ANOMALOUS CONDUCTIVITY OF INHOMOGENEOUS MEDIA IN A STRONG MAGNETIC FIELD

Yu. A. DREĬZIN and A. M. DYKHNE

Moscow Physico-Technical Institute

Submitted October 22, 1971

Zh. Eksp. Teor. Fiz. **63**, 242–260 (July, 1972)

Two problems concerning the conductivity of randomly inhomogeneous media in a strong magnetic field are studied in the case when the dimensions of the inhomogeneities are larger than the mean free path of the carriers. In the first problem the effect of inhomogeneity of the carrier concentration is investigated. It is shown that the transverse components of the effective conductivity tensor in a strong magnetic field H are proportional to $H^{-4/3}$; the inhomogeneities have no appreciable effect on the other components. The other problem concerns the magnetoresistance of polycrystals of metals with open Fermi surfaces (of the "space-mesh" and "goffered-cylinder" types). The asymptotic forms of the transverse magnetoresistance in strong magnetic fields are found. The existence of a size effect in the magnetoresistance of polycrystals is noted (this occurs for sample thicknesses that exceed not only the mean free path but also the dimensions of the crystallites).

1. INTRODUCTION

In a number of physical situations, the problem arises of calculating the effective conductivity of a randomly inhomogeneous medium, i.e., the conductivity connecting the volume averages of the current and field:

$$\langle \mathbf{j} \rangle = \hat{\sigma}^e \langle \mathbf{E} \rangle. \tag{1}$$

A more detailed formulation of this problem in the case when the dimensions of the inhomogeneities are greater than the mean free path will be described below. The "classical" objects of this problem are mixtures and polycrystals. As a rule, in these cases the inhomogeneities are not too great and, in order of magnitude, $\hat{\sigma}^e$ does not differ, e.g., from $\langle \hat{\sigma} \rangle$ (so that the problem of the theory is to determine $\hat{\sigma}^e$ more exactly than in order of magnitude).

In certain situations, however, the presence of inhomogeneities leads to a radical rearrangement of the current flow pattern, a characteristic feature being the presence of a large parameter to which this rearrangement is related. The class of problems of this type also includes the problems, considered in the present paper, of the calculation of the effective conductivity of an inhomogeneous medium in a strong magnetic field, when $\beta \equiv \omega_H/\nu \gg 1$ (ω_H is the Larmor frequency and ν is the collision frequency). One of these problems (described in Secs. 2 and 3) arises in the study of an isotropic medium with a nonuniform concentration of carriers; another (Secs. 4 and 5) arises in the study of the galvanomagnetic properties of polycrystals of metals with open Fermi surfaces. We note that the reasons for the anomalous conductivity (i.e., the sharp difference between $\hat{\sigma}^e$ and $\langle \sigma \rangle$) are completely different in the two cases and, to a considerable extent, these problems will be treated independently.

It is well known that in an isotropic medium, e.g., in a plasma, the conductivity tensor for $\beta \gg 1$ has the form (the z-axis is along **H**)

$$\hat{\sigma} = \sigma_0 \begin{pmatrix} \beta^{-2} & \beta^{-1} & 0 \\ -\beta^{-1} & \beta^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

The important point is that $\sigma_{xy} \gg \sigma_{xx}$ and $\sigma_{yy} \equiv \sigma_\perp$. Therefore, even for small inhomogeneities, when σ_0 and β fluctuate weakly ($\delta\sigma_0/\sigma_0 \ll 1$ and $\delta\beta/\beta \ll 1$), the fluctuation of σ_{xy} can be large compared with σ_{xx}. Perturbation theory is then inapplicable (the results of papers [1–4], in which the treatment of this problem was confined to the first perturbation-theory correction, are valid only if this correction is small). We note that, since for $\beta \gg 1$ the quantity $\sigma_{xy} = nec/H$, where n is the carrier concentration, e the carrier charge and c the speed of light, the fluctuations of σ_{xy} are induced by fluctuations of the carrier concentration.

The problem of the effective conductivity arises, e.g., in the study of the ionization instability in a low-temperature plasma.[5,6] Because of the development of this instability, the degree of ionization, and with it $\hat{\sigma}$ also, depends on the coordinates. This leads to a sharply inhomogeneous current flow (in their turn, the current inhomogeneities maintain the nonuniform ionization). The case of layer inhomogeneities was considered by Velikhov[7] and Rose.[8] In paper [9], an exact solution was obtained (under certain additional model assumptions) of the problem of the effective conductivity for the case of two-dimensional inhomogeneities lying along **H** (the assumption of two-dimensionality is justified, for example, by the fact that the thermal conduction and diffusion along the magnetic field smooth the inhomogeneities in this direction).

In the present paper, we examine the problem of the conductivity of a medium with three-dimensional inhomogeneities. As well as the application to a plasma, this problem is characteristic, e.g., for inhomogeneously alloyed semiconductors.[1,3,4]

The problem of the effective conductivity in a strong magnetic field also arises in the study of polycrystals, where the inhomogeneity is due to the random orientation of the crystallites. However, since for $\beta \gg 1$ the quantity $\sigma_{xy} = (n_e - n_h)ec/H$, where n_e and n_h are the concentrations of electrons and holes, σ_{xy} does not depend on the orientation of the crystallites with respect to the magnetic field. At the same time, in studying polycrystals, it is necessary to take into account specific features connected with the complicated dispersion law of the electrons. Namely, in the case of metals with an open Fermi surface, for certain orientations of the crystallites with respect to the magnetic field, the components σ_{xx} and σ_{yy} are appreciably greater than $\sigma_0\beta^{-2}$ (for most orientations, $\sigma_\perp \sim \sigma_0\beta^{-2}$). This is connected with the appearance for such orientations of open (or almost open) paths.[10]

Crystallites oriented in this way play an important role in the flow of current in strong magnetic fields, although the fraction C of such crystallites (for brevity, we shall call them the special crystallites) is usually small.

The problem of the magnetoresistance of polycrystals with allowance for open paths has been considered by a number of authors.[11–14] In the paper by Lifshits and Peschanskiĭ, this problem was solved for a thin sample (a wire) with one crystallite placed across a section. This makes it possible to average the resistance. In the paper by Stachowiak,[12] the problem was solved by a self-consistency method, the application of which, however, was in no way justified by the author. The same can be said of the paper by Ziman, in which, to determine $\hat{\sigma}^e$, a simple volume averaging of $\hat{\sigma}$ was performed. In the paper by Korzh,[14] an expansion was performed in the small concentration C of crystallites with open paths, only the first term of the expansion being taken into account. For the effective conductivity, he obtained the result $\sigma_\perp^e = \sigma_0\beta^{-2}(1 + \beta C)$. It was assumed that this result is also valid in the region $\beta C \gg 1$. It is not difficult to show, however, that the expansion parameter is effectively βC. Therefore, in the region $\beta C \gg 1$, it is necessary also to take into account the next terms. We shall be interested principally in the case of very strong magnetic fields, when $\beta C \gg 1$. In this case, the distribution of current is sharply nonuniform. In Sec. 4, we find, to within a numerical factor, the effective conductivity of a polycrystalline medium for different types of Fermi surface. We point out that we treated a particular case of this problem in a preliminary communication[15] by a method different from that applied in the present paper.

However, perhaps the greatest interest attaches to an unusual size effect, namely, the fact that the resistance of a polycrystalline sample displays a sharp dependence on its thickness in the direction of the magnetic field. The characteristic thicknesses at which the size effect appears exceed not only the mean free path, but also the dimensions of the crystallites. This effect is considered in Sec. 5.

Before proceeding to the main part of the paper, we shall formulate the problem of the effective conductivity more precisely.

We consider a stationary current flow through a conducting medium, at each point of which the conductivity tensor $\hat{\sigma}(\mathbf{r})$ is given ($\hat{\sigma}(\mathbf{r})$ is a random function of the coordinates); this tensor connects the local current density $\mathbf{j}(\mathbf{r})$ and the local electric field $\mathbf{E}(\mathbf{r})$

$$\mathbf{j}(\mathbf{r}) = \hat{\sigma}(\mathbf{r})\mathbf{E}(\mathbf{r}), \tag{3}$$

and j and E satisfy the equations

$$\operatorname{div}\mathbf{j} = 0, \quad \operatorname{rot}\mathbf{E} = 0. \tag{4}$$

Equations (3) and (4) can be combined into one equation for the potential:

$$\frac{\partial}{\partial x_i}\left(\sigma_{ik}\frac{\partial\varphi}{\partial x_k}\right) = 0. \tag{5}$$

Finding E and j from (3) and (4) and averaging them over the volume (or over an ensemble of the random functions $\hat{\sigma}(\mathbf{r})$), from (1) we find $\hat{\sigma}^e$.

We shall now discuss the restrictions under which our formulation of the problem of the effective conductivity is valid. If the conductivity depends only on the coordinates, then we must require that the mean free path l be much smaller than the characteristic dimensions a of the inhomogeneities in the conductivity. But if $\hat{\sigma}$ also depends on time (with a characteristic frequency ω), then to this we must add the conditions

$$\omega \ll \omega_0, \quad \omega \ll \sigma_0, \quad \omega \ll \frac{c^2}{4\pi\sigma_0 a^2}, \tag{6}$$

where ω_0 is the characteristic dispersion frequency of the conductivity and c is the speed of light. The first of these conditions (together with $l \ll a$) enables us to use Eq. (3), and the second and third enable us to use Eq. (4). We emphasize that the specific conduction mechanism is not important for us, if the conditions enumerated above are fulfilled: in particular, in the case of a plasma, the mechanism may be due to micro-turbulence. We note also that, although we shall everywhere speak of the conductivity, our treatment is also applicable to other analogous transport problems—thermal conduction and diffusion in an inhomogeneous medium, and so on.

