FLUCTUATION CHARACTERISTICS OF A DENSE PLASMA OF HIGH CURRENT DISCHARGES PRODUCED BY ELECTRIC EXPLOSION OF METALLIC WIRES

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Submitted February 22, 1972

Zh. Eksp. Teor. Fiz. 63, 844–853 (September, 1972)

The fluctuation characteristics of a high current discharge plasma produced by electric explosion of lithium wires in vacuum are investigated (discharge geometry—straight Z-pinch, W_{tr} = 22.5 kJ, $\tau_{disch} \approx 120$ μsec, I_{max} = 220 kA, initial wire diameter d_0 is 0.17 mm, 0.34 mm or 1.0 mm). Correlation reduction of the signals from magnetic sensing elements yielded the mean size, velocity of the inhomogeneities and energy spectrum of the fluctuations. In the second part of the paper the magnetohydrodynamics equations are analyzed; this leads to relations for plasma pulsations which are in good agreement with the experimental data. It is concluded that plasma fluctuations of the discharge are determined by superheat instabilities peculiar to an optically transparent plasma.

1. INTRODUCTION

THE plasma formed by the discharge of a capacitor bank (1800 μF, 5 kV) through a lithium wire does not possess a regular structure during the first quarter of the period of the discharge current (0–40 microseconds); it has inherent inhomogeneities of composition and luminescence whose location and size vary with time. This character of the discharge was indicated by the results of measurements of the local magnetic fields and the high-speed photographs of the discharge[1,2]. Study of the oscillograms of signals from magnetic probes placed inside the plasma shows that large oscillations of the magnetic field are present during the first 35–40 microseconds. The curves consist of aggregates of pulsations of various amplitudes and periods superimposed on one another without any visible regularity, i.e. the instantaneous values of the magnetic fields are irregular sums of three-dimensional pulsations (H_φ, H_z, H_r) of different orientation, amplitude, and frequency. Since electric and magnetic fields accompany the majority of the collective motions of the plasma, it is possible to investigate plasma motions with the aid of the magnetic fields associated with them.

Measurements show that the fields excited in the plasma are of random character, i.e., there is no reproducibility from discharge to discharge (from "shot" to "shot"). This is evidence of the turbulent state of the plasma and the fact that the hydrodynamic quantities (pressure p, velocity v, and temperature T) pulsate irregularly during the first 40 microseconds, changing with extreme irregularity in time and space. After the current maximum (t > 40 μsec) the turbulent state of the plasma passes to the quiescent state ("smooth," continuous character of signals from the magnetic probes), in which the hydrodynamic variables are connected by a definite functional relation, which we determined in previous papers[1].

In this paper we investigate the fluctuation characteristics of the considered discharge plasma both by the method of correlation reduction of the magnetic measurements and by dimensional analysis of the magnetohydrodynamic equations. Apart from its own value, the determination of the plasma fluctuation characteristics permits a more exact analysis of the integral characteristics of the discharge (the time of existence of the turbulent state of discharge constitutes an appreciable fraction of the discharge time, the latter being ~100 μsec.

2. CORRELATION REDUCTION OF MAGNETIC FIELD MEASUREMENTS

A. Formulation of the Problem

To determine the mean values of the physical quantities in a turbulent process it is necessary to employ the statistical description of turbulence. Since it is difficult in practice, and in particular in our experiment, to obtain the large number of measurements made in a large number of identical repeated trials for the determination of the mean values of the hydrodynamic fields, it seems realistic to utilize measurement data obtained in the course of only one (or several) trials. By the same token, it becomes necessary in practical calculations to replace the "averages over the ensemble" (mean values in the probability-theoretical sense) with the "averages over one realization" (directly observed averages). For such a convergence to be fulfilled for a random process, i.e., for the ergodic theorem to be valid

$$\lim_{t\to\infty} |\widetilde{x_{T_a}}(t) - \overline{x(t)}| = 0,$$

it is necessary and sufficient that the correlation function $B_{xx}(\tau)$ satisfy the condition (see[3])

$$\lim_{T_y\to\infty} \frac{1}{T_a}\int_0^{T_a} B_{xx}(\tau)\,d\tau = 0,$$

which is satisfied if $B_{xx}(\tau) \to 0$ as $\tau \to \infty$. We introduce here the following notation: x(t) is the value of the field pulsation, i.e., the difference between the individual values of the field and its mean value; $\overline{x(t)}$ is the probability-theoretical mean value of the random process; $\widetilde{x_{T_a}}(t)$ is the average over the realizations in a time T_a.

