}_1 - \tilde{b}_1^2) & 2^{1/2}\rho\tilde{b}_1^2 \\ -2^{1/2}\rho\tilde{b}_1^2 & 1 - \rho(\tilde{b}_0^2 - 2\tilde{b}_1^2) \end{pmatrix} \tag{A.7}$$

Hence, using (A.3), we easily obtain S_1. If we consider the process of elastic π^+d scattering, then in fact we need only the matrix element $(S_1)_{11}$. We can find analogously S_2, which is the sum of those terms in which the scattering begins with the second particle, and we obtain

$$(S_2)_{11} = (S_1)_{11}.$$

We thus arrive at a relation that generalizes formula (11):

$$f_{\pi d} = \frac{1}{1 + \mu/m_d}\int d\mathbf{r}\psi_d^2(\mathbf{r})\frac{1}{\delta}\Big\{2\tilde{b}_0 - 2\frac{e^{-2\varkappa r}}{r^2}(\tilde{b}_0^3 - 2\tilde{b}_0^2\tilde{b}_1 - \tilde{b}_0\tilde{b}_1^2 + 2\tilde{b}_1^3) + 2\frac{e^{i\mathbf{kr} - \varkappa r}}{r}\Big[(\tilde{b}_0^2 - 2\tilde{b}_1^2) - \frac{e^{-2\varkappa r}}{r^2}((\tilde{b}_0^2 - 2\tilde{b}_1^2)(\tilde{b}_0^2 - 2\tilde{b}_0\tilde{b}_1 - \tilde{b}_1^2) + 2\tilde{b}_1^4)\Big]\Big\}. \tag{A.8}$$

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