2. FLUCTUATIONS OF THE HALL COMPONENTS

As we have already indicated, for small fluctuations of the components, only the fluctuations of σ_{xy} are important and these can be large compared with $\sigma_\perp$ (in (2) there is an even larger component, σ_{zz}, but its small fluctuations are unimportant). In the following, therefore, we shall assume that only σ_{xy} fluctuates. If σ_{ik} is a symmetric tensor, then Eq. (5) can also be regarded as the equation of stationary diffusion, with φ corresponding to the density of diffusing material and σ_{ik} corresponding to the diffusion tensor. The advantage of this interpretation consists in the fact that diffusion in an inhomogeneous medium can be represented in a highly visualizable way by "following" the diffusion (the "Brownian motion") of one particle of the material (from a formal point of view, we are concerned with the well-known method of representing the solution of Eq. (5) in the form of a path integral[16]). If, however, there is an antisymmetric part in $\hat{\sigma}$ (as is the case in a magnetic field), then this interpretation is not applicable directly. In this case, we proceed as follows: we separate in $\hat{\sigma}$ the symmetric and antisymmetric parts and write (5) in the form

$$\frac{\partial}{\partial x_i}\left(\sigma_{ik}^s\frac{\partial\varphi}{\partial x_k}\right) + \frac{\partial}{\partial x_i}\left(\sigma_{ik}^a\frac{\partial\varphi}{\partial x_k}\right) = 0. \tag{7}$$

Performing the differentiation in the second term and noting that $\sigma_{ik}^a\partial^2\varphi/\partial x_i\partial x_k = 0$ (as a contraction of an antisymmetric and a symmetric tensor), we can rewrite this in the form

$$\frac{\partial}{\partial x_i}\left(\sigma_{ik}^s\frac{\partial\varphi}{\partial x_k}\right) - v_k\frac{\partial\varphi}{\partial x_k} = 0, \tag{8}$$

where

$$v_k = -\partial\sigma_{ik}^a / \partial x_i. \tag{9}$$

We see that if the antisymmetric part does not depend on the coordinates, then $v_k \equiv 0$ and its presence is not manifested at all in the equation for the potential. But if $v \neq 0$, then, making use of the fact that

$$\operatorname{div} \mathbf{v} = \partial^2\sigma_{ik}^a / \partial x_i \partial x_k \doteq 0,$$

we can write (8) in the form

$$\frac{\partial}{\partial x_i}\left(\sigma_{ik}^s \frac{\partial\varphi}{\partial x_k}\right) - \frac{\partial}{\partial x_i}(v_i\varphi) = 0. \tag{10}$$

This equation has the form div $\mathbf{Q} = 0$, where the flux $Q_i = -\sigma_{ik}^s \partial\varphi/\partial x_k + v_i\varphi$, and it can now be interpreted in the language of diffusion. Namely, it describes stationary transport under the influence of diffusion (the tensor of the diffusion coefficients is σ_{ik}^s) and of convection (with velocity $\mathbf{v}$). It is precisely the presence of the convective transport which leads, as we shall see, to the result that the transverse components of the tensor $\hat{\sigma}^e$ will differ appreciably from the corresponding components of $\langle\hat{\sigma}\rangle$. The reason for this lies in the fact that, as follows from (9) and (2), the velocity is proportional to β^{-1}, whereas $\sigma_\perp \sim \beta^{-2}$. Therefore, transport across the magnetic field proceeds for $\beta \gg 1$ (and for not too weak inhomogeneity) as a result of the convection.

We shall trace a more formal connection between the problem of calculating $\hat{\sigma}^e$ and its diffusion analog. We write the average current density in the form

$$\langle j_i\rangle = \frac{1}{V}\int [-\sigma_{ik}^s + (\langle\sigma_{ik}^a\rangle - \sigma_{ik}^a)]\frac{\partial\varphi}{\partial x_k}dV - \frac{1}{V}\int \langle\sigma_{ik}^a\rangle\frac{\partial\varphi}{\partial x_k}dV. \tag{11}$$

In the first integral, we integrate the term $(\langle\sigma_{ik}^a\rangle - \sigma_{ik}^a)\partial\varphi/\partial x_k$ by parts; the surface integral is small compared with the volume integral, and therefore

$$\langle j_i\rangle = \frac{1}{V}\int \left(-\sigma_{ik}^s\frac{\partial\varphi}{\partial x_k} + v_i\varphi\right)dV + \langle\sigma_{ik}^a\rangle\langle E_k\rangle. \tag{12}$$

The second term here gives the Hall current, which changes sign on change of direction of the magnetic field. As regards the first term, it cannot be asserted, generally speaking, that it gives only a conduction current. If, however, σ fluctuates symmetrically about its average value, then the integral term in (12) does indeed give only the conduction current. In fact, $\mathbf{v}$ becomes $-\mathbf{v}$ not only when $\mathbf{H} \to -\mathbf{H}$, but also when $\delta\sigma_{ik}^a \to -\delta\sigma_{ik}^a$ ($\delta\sigma_{ik}^a \equiv \sigma_{ik}^a - \langle\sigma_{ik}^a\rangle$), and, therefore, this latter transformation, which, by assumption, does not change the properties of the medium, can be used to compensate a change of direction of the magnetic field. In the general case, we can show that, for small fluctuations, the correction to the Hall current due to the first term is small compared with $\langle\sigma_{ik}^a\rangle\langle E_k\rangle$, i.e., $\sigma_{ik}^{e,a} \approx \langle\sigma_{ik}^a\rangle$ (this can be shown by means of perturbation theory; see Sec. 3). We also make the following remark. The dimensions of $\hat{\sigma}^s$ and $\mathbf{v}$ do not correspond to the dimensions of the physical diffusion coefficient and velocity (the same also applies to the "time" t; see below). However, no attention need be given to this, since in the end we shall obtain quantities of the correct dimensions, as we should.

We turn to the calculation of $\sigma_\perp^e$. If in the right-hand side of Eq. (8) we replace the zero by $\partial\varphi/\partial t$, thereby going over from the study of stationary diffusion to the study of the situation in time, then the effective diffusion coefficient can be found by "following" the diffusional motion of a particle over long times. Indeed, the effective diffusion coefficient can be found from the well-known formula

$$\sigma_\perp^e = \lim_{t\to\infty}\frac{\langle \mathbf{r}_\perp^2(t)\rangle}{4t}, \tag{13}$$

where $\mathbf{r}_\perp(t)$ is the displacement of the particle across the magnetic field in time t, and the brackets denote averaging over all possible diffusion paths. We shall study the case of three-dimensional isotropic fluctuations. Let a be the characteristic dimensions (correlation length) of the fluctuations, and let $\Delta \equiv \langle(\delta\sigma_{xy})^2\rangle^{1/2}/\langle\sigma_{xy}\rangle$ give the relative magnitude of the fluctuations. Then from (9) and (2) we obtain an estimate for the velocity

$$v \sim \sigma_0\Delta/\beta a. \tag{14}$$

We note that, in the case under consideration, despite the fact that the fluctuations are three-dimensional, the velocity $\mathbf{v}$ has only the components v_x and v_y, since $\hat{\sigma}^a$ contains only σ_{xy} and σ_{yx}. Then the current lines of the velocity field coincide with the intersections of the planes σ_{xy} = const with planes perpendicular to the magnetic field.

Thus, we must study the following picture: the particle moves under the influence of sharply anisotropic diffusion (the diffusion coefficient along the magnetic field is σ_0 and across the magnetic field is $\sigma_0\beta^{-2}$) and of a random field of velocities, the magnitude of which is determined by formula (14). If there were no transverse diffusion, then an effective transverse diffusion would be established as a result of the velocity field. If this effective diffusion turns out to be greater than the "bare" transverse diffusion, we can, in fact, disregard the latter. As we shall see, precisely such a situation is realized for sufficiently large β; therefore, below, we shall not take the bare transverse diffusion into account.

It might appear that the result for the effective diffusion could be written, in order of magnitude, in the form of a product of the velocity with the correlation length. This would give $va \sim \sigma_0\Delta/\beta$. This estimate is derived from the fact that, until the particle has moved through the correlation length, its path has a regular character and becomes similar to a random walk when its length appreciably exceeds the correlation length. In the case we are studying, these arguments are, however, incorrect, since, because of the rapid diffusion along z, the particle moves rapidly outside the correlation cell without having had time to be displaced appreciably in the transverse direction. In order of magnitude, the distance it succeeds in being displaced in the time of passage through the correlation cell is $s \sim v\tau_0$, where $\tau_0 \sim a^2/\sigma_0$; thus, $s \sim va^2/\sigma_0 \sim a\Delta\beta^{-1} \ll a$. Therefore, va is an upper bound for the effective diffusion.