In different physical random processes $B_{xx}(\tau) \to 0$ as $\tau \to \infty$; therefore, if the physical process is stationary (steady-state turbulent flow), then the probability-theoretical mean values of the hydrodynamic fields can be determined by averaging over a sufficiently large time interval.

Summing up what has been said above, we can draw a number of conclusions. From a single realization of a random process one can assess the probability-theoretical mean values (and thus considerably simplify the analysis of the process) if the following conditions are satisfied:

1. The averaging interval T_a must be sufficiently large, since a reliable determination of $\overline{x(t)}$ calls for time averaging over an interval T_a much longer than the corresponding correlation time defined by

$$T_{\text{corr}} = \frac{1}{B_{xx}(0)} \int_0^\infty B_{xx}(\tau)\, d\tau. \tag{1}$$

Then one can determine the degree of accuracy in the replacement of $\overline{x(t)}$ by $\widetilde{x_{T_a}}(t)$, i.e., the largest mean-squared error[3]:

$$\overline{|\widetilde{x_{T_a}}(t) - \overline{x(t)}|^2} \approx 2\frac{T_{\text{corr}}}{T_a} B_{xx}(0). \tag{2}$$

Thus, for example, for $T_a = 15\ \mu$sec and $T_{\text{corr}} = 3\ \mu$sec, we have

$$\frac{1}{B_{xx}(0)} \overline{|\widetilde{x_{T_a}}(t) - \overline{x(t)}|^2} \approx 0.4.$$

2. Since we are considering an essentially nonstationary stage of the discharge (from the instant of explosion of the wire to the establishment of laminar flow), the averaging interval T_a must be sufficiently short that the process can be regarded as stationary in that time interval. Physically, the process to which such a realization corresponds must be a steady-state process, i.e., all external conditions that produce the process (discharge current I, voltage U) and all the mean characteristics of the current (e.g. mean temperature $\overline{T}$, mean density $\overline{n}$) should, strictly speaking, be invariant in time. The chosen averaging interval T_a enabled us in practice to satisfy these two contradictory requirements, since in our case there is satisfied the condition $T_{\text{corr}} = 3\ \mu\text{sec} < T_a < 15\ \mu\text{sec} < T_{\text{av. motion}} = 100\ \mu$sec (i.e., an extremely irregular "pulsation" motion is superimposed on an average motion that varies comparatively slowly).

3. The turbulence we are considering must be homogeneous (the condition for the ergodic theorem). Since we are investigating a discharge contained within walls, we must conclude that our analysis of turbulence will be valid only if the dimensions of the plasma inhomogeneities are substantially less than the characteristic dimension of the discharge (for example, the chamber diameter, equal to 10 cm), i.e., the turbulence must be small-scale.

For ergodic random (i.e., stationary and metrically transitive) processes, the expression for the correlation function takes the form

$$B_{xy}(\tau) = \lim_{T_a \to \infty} \frac{1}{2T_y} \int_{-T_a}^{T_a} x^{(k)}(t)\, y^{(r)}(t+\tau)\, d\tau, \tag{3}$$

where $x^{(k)}$ is the realization of the k-th random process x(t) and $y^{(r)}$ is the realization of the r-th random process y(t). For ease in calculation we use the correlation coefficient of the stationary random process ("normalized correlation function")

$$R_x(\tau) = B_{xx}(\tau) \,/\, B_{xx}(0)$$

and the mutual correlation coefficient

$$R_{xy}(\tau) = B_{xy}(\tau) \,/\, [B_{xx}(0) B_{yy}(0)]^{1/2},$$

where $B_{xx}(0)$ and $B_{yy}(0)$ are the rms values of the steady state random processes x(t) and y(t). We recall that we are considering processes with zero mean values: $\overline{x(t)} = 0$, $\overline{y(t)} = 0$.

Once the correlation coefficient is calculated, we can calculate the correlation time T_{corr} (with the aid of formula (1)), which characterizes the process in the sense that at $\tau > \tau_{\text{corr}}$ the quantities x(t) and x(t + τ) can be regarded as uncorrelated in practice.

It will subsequently be shown that if the correlation functions are known we can determine the dimensions of the plasma inhomogeneities, their velocity, and other quantities.