This discussion prompts the idea of estimating $\sigma_\perp^e$ as $s^2/\tau_0 \sim \sigma_0\Delta^2\beta^{-2}$, since the path (or, more precisely, its projection on the xy-plane) conserves its regular character along lengths $\sim$s, and then "becomes tangled." However, as we shall show now, this estimate is also incorrect (in fact, it is a lower bound). The point is that, so long as the transverse displacement $r_\perp(t)$ of the particle is less than a, the particle, even though it passes rapidly through the correlation cell in its diffusion along z (so that the resulting displacement is small), then repeatedly returns to this cell. In this case, its displacements within the same cell should be added together "coherently," since they are strongly correlated. Frequent returns to the same cell will be repeated until the transverse displacement of the particle exceeds the correlation length.

We shall study in more detail the first stage of the motion of the particle (while $r_\perp(t) \ll a$). We can then neglect the dependence of the velocity on the transverse coordinates (i.e., put $v = v(z)$). This makes it possible to calculate in explicit form the dependence of $\langle r_\perp^2 \rangle$ on time. In fact, for a given motion z(t), we have for $\mathbf{r}_\perp(t)$

$$\mathbf{r}_\perp(t) = \int_0^t \mathbf{v}(z(t_1))\,dt_1. \tag{15}$$

Squaring this equality and averaging, we obtain

$$\langle \mathbf{r}_\perp^2(t)\rangle = \int_0^t\int_0^t \langle \mathbf{v}(z(t_1))\mathbf{v}(z(t_2))\rangle dt_1 dt_2. \tag{16}$$

Here, the angular brackets denote averaging both over all possible diffusion paths z(t) and over an ensemble of random fields of v. Since $\langle \mathbf{v}(z_1)\mathbf{v}(z_2)\rangle$ depends only on $z_1 - z_2$, we have

$$\langle \mathbf{v}(z(t_1))\mathbf{v}(z(t_2))\rangle = \int_{-\infty} \langle \mathbf{v}(0)\mathbf{v}(z)\rangle \Phi dz, \Phi = \frac{\exp\{-z^2/2\sigma_0|t_1-t_2|\}}{\sqrt{2\pi\sigma_0|t_1-t_2|}} \tag{17}$$

($\Phi\,dz$ gives the fraction of paths from the point $z = 0$ arriving at the segment $(z, z+dz)$ in time $|t_1 - t_2|$). Substituting this result into (16), we obtain

$$\langle \mathbf{r}_\perp^2(t)\rangle = \int_0^t\int_0^t\int_{-\infty}^{\infty} \langle \mathbf{v}(0)\mathbf{v}(z)\rangle \Phi dt_1\, dt_2\, dz. \tag{18}$$

The correlator $\langle \mathbf{v}(0)\mathbf{v}(z)\rangle$ differs essentially from zero for $z \leqslant a$. We shall be interested in $\langle r_\perp^2(t)\rangle$ for $t \gg a^2/\sigma_0$. In the integral (18), as we shall now convince ourselves, $|t_1 - t_2| \gg a^2/\sigma_0$ are important, and in this region the exponential can be replaced by unity. We then obtain

$$\langle \mathbf{r}_\perp^2(t)\rangle \sim \int_{-\infty}^{\infty} \langle \mathbf{v}(0)\mathbf{v}(z)\rangle dz \int_0^t\int_0^t \frac{dt_1\,dt_2}{\sqrt{2\pi\sigma_0|t_1-t_2|}} \sim \frac{t^{3/2}}{\sqrt{\sigma_0}}\int_{-\infty}^{\infty} \langle \mathbf{v}(0)\mathbf{v}(z)\rangle dz. \tag{19}$$

The integral in the latter expression, generally speaking, is of order $v^2 a$, and therefore

$$\langle \mathbf{r}_\perp^2(t)\rangle \sim v^2 a t^{3/2}/\sqrt{\sigma_0}. \tag{20}$$

We have found that, if v depends only on z, the mean square transverse displacement of the particle increases with time more rapidly than for diffusion (for diffusion, we would have $\langle \mathbf{r}_\perp^2(t)\rangle \sim t$).

This result can be given a simple interpretation. For simplicity, let the velocities have only an x-component and be equal in absolute magnitude, so that the pattern of the velocities has the form depicted in Fig. 1, where the layers have equal thickness, so that the velocity certainly does not change over the extent of one layer, and, in neighboring layers, can be oriented in either the same or opposite directions (i.e., in each layer, one or the other direction of the velocity is selected independently, with probability $\frac{1}{2}$). If all the velocities were in the same direction, the particle would be displaced by

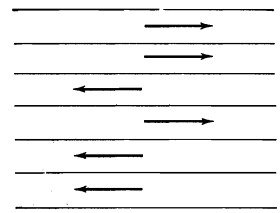


FIG. 1

a distance vt in time t. Since, however, the average velocity is equal to zero, the displacements in different layers are well compensated, and only a certain number δN of "uncompensated" layers contribute effectively. Therefore, the displacement of the particle is of order $vt\,\delta N/N$, where N is the total number of layers passed through in time t, $N \sim \sqrt{\sigma_0 t/a}$. Since the velocities in the layers are randomly distributed, we have $\delta N \sim N^{1/2}$ and $r_\perp(t) \sim vt/\sqrt{N} \sim t^{3/4} va\sigma_0^{-1/4}$, or, squaring, we obtain $r_\perp^2(t) \sim v^2 a t^{3/2}/\sqrt{\sigma_0}$, which coincides with (20). We note that, if the velocities in the layers alternated, then, as is easily seen, this would lead to $\langle r_\perp^2(t)\rangle \sim t$, since in this case $\delta N \sim 1$, independently of t.

We return to the question of the calculation of $\sigma_\perp^e$. The above treatment is valid so long as $r_\perp \leqslant a$, or, from (20), so long as $t \leqslant \tau$, where $\tau \sim \sigma_0^{1/3} a^{2/3}/v^{4/3}$. For longer times, the returns of the particle can be neglected (unlike one-dimensional diffusional motion, which is recurrent, three-dimensional diffusion is non-recurrent[16]). Therefore, the effective diffusion coefficient can be estimated as

$$\sigma_\perp^e \sim a^2/\tau \sim v^{4/3} a^{4/3}/\sigma_0^{1/3} \sim \sigma_0(\Delta/\beta)^{4/3}. \tag{21}$$

It remains to "recall" that, in fact, the calculated quantity is the effective conductivity. Comparing (21) with the bare transverse conductivity $\sigma_0\beta^{-2}$, we find that (21) is valid for $\beta\Delta^2 \gg 1$. In the opposite limiting case, the inhomogeneities do not lead to anomalous conductivity, and $\sigma_\perp^e \sim \sigma_0\beta^{-2}$.

The derivation given cannot, of course, pretend to be rigorous. We shall give an account here of certain considerations which reinforce the result obtained. We assumed in deriving (21) that the random function $\hat\sigma(\mathbf{r})$ can be crudely characterized by the magnitude and scale of the fluctuations. But if $\hat\sigma(\mathbf{r})$ has a more complicated structure, e.g., possesses an entire spectrum of scales, then it is not clear in advance whether the result obtained is conserved in this case. We shall show that the result is stable to such a modification. For simplicity, let there be two scales: a_1 and a_2, with $a_1 \ll a_2$. Then it is natural to proceed in the following way, namely, to find the effective conductivity $\hat\sigma^{e\prime}$ first with respect to the scales L, $a_1 \ll L \ll a_2$. The inhomogeneities of $\hat\sigma^{e\prime}$ will now have only the scale a_2. Then, treating $\hat\sigma^{e\prime}$ as the local conductivity tensor, we find $\hat\sigma^e$ (with respect to scales greater than a_2). For fluctuations of σ_{xy} that are not too small on the scales a_1, the diagonal components of $\hat\sigma^{e\prime}$ will be $\sim\beta^{-4/3}$, while the Hall components of $\hat\sigma^{e\prime}$ will be equal to the averages (over the scale L) of the Hall components of $\hat\sigma(\mathbf{r})$. The important point is, as can be seen from the derivation of (21), that the transverse diffusion is unimportant. Therefore, the effective diffusion due to the scales a_2 will also be

$\sim\beta^{-4/3}$. As a result, the convective diffusion will proceed as a result of coarse or fine scales (depending on where the magnitude of the inhomogeneities is greater), but in every case the asymptotic form for sufficiently large β will be $\sim\beta^{-4/3}$.

A further remark concerns the calculation of the characteristic time τ in which the particle is displaced over the correlation length in the transverse direction. We have found τ from the equality $\langle \mathbf{r}_\perp^2(\tau)\rangle \sim a^2$, where $\langle \mathbf{r}_\perp^2(t)\rangle$ is determined by formula (20). It is not clear in advance, however, whether the same order of magnitude for τ will be obtained if we determine it, e.g., from the equality $\langle \mathbf{r}_\perp^4(\tau)\rangle \sim a^4$. Calculations (for brevity, we shall not give them) analogous to those carried through in the derivation of (20) give a positive answer to this question (the same applies to the higher moments $\langle \mathbf{r}_\perp^{2n}(t)\rangle$ also).

The diffusion analogy can also be used in the case when the inhomogeneities are two-dimensional (extended along the magnetic field). A study of this case leads to the formula

$$\sigma_\perp^e \sim \sigma_0\Delta / \beta. \tag{22}$$

This result coincides with that obtained in [9] on the basis of an exact solution for a two-phase system with equal concentrations of the two phases.