B. Reduction of Results. Results of the Measurements

In this experiment (the setup and procedure are described in[1]) the local fields were measured with miniature magnetic pickups consisting of single layer-coils of ~ 3 mm diameter. In one trial it was possible to measure dH(t)/dt (or H(t)) at four different points of the discharge plasma, while varying the orientation of the probes permitted us to measure the different field components (H_φ, H_r, H_z). The signals from the pickups were recorded with the aid of an S1-33 oscilloscope on photographic film. Typical oscillograms of the magnetic field intensity derivatives are shown in Fig. 1 (henceforth H′ will designate the derivative with respect to time). The experimental data were tabulated (at intervals determined by the maximum fluctuation frequency and the bandwidth of the recording apparatus) and were fed to a computer along with the corresponding computation programs.

With the aid of the computer we calculated the norm-

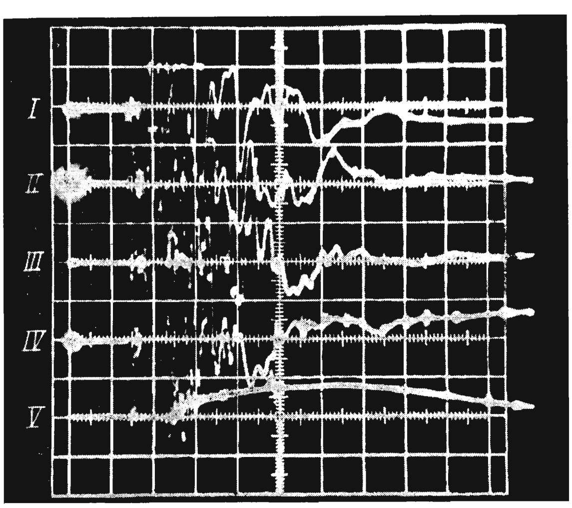

FIG. 1. Oscillograms of signals from magnetic probes. Each division equals 10 microseconds. Traces: I, IV—$H'_\varphi(t)$, II—$H'_r(t)$, III—$H'_z(t)$, v—total discharge current I(t).

alized autocorrelation functions $R_x(\tau)$ of the magnetic field fluctuation, the mutual correlation function $R_{xy}(\tau)$ between different points of the plasma, and the fluctuation spectrum. It should be noted that the time delay of the measuring apparatus ($f_{lim} \approx 10^7$ Hz) and the finite size of the pickups (~ 3 mm) limits the possibility of measuring the small-scale and high-frequency components of the turbulence. The averaging in our experiments was therefore over the time interval $\tau = 1/f_{lim} \approx 10^{-7}$ sec and over a region of space having linear dimensions of the order of 3 mm.

The averaging interval T_a was chosen to be 15 μsec, approximately in the middle of the first quarter-period of the discharge. This satisfied the two aforementioned conditions on the choice of T_a. The autocorrelation coefficient underwent almost no change when T_a was varied by ± 5 μsec, which indicates that the choice of the interval T_a was a reasonable one. It should also be noted that the correlation functions agree for different identical trials (the same initial wire diameter, energy input, wire material), i.e., for different realizations of the random process. This indicates that not only are the mean discharge parameters ($I(t)$, $U(t)$, $\overline{T}$, $\overline{n}$)[4] reproduced from one trial to the next, but the statistical properties of the fluctuation of the initial phase of the discharge and the character of turbulent motion are all preserved.

As noted earlier, the accuracy of the replacement of $\overline{x(t)}$ by $\widetilde{x_{T_a}(t)}$ is about 40% in our case.

Figure 2 shows the autocorrelation coefficient of the derivative of the azimuthal component of the magnetic field R_x ($x \equiv H'_\varphi$) for wires of initial diameter 0.17 mm. An analogous functional form is obtained for the axial and radial field components.

Figure 3 shows the autocorrelation function of the fluctuations of a helium-neon laser beam that passes through a plasma (the oscillograms for the calculation were taken from[5], where measurements of light passing through a lithium plasma were made at the same lithium discharge parameters as in our case). It is seen that the plots in Figs. 2 and 3 are similar. Since the fluctuations of the transmitted laser beam are mainly determined by the plasma-density fluctuations, the good agreement of the autocorrelation functions indicates that the magnetic-field fluctuations correspond to fluctuations in the plasma interval parameters.