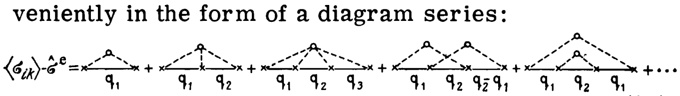
3. DIAGRAM TECHNIQUE IN THE EFFECTIVE CONDUCTIVITY PROBLEM

Since the treatment based on the diffusion analogy was of a largely qualitative nature, it is of interest to obtain a solution by using a perturbation-theory diagram technique. However, in the region of interest to us, the perturbation-theory series diverges. As we shall see, the so-called strong-coupling case is realized, in which diagrams of different topological structures are equally important. Nevertheless, it is possible to confirm (21) and (22) from an examination of the diagrammatic series.

The diagram technique for the effective conductivity problem is obtained if in Eq. (5) we represent the potential in the form $\varphi = -\langle E_i\rangle x_i + \tilde{\varphi}$ ($\langle\tilde{\varphi}\rangle = 0$) and σ_{ik} in the form $\sigma_{ik} = \langle\sigma_{ik}\rangle + \tilde{\sigma}_{ik}$. Then, assuming $\tilde{\sigma}_{ik}$ to be the perturbation and solving (6) for $\tilde{\varphi}$ by successive iterations in $\tilde{\sigma}_{ik}$, we obtain an expression for $\tilde{\varphi}$ in the form of a series. Substituting this series into the formula

$$\langle j_i\rangle = -\frac{1}{V}\int \sigma_{ik}\frac{\partial\varphi}{\partial x_k}\,\partial V$$

and averaging over an ensemble of random fields of σ, we find an expansion for $\hat{\sigma}^e$, which can be written conveniently in the form of a diagram series:



(23)

The correspondence rules here are as follows: a line with wave vector $\mathbf{q}$ corresponds to the factor

$$\frac{q_iq_k}{(\mathbf{q}\langle\hat{\sigma}\rangle\mathbf{q})} \equiv \frac{q_iq_k}{\sigma_0(q_z^2+\beta^{-2}q_\perp^2)}$$

and a cross between lines $\mathbf{q}_1$ and $\mathbf{q}_2$ corresponds to the factor $(-\tilde{\sigma}_{ik}^{\mathbf{q}_2-\mathbf{q}_1})$. The dashed lines linking several crosses and converging at the unshaded circles imply that the factors $\tilde{\sigma}_{ik}^{\mathbf{q}_2-\mathbf{q}_1}$ corresponding to these crosses appear in the irreducible correlator (cf., e.g., [17]). Integrations are performed over all the intermediate wave vectors, and contractions over repeated tensor indices. Thus, (23) has the following expanded form:

$$\begin{aligned}
\langle\sigma_{ik}\rangle - \sigma_{ik}^e &= \int \frac{\langle\tilde{\sigma}_{ip}^{\mathbf{q}_1}\tilde{\sigma}_{sk}^{-\mathbf{q}_1}\rangle_0 q_{1p}q_{1s}}{\sigma_0(q_{1z}^2+\beta^{-2}q_{1\perp}^2)}\,d^3q_1 \\
&- \int \langle\tilde{\sigma}_{ip}^{\mathbf{q}_1}\tilde{\sigma}_{mn}^{\mathbf{q}_2-\mathbf{q}_1}\tilde{\sigma}_{sk}^{-\mathbf{q}_2}\rangle_0 q_{1p}q_{1m}q_{2n}q_{2s} \\
&\times [\sigma_0(q_{1z}^2+\beta^{-2}q_{1\perp}^2)\sigma_0(q_{2z}^2+\beta^{-2}q_{2\perp}^2)]^{-1}\,d^3q_1\,d^3q_2 \\
&+ \int \langle\tilde{\sigma}_{ip}^{\mathbf{q}_1}\tilde{\sigma}_{mn}^{\mathbf{q}_2-\mathbf{q}_1}\tilde{\sigma}_{rq}^{\mathbf{q}_3-\mathbf{q}_2}\tilde{\sigma}_{sk}^{-\mathbf{q}_3}\rangle_0 q_{1p}q_{1m}q_{2n}q_{2r}q_{3q}q_{3s} \\
&\times [\sigma_0(q_{1z}^2+\beta^{-2}q_{1\perp}^2)\sigma_0(q_{2z}^2+\beta^{-2}q_{2\perp}^2)\sigma_0(q_{3z}^2+\beta^{-2}q_{3\perp}^2)]^{-1}\,d^3q_1\,d^3q_2\,d^3q_3 \\
&+ \int \langle\tilde{\sigma}_{ip}^{\mathbf{q}_1}\tilde{\sigma}_{rq}^{-\mathbf{q}_1}\rangle_0\langle\tilde{\sigma}_{mn}^{\mathbf{q}_2-\mathbf{q}_1}\tilde{\sigma}_{sk}^{\mathbf{q}_1-\mathbf{q}_2}\rangle_0 q_{1p}q_{1m}q_{2n}q_{2r}(q_{2q}-q_{1q})(q_{2s}-q_{1s}) \\
&\times[\sigma_0(q_{1z}^2+\beta^{-2}q_{1\perp}^2)\sigma_0(q_{2z}^2+\beta^{-2}q_{2\perp}^2)\sigma_0[(q_{2z}-q_{1z})^2 \\
&+\beta^{-2}(q_2-q_1)_\perp^2]]^{-1}\,d^3q_1d^3q_2 + \int\langle\tilde{\sigma}_{ip}^{\mathbf{q}_1}\tilde{\sigma}_{sk}^{-\mathbf{q}_1}\rangle_0\langle\tilde{\sigma}_{mn}^{\mathbf{q}_2-\mathbf{q}_1}\tilde{\sigma}_{rq}^{\mathbf{q}_1-\mathbf{q}_2}\rangle_0 \\
&q_{1p}q_{1m}q_{2n}q_{2r}q_{1q}q_{1s}[[\sigma_0(q_{1z}^2+\beta^{-2}q_{1\perp}^2)]^2\sigma_0(q_{2z}^2+\beta^{-2}q_{2\perp}^2)]^{-1}\,d^3q_1\,d^3q_2.
\end{aligned}$$

Here, the angular brackets with the zero subscript $\langle\ldots\rangle_0$ denote an irreducible correlator. We note that reducible diagrams do not appear in the expansion (23). In (23), in all the diagrams, the incoming wave vector is equal to zero. In the following, we shall need the quantity $\hat{\sigma}^e(\mathbf{q})$ ($\hat{\sigma}^e = \hat{\sigma}^e(0)$), which is introduced formally and defined by the expansion (23) with the incoming wave vector equal to $\mathbf{q}$ in the diagrams.

We proceed to select and estimate the diagrams. We shall consider the simplest diagram (the first in (23)). We shall be interested in $\sigma_\perp^e$ (e.g., σ_{xx}^e). The corresponding expression has the form

$$\langle\sigma_{xx}\rangle - \sigma_{xx}^e = \int\frac{\langle\tilde{\sigma}_{xi}^{\mathbf{q}}\tilde{\sigma}_{jx}^{-\mathbf{q}}\rangle_0 q_iq_j d^3q}{\sigma_0(q_z^2+\beta^{-2}q_\perp^2)}. \tag{24}$$

We shall study the terms corresponding to different i and j separately. We start from the fluctuations of $\tilde{\sigma}_{xy} \sim \sigma_0\Delta/\beta$ ($\Delta \ll 1$ is the relative magnitude of the fluctuations). We shall define the function $D(\mathbf{q})$ from the relation $\langle\tilde{\sigma}_{xy}^{\mathbf{q}}\tilde{\sigma}_{xy}^{-\mathbf{q}}\rangle_0 = \sigma_0^2\Delta^2\beta^{-2}D(\mathbf{q})$. If a is the correlation length of the fluctuations, then $D(\mathbf{q}) \approx a^3$ for $q \leqslant a^{-1}$ and $D(\mathbf{q}) \approx 0$ for $q \gg a^{-1}$. The integral (24) is equal to

$$I = -\frac{\sigma_0\Delta^2}{\beta^2}\int\frac{D(\mathbf{q})q_y^2d^3q}{(q_z^2+\beta^{-2}q_\perp^2)}. \tag{25}$$

For $\beta \gg 1$, the small $q_z \sim a^{-1}\beta^{-1}$ are important. Therefore, the integration over q_z can be performed explicitly, after which we obtain

$$I = -\frac{\sigma_0\Delta^2\pi}{\beta}\int D(q_\perp)\frac{q_y^2}{q_\perp}\,d^2q_\perp;$$

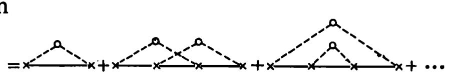
the remaining integral is of order unity, and, omitting numerical factors, we obtain $I \sim \sigma_0\Delta^2/\beta$. For $\beta\Delta^2 \gg 1$, this correction is greater than $\langle\sigma_{xx}\rangle$. Since the fluctuations of σ_{xx} are β times smaller than the fluctuations of σ_{xy}, for small fluctuations the correction due to $\tilde{\sigma}_{xx}$ is small. We note that, if $\tilde{\sigma}_{xz}$ and $\tilde{\sigma}_{yz}$ are non-zero (as, e.g., in metals, where in order of magnitude $\sigma_{xz} \sim \sigma_{yz} \sim \sigma_0\beta^{-1}$), then the correction to σ_{xx}^e due to the fluctuations of σ_{xz} and σ_{yz} is also small. Indeed, this correction differs from (25) in that q_z^2 appears together with q_y^2 in the numerator; then the integration over q_z is no longer singular and the correction is $\sim\beta^{-2}$. For the same reason, small fluctuations of σ_{zz} are unimportant.