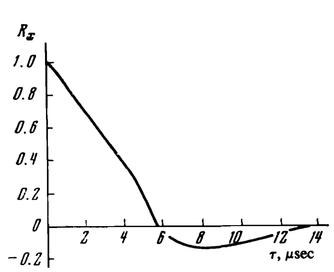

FIG. 2

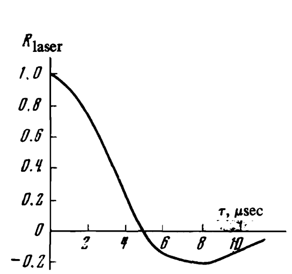

FIG. 3

FIG. 2. Autocorrelation coefficient of the derivative of the azimuthal magnetic field component R_x ($x \equiv H'_\varphi$) for a lithium discharge with initial wire diameter $d_0 = 0.17$ mm.

FIG. 3. Autocorrelation coefficient of the brightness fluctuation of the laser beam illuminating the plasma.

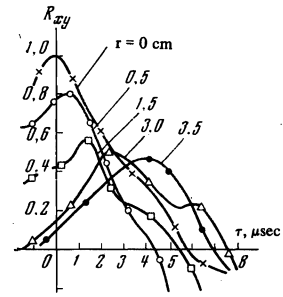

FIG. 4

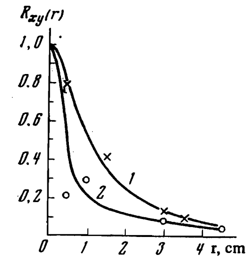

FIG. 5

FIG. 4. Mutual correlation coefficient $R_{xy}(\tau)$ ($xy \equiv H'_1 H'_2$) of signals from magnetic probes separated by different distances r.

FIG. 5. Mutual correlation coefficient $R_{xy}(r)$ ($xy \equiv H'_1 H'_2$) as a function of the distance between probes: $1-d_0 = 0.31$ mm, $2-d_0 = 0.17$ mm.

Figure 4 shows a family of mutual-correlation functions of the signals from two magnetic probes at different distances between probes. As the distance increases, the maximum of the mutual correlation function $R_{max}(\tau)$ decreases.

Figure 5 shows the mutual correlation coefficient $R_{xy}(r)$ of signals from two magnetic probes as functions of the distance between probes (x and y correspond to values of the magnetic field at different points in the plasma). As expected, the correlation coefficient decreases with increasing distance. From Fig. 5 it is possible to determine the dimension of the turbulence, i.e., the characteristic scale of the field of the fluctuating quantities, over which significant correlation still obtains between field values at two points[3]:

$$L = \frac{1}{B(0)} \int_0^\infty B_{xy}(r)\,dr = \int_0^\infty R_{xy}(r)\,dr. \quad (4)$$

The inhomogeneity dimension thus determined is 0.9 cm for the plasma formed by the explosion of a lithium wire of initial diameter $d_0 = 0.17$ mm, and 1.3 cm for $d_0 = 0.31$ mm.

Figure 6 shows $H'_z(t)$ oscillograms for $d_0 = 1.02$ mm (a) and $d_0 = 0.17$ mm (c) (the distance from the center of the pickup to the discharge-chamber axis was 3 cm in both cases), from which it is seen that pulsation amplitude increases and the pulsation spectrum broadens as the number of particles in the discharge is decreased. Analysis of the results presented in Fig. 5 and Fig. 6 shows that the autocorrelation function becomes more level as the mass of the wire increases; consequently, as the optical density of the discharge increases, the dimensions of the inhomogeneities increase (whereas the discharge characteristics $I(t)$ and $U(t)$ remain almost unchanged).

According to the general concepts, the excitation of turbulent currents is due to instability[6]. As we proceed to thicker wires, i.e., to an optically denser, "black" plasma, the superheat instability mechanism characteristic of transparent plasmas becomes weaker[7]. We can therefore conclude that the suppression of turbulence in the case of "thick" wires is connected with the suppression of the superheat instability, that determines the turbulence of the plasma in our case.

If the plasma inhomogeneities move with a mean

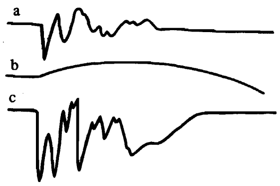

FIG. 6

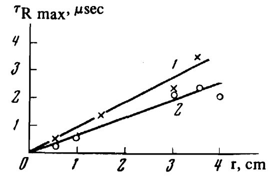

FIG. 7

FIG. 6. Oscillograms of $H'_z(t)$ for a–d_0 = 1.02 mm, b–oscillogram of total current I(t), c–d_0 = 0.17 mm.