It is not difficult to convince oneself that these conclusions are also valid for corrections of higher orders. In addition, by estimating the last two diagrams in (23) (they are both $\sim\sigma_0\Delta^4$), we convince ourselves that the intersection of the dashed lines has no effect on the estimation of the diagrams. We shall show that diagrams containing irreducible correlators of higher orders are small compared with diagrams containing pair correlators (e.g., the third term in (23) is smaller than both the following terms). In fact, the number of "singular" denominators in diagrams of both types is the same, whereas the number of integrations (over q_z) is greater in the diagrams with irreducible correlators of higher orders, and this leads to the appearance of extra factors β^{-1}.

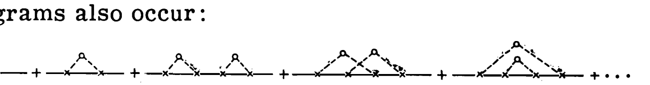
Thus, retaining in each order only the principal terms, arrive at an expansion in which $\tilde{\sigma}_{xy}$ (and $\tilde{\sigma}_{yx}$) corresponds to the crosses and all the correlators are pair correlators. For the following, it is convenient to redefine the factors associated with the crosses and lines. For this we multiply the expansion of $\langle\sigma_{ik}\rangle - \sigma^e_{ik}(q)$ by q_iq_k and assume that with each line is associated a factor $G_0(q) = 1/(q\langle\hat{\sigma}\rangle q)$, and with a cross between the lines q_1 and q_2, a factor $(q_{1x}q_{2y} - q_{1y}q_{2x})\tilde{\sigma}_{xy}^{q_2-q_1}$ (we are not changing the notation in the diagrams). Thus, for

$$\Pi(q) \equiv (q_i\langle\sigma_{ik}\rangle q_k) - (q_i\sigma^e_{ik}(q)q_k)$$

we obtain

$\Pi(q) =$  (26)

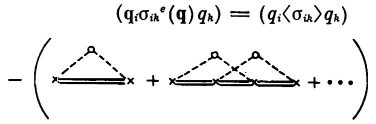
Further, we shall introduce the exact Green's function $G(q)$, defining it by means of a series in which reducible diagrams also occur:

$G(q) =$ 

G and Π are related by Dyson's equation $G^{-1}(q) = G_0^{-1}(q) - \Pi(q)$, whence, from the definitions of Π and G_0, we obtain $G^{-1}(q) = (q_i\sigma^e_{ik}(q)q_k)$.

We can now set up a closed equation for $\sigma^e(q)$ (even if with an infinite number of terms). For this, we postulate that in the series for Π only the compact diagrams, from which it is impossible to cut off part of the diagram by means of two cuts through G_0-lines, are retained. It is easy to understand that the discarded diagrams will be taken into account exactly if in the remaining diagrams we associate not G_0 but the exact Green's function G, with all solid lines (in the diagrams, G will be depicted by a double line). Thus, we obtain

$$(q_i\sigma^e_{ik}(q)q_k) = (q_i\langle\sigma_{ik}\rangle q_k)$$

 (27)

We shall first study an abbreviated equation, which we obtain by retaining only the first diagram in the whole series in the brackets. Written out explicitly, this equation has the form*

$$(q_i\sigma^e_{ik}(q)q_k) = (q_i\langle\sigma_{ik}\rangle q_k) + \int d^3q_1 \frac{[qq_1]_z\langle\tilde{\sigma}_{xy}^{q-q_1}\tilde{\sigma}_{xy}^{q_1-q}\rangle_0}{(q\hat{\sigma}^e(q)q)}.$$

Since we are not taking fluctuations of σ_{zz} into account, we have $\sigma^e_{zz}(q) = \langle\sigma_{zz}\rangle = \sigma_0$, so that only the function $\sigma^e_\perp(q)$ remains unknown. Substituting $(q\sigma^e(q)q) = \sigma_0q_z^2 + \sigma^e_\perp(q)q_\perp^2$ into (27), we obtain

$$\sigma_\perp^e(q) = \langle\sigma_\perp\rangle + \int d^3q_1 \frac{q_{1\perp}^2\sin^2\psi\langle\tilde{\sigma}_{xy}^{q-q_1}\tilde{\sigma}_{xy}^{q_1-q}\rangle_0}{\sigma_0q_{1z}^2 + \sigma_\perp(q_1)q_{1\perp}^2}, \qquad (28)$$

where ψ is the angle between q and q_1.

We shall study Eq. (28) for $q_z = 0$. We shall assume (confirming it later by the result) that $\sigma^e_\perp(q) \ll \sigma_0$. Then the small $q_{1z} \sim q_{1\perp}(\sigma^e_\perp/\sigma_0)^{1/2}$ are important in the integration over q_{1z}. Therefore, the integration over q_{1z} can be performed explicitly. This gives

$$\sigma_\perp^e(q) = \langle\sigma_\perp\rangle + \frac{\pi}{\sqrt{\sigma_0}}\int d^2q_{1\perp}\frac{q_{1\perp}\sin^2\psi\langle\tilde{\sigma}_{xy}^{q-q_1}\tilde{\sigma}_{xy}^{q_1-q}\rangle}{\sqrt{\sigma_\perp^e(q_1)}}. \qquad (29)$$

Since

$$\langle\tilde{\sigma}_{xy}^{q-q_1}\tilde{\sigma}_{xy}^{q_1-q}\rangle = \sigma_0^2\Delta^2\beta^{-2}D(q-q_1),$$

Equation (29) connects $\sigma^e_\perp(0)$ with the values of $\sigma^e_\perp(q)$, $q \leqslant a^{-1}$. In order to determine the order of magnitude of $\sigma^e_\perp(q)$ in this region, we shall assume that $\sigma^e_\perp(q) \approx \text{const}$ for $q \leqslant a^{-1}$. Bearing in mind that

$$\int d^2q_{1\perp}\,q_{1\perp}\sin^2\psi D(q_\perp) \sim 1:$$

we obtain

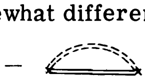
$$\sigma_\perp^e \sim \langle\sigma_\perp\rangle + \sigma_0^{3/2}\Delta^2/\beta^2\sqrt{\sigma_\perp^e}.$$

For $\beta\Delta^2 \gg 1$, the first term in the right-hand side can be neglected and, finally,

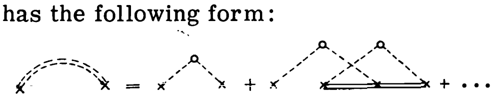
$$\sigma_\perp^e \sim \sigma_0(\Delta/\beta)^{4/3}. \qquad (30)$$

This result was obtained earlier from a qualitative treatment of the problem.

We recall that (30) was obtained as the result of solving the "truncated" equation (27). We must now show that this result is not changed by taking the remaining terms of the series in (27) into account. For this, we rewrite (27) in a somewhat different form:

$(q_i\sigma^e_{ik}(q)q_k) = (q_i\langle\sigma_{ik}\rangle q_k) -$  , (31)

where with the dashed double line we associate the factor $\Gamma(q, q_1)$ (the exact vertex), which in diagrammatic notation has the following form:

 (32)

and in explicit form is*

$$\Gamma(q,q_1) = [qq_1]_z\langle\tilde{\sigma}_{xy}^{q_1-q}\tilde{\sigma}_{xy}^{q_1-q}\rangle_0[q_1q]_z + \int[qq_1]_z\langle\tilde{\sigma}_{xy}^{q_1-q}\tilde{\sigma}_{xy}^{q-q_1}\rangle_0$$
$$\times\langle\tilde{\sigma}_{xy}^{q_2-q_1}\tilde{\sigma}_{xy}^{q_1-q_2}\rangle_0[q_1q_2]_z[q_2, q_2+q-q_1]_z(q_2+q-q_1, q]_z$$
$$\times[(q_2\hat{\sigma}^e(q_2)q_2)((q_2+q-q_1)\hat{\sigma}^e(q_2+q-q_1)(q_2+q-q_1))]^{-1}d^3q_2 + \ldots$$

Our approximation consisted in using the bare vertex (the first term in (32)) in place of the exact vertex in the exact equation (31). But, in deriving (30), we used only the "coarse" properties of the vertex, namely, the dimensions of the region in which it is appreciably non-zero and its order of magnitude in this region. If these "coarse" properties do not change when the next terms of the expansion are taken into account, then formula (30) also remains true. We note that, as can be seen from the derivation of (29), it is sufficient to confine ourselves to the study of $\Gamma(q, q_1)$ for q_z, $q_{1z} = 0$. We shall estimate the correction $\Gamma^{(2)}$ to $\Gamma(q, q_1)$ due to

*$[qq_1] \equiv q \times q_1$.

the second term in (32). Omitting factors of order unity, we obtain for it

$$\Gamma^{(2)} \sim \frac{\sigma^4\Delta^4}{\beta^4}\int d^3q_2 D(\mathbf{q}-\mathbf{q}_1)D(\mathbf{q}_1-\mathbf{q}_2)[\mathbf{q}\mathbf{q}_1]_z[\mathbf{q}_1\mathbf{q}_2]_z$$
$$\times[\mathbf{q}_2,\mathbf{q}-\mathbf{q}_1]_z[\mathbf{q}_2-\mathbf{q}_1,\mathbf{q}]_z\{(\sigma_0 q_{2z}^2+\sigma_\perp^e q_{2\perp}^2)\cdot(\sigma_0 q_{2z}^2+\sigma_\perp^e(q_2+q-q_1)_\perp^2\}^{-1}.$$

Here the integration over q_{2z} is singular (the small q_{2z} are important) and can be performed as previously. Since we are interested only in an estimate, and the subsequent integration over $q_{2\perp}$ is non-singular ($q_{2\perp} \sim a^{-1}$ are important), we obtain, putting all $q_\perp \sim a^{-1}$,

$$\Gamma^{(2)} \sim \frac{\sigma_0^2\Delta^4 D(\mathbf{q}-\mathbf{q}_1)}{\beta^4}\int\frac{D(\mathbf{q}_1-\mathbf{q}_2)a^{-5}d^2q_{2\perp}}{(\sigma_\perp^e)^{3/2}(\sigma_0)^{-3/2}} \sim \frac{\sigma_0^2\Delta^4 a^{-1}}{\beta^2}. \quad (33)$$

Here we have used formula (30) for $\sigma_\perp^e$. On the other hand, by estimating the bare vertex in the region q, $q_1 \sim a^{-1}$, we obtain that $\Gamma^{(2)}$ has the same order of magnitude as the bare vertex. It is not difficult to show that the situation remains the same in the next approximations. We shall see that, although the corrections to the bare vertex are not small, being of the same order they do not change the "coarse" properties of the vertex. Therefore, (30) also remains valid.