FIG. 7. Dependence of $\tau_{R_{max}}$ on the distance between probes: 1–d_0 = 0.31, 2–d_0 = 0.17.

velocity v, then the equality $|v| = v/\tau_{R_{max}}$ is valid, where r is the distance between probes and $\tau_{R_{max}}$ is the time shift corresponding to the maximum value of the mutual correlation function[8]. We can therefore determine the velocity of the plasma inhomogeneities from the plot of $\tau_{R_{max}} = f(r)$ in Fig. 7. This velocity is equal to ~ 10 km/sec and 7 km/sec for d_0 = 0.17 mm and d_0 = 0.31 mm, respectively. It is interesting to note that these values are equal to the speed of isothermal sound in the plasma, calculated for the averaged parameters of the investigated plasma and given in[1].

3. ESTIMATE OF PLASMA-INHOMOGENEITY PARAMETERS

The individual realizations of the fields of turbulent motion satisfy the magnetohydrodynamic equations (see Eqs. (5) to (8) below). However, the equations of turbulent motion, obtained by separating the regular and pulsation parts, always prove to be indeterminate (i.e., they contain more variables than equations), and in turbulence theory it is impossible to reduce the problem to one of finding a unique solution determined by known initial and boundary conditions. To find the inhomogeneity parameters of the mean values of the pulsations it is therefore necessary to adduce additional considerations. In this case we use dimensionality considerations based on the separation of those physical parameters which affect the stationary regime in the turbulent flow under consideration. The characteristics of the turbulence, obviously, depend on only a small number of physical parameters. In the present problem such dimensional parameters are: E (the electric field intensity), σ_0 (a quantity characterizing the conductivity of the plasma: $\sigma_0 = \sigma/T^{3/2}$ where σ is the conductivity), and Q_0 (a quantity characteristic of the emissivity of the plasma: $Q_0 = q/\rho^{\alpha}T^{\alpha}$, where q is the energy flux of the radiation from a unit volume). The quantities E and σ characterize the energy input to a plasma volume with definite T and ρ, while Q_0 characterizes the energy dissipation by radiation from that volume.

We write the equations of magnetohydrodynamics in the following form:

$$\partial\rho / \partial t + \operatorname{div} \rho \mathbf{v} = 0, \qquad (5)$$

$$\rho \frac{d\mathbf{v}}{dt} = -\operatorname{grad} p + \frac{1}{c}[\mathbf{jH}], \qquad (6)^*$$

$$A\rho T \frac{d}{dt} \ln \frac{T}{\rho^{\gamma-1}} = \mathbf{jE} - Q_0\rho^{\alpha}T^{\beta}, \qquad (7)$$

$$\operatorname{rot} \mathbf{H} = 4\pi c^{-1}\mathbf{j}. \qquad (8)$$

In these equations j is the current density, c is the speed of light, H is the magnetic field intensity, T is the temperature of the plasma, ρ is the density, and $d/dt = (\partial/\partial t) + \mathbf{v}\partial/\partial\mathbf{r}$.

The second term on the right-hand side of (7) determines the energy losses due to volume radiation; we shall use henceforth the values $\alpha = 2$ and $\beta = 1/2$, since they are just right for the volume radiation losses of a lithium plasma.

In the analysis of Eqs. (5) to (8) it was assumed that $H \approx jx/c$, i.e., the magnetic field is determined by the local current. It was also assumed that one can exclude induction effects due to the motion of the plasma ($E \gg vB/c$) and that the electric field is determined only by the component due to the ohmic resistance of the plasma ($E \gg c^{-1}(\partial/\partial t) \int H dx$)[1]. Equations (5) to (8) yield the following relations for the parameters of the pulsations of the hydrodynamic quantities:

$$t \approx A\sigma_0^{-1/2} Q_0^{-1/2} E^{-1} \text{ sec}, \quad T \approx c^{4/5}A^{-4/5}\sigma_0^{-1/5} Q_0^{1/5} E^{2/5} \text{ eV}, \qquad (9)$$