We shall also study the case of two-dimensional inhomogeneities extended along the magnetic field (along the z-axis). The two-dimensionality means that all the correlators (in x-space) are independent of z, and in the Fourier representation contain $\delta(q_z)$. Therefore, in the diagrams, the integration over q_z is performed automatically and reduces to the result that we must put $q_z = 0$ everywhere (and perform the integration only over $q_\perp$). Then $G_0(q) = \beta^2/\sigma_0 q_\perp^2$. As before, the fluctuations of σ_{xy} are the most important. We note that in the diagrams there are no singular integrations. Because of this, generally speaking, we cannot neglect diagrams containing irreducible many-point correlators. However, allowance for these correlators presents essentially no difficulties (cf. [17]) and does not change the results. Therefore, we confine ourselves for simplicity to the case of a Gaussian field, for which the irreducible many-point correlators are equal to zero. As previously, the order of a diagram does not depend on the way in which the dashed lines intersect. We note that, as is shown by an estimate of the diagrams, the expansion is made in the parameter $\beta^2\Delta^2$ (and not, as before, in $\beta\Delta^2 \gg 1$). For $\beta^2\Delta^2 \gg 1$, it is necessary to take all the diagrams into account. By analogy with the derivation of (30), we study the abbreviated equation (27). In explicit form (we omit the symbols $\perp$ from q) it is

$$\sigma_\perp^e(\mathbf{q}) = \langle\sigma_\perp\rangle + \frac{\sigma_0^2\Delta^2}{\beta^2}\int\frac{d^2q_1 D_2(\mathbf{q}-\mathbf{q}_1)\sin^2\psi}{\sigma_1^e(\mathbf{q}_1)}. \quad (34)$$

In order to find the order of magnitude of $\sigma_\perp^e(q)$ for $q \leqslant a^{-1}$ (in particular, the quantity $\sigma_\perp^e(0)$ of interest to us), we shall estimate the integral in (34), assuming that $\sigma_\perp^e \approx$ const in this region. We obtain $\sigma_\perp^e = \langle\sigma_\perp\rangle + \sigma_0^2\Delta^2/\beta^2\sigma_\perp^e$, whence, for $\beta\Delta \gg 1$, we have $\sigma_\perp \sim \sigma_0\Delta/\beta$. We must now check that the discarded terms in (27) do not change this estimate. For this, we must estimate the correction to the vertex. This estimate (which is performed completely analogously to that of (33)) shows that the situation remains the same as in the case of three-dimensional fluctuations—the corrections to the vertex are found to be of the same order as the bare vertex.

4. PATTERN OF THE CURRENTS IN A POLYCRYSTAL AND CALCULATION OF THE EFFECTIVE CONDUCTIVITY

It is known[10] that in a strong magnetic field ($\beta \gg 1$) the form of the conductivity tensor of a crystal in the case of a metal with an open Fermi surface depends substantially on the orientation of the crystallite with respect to the magnetic field. Namely, if the orientation is such that the open paths make no contribution to the conductivity, the asymptotic form is (the z-axis is along H):

$$\hat{\sigma} = \sigma_0\begin{pmatrix} a_{11}\beta^{-2} & a_{12}\beta^{-1} & a_{13}\beta^{-1} \\ -a_{12}\beta^{-1} & a_{22}\beta^{-2} & a_{23}\beta^{-1} \\ -a_{13}\beta^{-1} & -a_{23}\beta^{-1} & a_{33}\end{pmatrix}. \quad (35)$$

But if the orientation is such that the open paths contribute to the conductivity, the asymptotic form of $\hat{\sigma}$ is

$$\hat{\sigma} = \sigma_0\begin{pmatrix} a_{11}' & a_{12}'\beta^{-1} & a_{13}'\beta^{-1} \\ -a_{12}'\beta^{-1} & a_{22}' & a_{23}'\beta^{-1} \\ -a_{13}'\beta^{-1} & -a_{23}'\beta^{-1} & a_{33}'\end{pmatrix}. \quad (36)$$

In these formulas, a_{ik}, $a_{ik}' \to$ const as $\beta \to \infty$ ($\hat{\sigma}$ has the form (36) if there are two directions of the open paths; if, as is more often the case, there is only one direction of openness, then one of the transverse diagonal components of $\hat{\sigma}$ tends to a constant as $\beta \to \infty$, and the other is $\sim\beta^{-2}$. As will be clear from the following, both these cases lead to qualitatively equivalent results.)

The fraction C of the special crystallites, in which $\hat{\sigma}$ has the form (36), is determined by that fraction of the area on the stereographic projection which corresponds to magnetic-field directions leading to open paths, and, as mentioned in the Introduction, is usually small.

Thus, because of the appearance of open paths, σ_{xx} and σ_{yy} can fluctuate strongly, and this leads to a sharply inhomogeneous current pattern and, as will be shown below, to a significant difference between $\hat{\sigma}^e$ and $\langle\hat{\sigma}\rangle$. In addition to these sharp fluctuations, there are also fluctuations induced by the ordinary anisotropy. We shall neglect the latter, however, since we are interested only in estimating $\hat{\sigma}^e$.

The fluctuations of the remaining components of $\hat{\sigma}$ are small. As was shown in Sec. 3, the small fluctuations of σ_{xz}, σ_{yz} and σ_{zz} are unimportant in the estimation of $\hat{\sigma}^e$. As regards the component σ_{xy}, of which even small fluctuations could turn out to be important in the estimation of $\hat{\sigma}^e$, since it is determined by the quantity $n_e - n_h$, which does not depend on the orientation, fluctuations of σ_{xy} are absent. More precisely, the situation is as follows: for orientations for which open paths do not appear, the value of σ_{xy} does not depend on the orientation and is equal to $(n_e - n_h)ec/H$; in the special crystallites, however, σ_{xy} differs from this value, but since $\sigma_\perp \gg \sigma_{xy}$ in the special crystallites, the quantity σ_{xy} plays no role in them, and we shall assume for simplicity that $\sigma_{xy} = (n_e - n_h)ec/H$ everywhere. Therefore, we shall consider a model of a polycrystalline medium in which the conductivity tensor has the form:

a) in the special crystallites

$$\hat{\sigma}=\begin{pmatrix}\sigma_1 & (n_e-n_h)ec/H & 0\\ -(n_e-n_h)ec/H & \sigma_1 & 0\\ 0 & 0 & \sigma_0\end{pmatrix}; \qquad (37)$$

b) in the other crystallites forming the "background,"

$$\hat{\sigma}=\begin{pmatrix}\sigma_0\beta^{-2} & (n_e-n_h)ec/H & 0\\ -(n_e-n_h)ec/H & \sigma_0\beta^{-2} & 0\\ 0 & 0 & \sigma_0\end{pmatrix}. \qquad (38)$$

We note also that the smallness of C is governed by the narrowness of the connecting necks linking portions of the Fermi surface in neighboring cells of the reciprocal lattice. The narrowness of the connecting necks also leads to the result that the transverse conductivity of the special crystallites $\sigma_1 \ll \sigma_0$, since, even when there are special paths, they form a small fraction of all the paths. A more detailed treatment shows that $\sigma_1 \sim \sigma_0 C$ (so that the asymptotic form (36) starts from $\beta \sim C^{-1/2}$).

Up to this point, we have been concerned with the case when there is a finite fraction of directions corresponding to open paths. This case corresponds, e.g., to a Fermi surface of the "space-mesh" type. But if the Fermi surface is a "goffered cylinder," then, in this case, there are only separate lines corresponding to open paths on the stereographic projection. However, if the direction of H is close to these lines, then the paths, although closed, are of great length, and the conductivity is anomalously large. As we shall see, in this case, $\sigma_\perp^e$ also differs significantly from $\langle\sigma_\perp\rangle$.

It follows from Eq. (5) that, if the antisymmetric part σ_{ik}^a is constant, it drops out completely from the equation for the potential, so that (5) can be rewritten in the form

$$\frac{\partial}{\partial x_i}\left(\sigma_{ik}{}^s\frac{\partial\varphi}{\partial x_k}\right)=0. \qquad (39)$$

Moreover, since

$$\langle j_i\rangle=-\frac{1}{V}\int\sigma_{ik}\frac{\partial\varphi}{\partial x_k}dV=-\frac{1}{V}\int\sigma_{ik}{}^s\frac{\partial\varphi}{\partial x_k}dV+\sigma_{ik}{}^a\langle E_k\rangle,$$

the antisymmetric part of the effective conductivity tensor is equal to σ_{ik}^a, and in determining $\sigma_{ik}^{e,s}$ we can assume the local conductivity tensor to be symmetric (this remark is due to Korzh[14]).

We shall begin the calculation of $\hat{\sigma}^e$ from an examination of the situation arising in the limit $\beta = \infty$. In this case, the transverse conductivity of the background is equal to zero. It is clear, however, that the effective transverse conductivity of the polycrystalline medium is then non-zero. We shall elucidate this statement by describing the form of a typical current line. Since the transverse conductivity of the "background" is equal to zero, the current line passes through the background parallel to H, until it "encounters" one of the special crystallites, at which it proceeds in the direction of $\langle \mathbf{E}\rangle$ ($\langle \mathbf{E}\rangle \perp \mathbf{H}$). Having passed through the special crystallite, through a distance of the order of the dimensions a of the crystallites, the current line again goes through the background parallel to H, until it again encounters one of the special crystallites. The current line goes equally often along H and against H, so that the mean current flows in the direction of $\langle \mathbf{E}\rangle$.

To estimate $\sigma_\perp^e$, we shall make use of the diffusion analogy—the technique described at the beginning of Sec. 3. Namely, we shall consider a particle diffusing through an inhomogeneous medium, with $\hat{\sigma}^s$ (given by the formulas (37) and (38) with $n_e = n_h$) playing the role of the diffusion coefficient, so that the transverse diffusion coefficient is non-zero only in the special crystallites. For an estimate of the effective transverse diffusion coefficient, we must find $\lim \langle \mathbf{r}_\perp^2(t)\rangle/t$ for $t \to \infty$, where $\mathbf{r}_\perp(t)$ is the displacement of the particle in the transverse direction in time t, and the angular brackets denote averaging over all possible diffusion paths.

It is clear that, however much time the particle spends in one special crystallite, its transverse displacement will be limited by the dimensions of this crystallite. In order to be displaced by a large distance in the transverse direction, the particle must, in diffusing along the magnetic field, reach the next special crystallite. The distance, between the two special crystallites, through which the particle must pass can be estimated (by analogy with the mean free path) as

$$\mathscr{L} \sim 1/na^2 \sim a/C, \qquad (40)$$

where n is the number of special crystallites in 1 cm^3 (clearly, $C \sim na^3$). The time τ in which the particle diffuses over this distance can be estimated from the formula $\tau \sim \mathscr{L}^2/\sigma_0 \sim a^2/C^2\sigma_0$. In this time, the particle will repeatedly return to the crystallite from which it began its motion. The total time τ_1 spent in this crystallite is estimated from the formula

$$\tau_1 \sim a\tau/\mathscr{L} \sim a^2/C\sigma_0. \qquad (41)$$

Here, the first estimate corresponds to the fact that the particle spends an approximately equal time in each of the segments of length a (within the limits of the "mean free path" $\mathscr{L}$).

It is then necessary to distinguish two cases, according to the magnitude of σ_1. If σ_1 is not too small, then, by the time τ, the transverse displacement of the particle, estimated from the formula $r_\perp(\tau) \sim \sqrt{\sigma_1\tau_1}$, will be greater than the dimensions a of the crystallite. This means that the transverse displacement of the particle in time τ will really be of order a. As can be seen from (41) this case is realized for $\sigma_1 \geqslant \sigma_0 C$. But if the opposite inequality is fulfilled, the particle, not having had time to "feel" the boundaries of one special crystallite, goes on to the next. We shall estimate the effective diffusion coefficient (effective conductivity) from the formula $\sigma_\perp^e \sim r_\perp^2(\tau)/\tau$; this gives

$$\sigma_\perp{}^e \sim \sigma_0 C^2 \text{ for } \sigma_1 \gtrsim \sigma_0 C, \qquad \sigma_\perp{}^e \sim \sigma_1 C \text{ for } \sigma_1 \lesssim \sigma_0 C. \qquad (42)$$

If $\beta \neq \infty$ (but $\beta \gg 1$), then, along with the transverse diffusion through the special crystallites, transverse diffusion through the background also occurs, with diffusion coefficient $\sigma_0\beta^{-2}$. Comparing this with (42), we find that conduction through the special crystallites is the principal mechanism for $\beta C \gg 1$ in the case $\sigma_1 \geqslant \sigma_0 C$ and for $C\beta^2\sigma_1/\sigma_0 \gg 1$ in the case $\sigma_1 \leqslant \sigma_0 C$. We also give simple interpolation formulas valid in the whole region $\beta \gg 1$:

$$\sigma_\perp{}^e \sim \sigma_0 C^2 + \sigma_0/\beta^2, \quad \sigma_1 \gtrsim \sigma_0 C, \qquad \sigma_\perp{}^e \sim \sigma_1 C + \sigma_0/\beta^2, \quad \sigma_1 \lesssim \sigma_0 C. \qquad (43)$$

In the case when the Fermi surface is a "corrugated cylinder," the situation is more complicated. If the magnetic-field direction lies close to the plane perpendicular to the openness direction and forms an angle ϑ with it, the length of the paths is increased by a factor of ϑ^{-1} compared with the length of the paths corresponding to a magnetic field along the openness direction. The conductivity then increases as the square of the path length, until this length becomes comparable with the mean free path; therefore,

$$\sigma_\perp(\vartheta) = \begin{cases} \sigma_0/\beta^2\vartheta^2, & \beta^{-1} \ll \vartheta \ll 1, \\ \sigma_0 & \vartheta \ll \beta^{-1}. \end{cases} \quad (44)$$

The fraction of crystallites oriented such that the magnetic-field direction forms an angle between ϑ and $\vartheta + d\vartheta$ with the plane perpendicular to the openness direction is equal to $\sim d\vartheta$. We shall study the diffusional motion of a particle in such a medium. Since the particle diffuses rapidly along z, the fraction of time spent by the particle in crystallites with angles between ϑ and $\vartheta + d\vartheta$ is proportional to their concentration $d\vartheta$. Therefore, in time t, the particle will spend a time $\sim t\,d\vartheta$ in crystallites with angles between ϑ and $\vartheta + d\vartheta$. It would seem that we could write $r_\perp^2(t, \vartheta) \sim \sigma_\perp(\vartheta) t\, d\vartheta$ for the square of the transverse displacement obtained in these crystallites, and, since all the displacements are independent, find the square of the total displacement in time t as $t\int_0^{\pi/2} \sigma_\perp(\vartheta) d\vartheta$. This argument, however, does not take into account the fact that in crystallites with sufficiently small angles (with large conductivity), a particle diffusing rapidly across the magnetic field "feels" the boundaries of this crystallite before it reaches the next crystallite with a sufficiently good conductivity. Thus, the square of its transverse displacement will not increase linearly with the time spent in this crystallite, but will be limited by the square of the crystallite dimensions. We therefore proceed as follows. We divide all the crystallites into two classes: those in which the particle does not have time to "feel" the boundary, and the remainder, in which the displacement is limited by the dimensions of the crystallite. Let ϑ_0 be the angle delimiting these classes, so that $\vartheta \geqslant \vartheta_0$ corresponds to the first class and $\vartheta \leqslant \vartheta_0$ to the second (of course, this division is only nominal, and ϑ_0 is determined only to within a factor of order unity). Then, for the square of the particle displacement after time t, we can write

$$r_\perp^2(t) \sim t\int_{\vartheta_0}^{\pi/2} \sigma_\perp(\vartheta)\, d\vartheta + \left(\text{contribution from the crystallites with } \vartheta \leqslant \vartheta_0\right).$$

We shall assume (this will be confirmed by the result) that $\beta^{-1} \ll \vartheta_0 \ll 1$, so that $\sigma_\perp(\vartheta) \sim \sigma_0/\beta^2\vartheta^2$ (cf. (44)). Dividing $r_\perp^2(t)$ by t and writing down the contribution to $\sigma_\perp^e$ from the crystallites with $\vartheta \leqslant \vartheta_0$ as $\sigma_0 C^2$ where C $\sim \vartheta_0$ is their concentration, we obtain

$$\sigma_\perp^e \sim \sigma_0/\beta^2\vartheta_0 + \sigma_0\vartheta_0^2. \quad (45)$$

It remains to find ϑ_0. We note that it follows from the derivation of (45) that crystallites with $\vartheta \sim \vartheta_0$ play the principal role. We can find ϑ_0 by requiring that our division indeed correspond to the fact that at $\vartheta \sim \vartheta_0$ the effect of the boundaries of the crystallites on the diffusion begins to appear. This gives (cf. the derivation of (42)) $\sigma_\perp(\vartheta_0) \sim \sigma_0 C(\vartheta_0)$, where $C(\vartheta_0) \sim \vartheta_0$ is the concentration of crystallites with $\vartheta \leqslant \vartheta_0$, whence, using (44), we find $\vartheta_0 \sim \beta^{-2/3}$. Finally, we obtain

$$\sigma_\perp^e \sim \sigma_0/\beta^{4/3}, \quad \beta \gg 1. \quad (46)$$

Up to this point, we have considered the case of a slightly goffered cylinder. The results are easily extended to the case when the neck of the goffered cylinder is narrow, so that the ratio ν of the neck diameter to the dimensions of the reciprocal-lattice cell is small, $\nu \ll 1$. We give the result for this case:

$$\sigma_\perp^e \sim \sigma_0\nu^2/\beta^{4/3}. \quad (47)$$

By analogy with (43), by adding $\sigma_0\beta^{-2}$ to this expression, we obtain an interpolation formula valid in the whole region $\beta \gg 1$.