$$\rho \approx c^{2/5}A^{-2/5}\sigma_0^{2/5} Q_0^{-2/5} E^{6/5} \text{ g/cm}^3, \quad x \approx c^{2/5}A^{11/10}\sigma_0^{-3/5} Q_0^{-2/5} E^{-4/5} \text{ cm},$$

$$v \approx c^{2/5}A^{1/10}\sigma_0^{-1/10} Q_0^{1/10} E^{1/5} \text{ cm/sec}, \quad p \approx c^{6/5}A^{-1/5}\sigma_0^{1/5} Q_0^{-1/5} E^{8/5} \text{ erg/cm}^3,$$

where $A = 5 \times 10^{11}$ eV-cm²/sec² and $c = 3 \times 10^{10}$ cm/sec. Here x is the mean dimension of the inhomogeneities, T is the mean temperature drop, and v is the mean velocity of the inhomogeneities.

In the numerical calculations we took the values of E and Q_0 obtained experimentally for the investigated lithium discharge of[1] at an approximate time 25 μsec and a wire with d_0 = 1.17 mm. These were E = 70 V/cm and $Q_0 = 24 \times 10^{24}$ cm^5-sec^{-3}-g-eV$^{1/2}$. The conductivity $\sigma = 4 \times 10^{13}$ sec^{-1}eV$^{3/2}$ was taken from a theoretical paper[9]. Finally, we find (for d_0 = 0.17 mm and an instant of time 25 μsec): $t \approx 2 \times 10^{-6}$ sec, T = 3.1 eV, and $\rho = 1.7 \times 10^{-5}$ g/cm³, which correspond to a particle density in the lithium plasma $n = 1.7 \times 10^{18}$ cm^{-3}, x = 3 cm, p = 20 atm, and v ≈ 23 km/sec.

It is seen from the obtained data that dimensional analysis gives for x, t and v results that are close to the experimental ones. We recall that the time (increment) of instability development, determined from high-speed photographs, was approximately 2–5 μsec[2]; the dimensions of the inhomogeneities and their velocities prove to be close to the corresponding values obtained from correlation analysis and are equal to 0.9 cm and 10 km/sec (for d_0 = 0.17 mm). Thus one can assume that our analysis gives approximately correct values for the other parameters of the turbulent flow of the investigated plasma as well. The fluctuations of the temperature, density, and pressure of the plasma do not differ greatly (by a factor on the order of 2) from the averaged parameters previously determined in[1,4]. Thus, plasma turbulization does not lead, in our case to qualitative changes in the properties of the discharge (to an essential change in the mean ion charge, an anomalous change of the conductivity, a change in the radiation properties). This is confirmed by the fact that the general discharge characteristics (total current I(t), electrode potential U(t), total light flux Φ(t)) have a smooth and continuous character.

*[jH] ≡ j × H.

4. CONCLUSION

Our investigation (a correlation reduction of the oscillograms of the local magnetic fields together with an analysis of the magnetohydrodynamic equations) of a lithium plasma obtained by exploding a wire in a vacuum has shown that the discharge is made turbulent during the first quarter-period, while the inhomogeneities have a mean linear dimension of about 1 cm and move in the plasma with the speed of sound, while the density and temperature fluctuations in the plasmoids can become appreciable (of the order of the mean values of these parameters in the plasma, $\rho \approx 1.7 \times 10^{-5}$ g/cm^3, T ≈ 3 eV). With increasing discharge mass (initial wire thickness), the turbulence of the plasma decreases, the dimensions of the plasmoids increase, and their velocity falls off.

In general outline, the picture of discharge development is the following. After the explosion of the wire, a conducting channel of inhomogeneous structure is formed at the center of the chamber at the initial instant of time. The electric field E applied to the discharge, and the flowing current I, heat up the plasma and the discharge column expands.

In a dense, optically transparent plasma ($n = 10^{17}-10^{18}$ cm^{-3}, T ≈ 3 eV), there arise instabilities that initiate and maintain turbulization of the discharge. Decay and dissipation of the turbulence set in at about 40 μsec as a consequence of instability damping. One can assume that the line emission of the plasma is responsible for the damping of the superheat instabilities. Although it does not materially determine the radiation of the investigated discharge as a whole (the line emission energy constitutes about 10% of the total radiation energy), the line emission can stabilize plasma instabilities, as attested to by calculations made for analogous discharges at the Institute of Applied Mathematics[9].

The authors thank I. A. Kosykh, whose computer program they used to calculate the correlation function.

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Translated by J. G. Adashko
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