The usual experimental setup in the study of the galvanomagnetic characteristics of metals (in a long sample, the current direction is fixed and the field along the current is measured) corresponds to the measurement of $\rho_\perp$—the transverse component of the resistivity tensor. To determine $\rho_\perp^e$, we must invert the tensor $\hat{\sigma}^e$:

$$\rho_\perp^e = \sigma_\perp^e / [(\sigma_\perp^e)^2 + (\sigma_{xy}^e)^2].$$

The result depends on whether we are dealing with the case $\sigma_{xy} = 0$ (for an equal number of electrons and holes) or with the case $\sigma_{xy} \sim \sigma_0/\beta$ (for $n_e \neq n_h$). We give a series of results for $\rho_\perp^e$ in the various cases ($\beta \gg 1$, $\rho_0 \equiv \sigma_0^{-1}$):

a) a "three-dimensional grid" Fermi surface, $\sigma_1 \geqslant \sigma_0 C$, $n_e \neq n_h$:

$$\rho_\perp^e \sim \rho_0[1 + (\beta C)^2], \quad \beta \lesssim C^{-2}, \qquad \rho_\perp^e \sim \rho_0 C^{-2}, \quad \beta \gtrsim C^{-2}; \quad (48)$$

b) the same, but with $n_e = n_h$:

$$\rho_\perp^e \sim \rho_0/(C^2 + \beta^{-2}); \quad (49)$$

c) a "three-dimensional grid" Fermi surface, $\sigma_1 \leqslant \sigma_0 C$, $n_e \neq n_h$:

$$\rho_\perp^e \sim \rho_0\left(\frac{\sigma_1 C\beta^2}{\sigma_0} + 1\right), \quad \beta \lesssim \frac{\sigma_0}{\sigma_1 C}, \qquad \rho_\perp^e \sim (\sigma_1 C)^{-1}, \quad \beta \gtrsim \sigma_0/\sigma_1 C; \quad (50)$$

d) the same, but with $n_e = n_h$:

$$\rho_\perp^e = \rho_0 \Big/ \left(\frac{\sigma_1}{\sigma_0}C + \beta^{-2}\right); \quad (51)$$

e) a "corrugated-cylinder" Fermi surface, $n_e \neq n_h$:

$$\rho_\perp^e \sim \rho_0(1 + \nu^2\beta^{2/3}); \quad (52)$$

f) the same, but with $n_e = n_h$:

$$\rho_\perp^e \sim \rho_0 \frac{\beta^2}{\beta^{2/3}\nu^2 + 1}. \quad (53)$$

5. SIZE EFFECT IN THE MAGNETORESISTANCE OF POLYCRYSTALLINE SAMPLES

It follows from the current-flow pattern in a polycrystalline sample, which was discussed in the derivation of (42), that a typical current line consists mainly of parts parallel to the magnetic field, the length $\mathscr{L} \sim a/C$ of these parts being greater than the dimensions of the crystallites. What happens when the sample thickness in the direction of the magnetic field is less than $\mathscr{L}$?

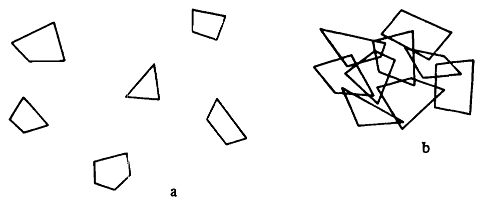


FIG. 2

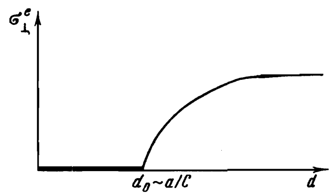


FIG. 3

We again start from the case $\beta = \infty$. Obviously, for the mechanism described above for conduction in a polycrystalline sample to be possible, it is necessary that the projections of the special crystallites on the plane perpendicular to H overlap on average sufficiently for paths to exist which pass through the projections of the special crystallites and go from one electrode to the other. For a small sample thickness d, the fractional area occupied by projections of the special crystallites is small, there are no such paths, and $\sigma_\perp^e = 0$ (Fig. 2a). For large thickness, the projections of the special crystallites begin to overlap, the conductivity is non-zero (Fig. 2b) and, finally, for $d \gg \mathscr{L}$ is given by formula (42). To make such a conclusion, it is important that both the transverse dimensions be large; if one of them (the sample width) were small, then over a large distance we would certainly find a fluctuation in the distribution of the crystallites which would serve as a break in the chain formed by the special crystallites. The dependence of $\sigma_\perp^e$ (for $\beta = \infty$) on the thickness is shown in Fig. 3. The change of conductivity with thickness has the character of a phase transition: the conductivity is equal to zero for $d \leq d_0 \sim \mathscr{L} \sim a/C$ and is non-zero for $d > d_0$. Here, d_0 is the thickness at which overlap of the projections of the special crystallites first occurs. The thickness at which $\sigma_\perp^e$ takes up the asymptotic form (42) is also of order $\mathscr{L}$. For $\beta \neq \infty$ (but $\beta C \gg 1$), the change of the conductivity loses its sharp character and is finally smeared out for $\beta C \leqslant 1$.

The size effect has somewhat different features in the case when the Fermi surface is a "corrugated cylinder." As we have seen in Sec. 4, in this case the principal role is played by crystallites with angles $\vartheta \leqslant \beta^{-2/3}$, of which the concentration $C \sim \beta^{-2/3}$. Therefore, $\mathscr{L} \sim a/C \sim a\beta^{2/3}$. So long as $\mathscr{L} \ll d$, the sample can be regarded as an infinite medium. With increase of the magnetic field $\mathscr{L}$ becomes comparable with the sample thickness. This occurs for $\beta_0 \sim (d/a)^{3/2}$. For $\beta \gg \beta_0$, the main role is played by crystallites with angles $\vartheta_0 \sim \beta^{-2/3}$, since there are few crystallites with $\vartheta_0 \sim \beta^{-2/3}$ (which dominate in the conductivity of an infinite medium) over the sample thickness. The conductivity of the sample can now be estimated from the second of the formulas (42) with $C \sim \beta_0^{-2/3}$ and $\sigma_1 \sim \sigma_0/\vartheta_0^2\beta^2 \sim \sigma_0\beta_0^{4/3}/\beta^2$. This gives

$$\sigma_\perp^e \sim \sigma_0\beta_0^{2/3}/\beta^2, \quad \beta \gg \beta_0 \sim (d/a)^{3/2}.$$

We note that, for $\nu \approx 1$ (as was assumed here), the size effect can be observed for $\beta \gg 1$.

[1] C. Herring, J. Appl. Phys. 31, 1939 (1960).
[2] S. Yoshikawa and D. J. Rose, Phys. Fluids 5, 334 (1962).
[3] V. A. Kudinov and B. Ya. Moĭzhes, Fiz. Tverd. Tela 7, 2309 (1965) [Sov. Phys.-Solid State 7, 1868 (1966)].
[4] Yu. M. Gal'perin and B. D. Laĭkhtman, Fiz. Tverd. Tela 13, 2102 (1971) [Sov. Phys.-Solid State 13, 1760 (1972)].
[5] E. P. Velikhov, A. M. Dykhne, and I. Ya. Shipuk, in: Ionization Phenomena in Gases (Proceedings of the Sixth International Conference, Belgrade, 1965).
[6] A. A. Vedenov and A. M. Dykhne, Electricity from MHD (Vienna) 2, 335 (1966).
[7] E. P. Velikhov, Proceedings of the Conference on MHD, Newcastle, 1962.
[8] D. J. Rose, Phys. Fluids 5, 9 (1965) [sic!]
[9] A. M. Dykhne, Zh. Eksp. Teor. Fiz. 59, 110 (1970) [Sov. Phys.-JETP 32, 63 (1971)].
[10] I. M. Lifshitz and M. I. Kaganov, Usp. Fiz. Nauk 87, 389 (1965) [Sov. Phys.-Uspekhi 8, 805 (1966)].
[11] I. M. Lifshitz and V. G. Peschanskiĭ, Zh. Eksp. Teor Fiz. 35, 1251 (1958) [Sov. Phys.-JETP 8, 875 (1959)].
[12] H. Stachowiak, Bull. Acad. Pol. Sci., Ser. Math. 15, 677 (1967)].
[13] J. M. Ziman, Phil. Mag. 3, 1117 (1958).
[14] S. A. Korzh, Zh. Eksp. Teor. Fiz. 59, 510 (1970) [Sov. Phys.-JETP 32, 280 (1971)].
[15] Yu. A. Dreizin and A. M. Dykhne, ZhETF Pis. Red. 14, 101 (1971) [JETP Lett. 14, 66 (1971)].
[16] E. B. Dynkin and A. A. Yushkevich, Teoremy i zadachi o protsessakh Markova (Theorems and Problems on Markovian Processes), Nauka, M., 1967.
[17] V. M. Finkel'berg, Zh. Eksp. Teor. Fiz. 53, 401 (1967) [Sov. Phys.-JETP 26, 268 (1968)].

Translated by P. J. Shepherd
27