SOVIET PHYSICS JETP VOLUME 36, NUMBER 4 APRIL, 1973

# *ELECTRON-POSITRON PAIR PRODUCTION BY A COULOMB CENTER LOCATED IN A CONSTANT FIELD*

N. B. NAROZHNYĬ and A. I. NIKISHOV

P. N. Lebedev Physics Institute, USSR Academy of Sciences

Submitted March 7, 1972

Zh. Eksp. Teor. Fiz. 63, 1135–1141 (October, 1972)

The probability for electron-positron pair production by the superposition of a Coulomb field and a constant crossed field ($E = H$, $\mathbf{E}\cdot\mathbf{H} = 0$) is determined. The crossed field is taken into account exactly, and the Coulomb field is treated in the first Born approximation. Although each of the fields separately does not create pairs (we assume that the Z of the Coulomb field is very much smaller than 137), superposition of these fields leads to appreciable pair production when the intensity of the crossed field $E \gtrsim m^2c^3/\hbar e$.

## 1. INTRODUCTION

WE considered pair production by an intense electromagnetic wave on a heavy particle, which did not interact with the field of the wave, in 1964 (unpublished, see [1]). A similar investigation was carried out independently by Yakovlev.[2] A series of articles by the authors of [3,4] has been devoted to the problem of the pair production of particles moving in a constant magnetic field. However, in spite of the rather large number of articles on this question, up till now the important case of electron-positron pair production by a Coulomb center located in a constant crossed field (a homogeneous field with $E = H \equiv B$, $\mathbf{E}\cdot\mathbf{H} = 0$) has not been investigated in detail. Two reasons account for the importance of this case: in the first place, for an ultrarelativistic particle (not traveling along the direction of the field) a constant field of general form looks like (with a large degree of accuracy) a crossed field in the particle's rest frame; in the second place, thanks to the large value of the ratio of the proton mass to the electron mass, one would expect the approximation in which the proton mass (mass of the nucleus) is assumed to be infinite to have a considerable range of validity.

It should be noted that the problem considered here has also recently gained considerable importance from the point of view of astrophysical estimates of the role of spontaneous pair production in the energy losses of heavy particles moving in intense electromagnetic fields. Thus, the energy region in which the major mechanism of energy loss is not bremsstrahlung but pair production is estimated in the work by Prylutsky and Rosental [5] for the case of protons moving in a magnetic field. These authors have also considered various consequences of pair production by ultrahigh-energy protons in the magnetic field of the sun and of the planets. On the other hand, knowledge of the detailed behavior of the probability for pair production by protons in the intermediate region $\zeta \sim 1$, where $\zeta = B/B_0$ ($B_0 = m^2/e = 1.25 \times 10^{13}$. Heaviside units is the intensity of the so-called critical field), is essential for estimates of the mechanism responsible for the acceleration of protons and nuclei in the shell of pulsars. G. B. Khristiansen has called our attention to this fact.

The differential and total probabilities for pair production by a Coulomb center located in a constant f crossed field are calculated in the present article. The total probability, which is represented in the form of a double integral, depends on the single parameter $\zeta$. The limiting cases of weak ($\zeta \ll 1$) and strong ($\zeta \gg 1$) fields are investigated, and the results of numerical calculations in the intermediate region are presented. Our results for the strong-field case qualitatively agree with the results obtained by other authors.[2-4]

In parallel with our work, Ritus [6] has considered the same problem, starting from the formula which he derived for the mass operator of a charged particle in a crossed field. We are happy to report that the expressions for the total probability agree.

## 2. THE PROBABILITY OF PAIR PRODUCTION

Let us write the 4-vector potential of the crossed field in the form[1)] $a_\mu kx$, where $ka = k^2 = 0$. Using the method of calculating the probabilities for quantum processes in such fields (compare with [7]), we obtain the following expression for the square of the matrix element for pair production, summed over the polarizations of the particles:

$$\sum_{s,s'}|M_{p'p}|^2 = \frac{8e^2}{k_0^2 p_0 p_0'}(4\beta)^{-2/3}\left\{\left[\left(A, p+\frac{\alpha}{8\beta}ea\right)\left(A, p'-\frac{\alpha}{8\beta}ea\right) - \frac{q^2}{4}A^2 - 2\beta(kq)\frac{\sigma m^2}{e^2a^2}A^2\right]\Phi^2(y) - (4\beta)^{-2/3}[e^2(aA)^2 + 2\beta(kq)A^2]\Phi'^2(y)\right\}. \quad (1)$$

Here $p'$, $p$, and $q$ are the 4-momenta respectively of the electron, positron, and virtual photon,

$$\alpha = e\left(\frac{ap}{kp} - \frac{ap'}{kp'}\right), \quad \beta = -\frac{e^2a^2}{8}\left(\frac{1}{kp} + \frac{1}{kp'}\right) = -\frac{e^2a^2(kq)}{8(kp)(kp')},$$

$$y = (4\beta)^{2/3}\frac{\sigma m^2}{e^2a^2}, \quad \sigma = 1 + \frac{q^2}{m^2}\frac{\chi'}{\varkappa} - \frac{q^2}{m^2}\frac{\chi'^2}{\varkappa^2} + \tau^2,$$

$$\tau = \frac{eF_{\mu\nu}^* p_\mu p_\nu'}{m^4\varkappa}, \quad \chi' = \frac{[(eF_{\mu\nu}p_\nu')^2]^{1/2}}{m^3}, \quad \varkappa = \frac{[(eF_{\mu\nu}q_\nu)^2]^{1/2}}{m^3}$$

$$F_{\mu\nu}^* = \frac{i}{2}\varepsilon_{\mu\nu\sigma\lambda}F_{\sigma\lambda}, \quad A_\mu = A_\mu' - \frac{kA'}{kq}q_\mu, \quad (2)$$

1) We shall use the units $\hbar = c = 1$, $e^2/4\pi = \alpha = 1/137$, and the notation $ab \equiv (ab) = \mathbf{a}\cdot\mathbf{b} - a_0b_0$.

where the vector $A'_\mu = A'_\mu(q)$ is related to the Fourier transform of the Coulomb potential $A''_\mu(q)$ by the relation

$$A_\mu''(q) = 2\pi\delta(q_0)A_\mu'(q) = 2\pi\delta(q_0)(0,\,0,\,0,-iZe/q^2). \tag{3}$$

The Airy function $\Phi(y)$ is defined by the relation

$$\Phi(y) = \int_0^\infty dt\cos\left(\frac{t^3}{3}+yt\right), \qquad \Phi'(y) = \frac{\partial\Phi(y)}{dy}. \tag{4}$$

The total probability for pair production is given by

$$W = \int\sum_{s,s'}|M_{p'p}|^2\frac{d^3p'\,d^3p}{(2\pi)^6}. \tag{5}$$

In what follows it is convenient to use the coordinate system in which the 4-vectors $a_\mu$ and $k_\mu$ have the components

$$a_\mu = (a,\,0,\,0,\,0), \qquad k_\mu = (0,\,0,k_0,\,ik_0). \tag{6}$$

In this connection the vectors $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{E}\times\mathbf{H}$ of the crossed field will point along the 1-, 2-, and 3-axes, respectively, and $E = H = ak_0 \equiv B$. The quantum numbers of the Volkov solution of the Dirac equation[7] are $p_1$, $p_2$, and $p_-$. Furthermore, the components $p_3$ and $p_0$ of the 4-vector $p_\mu$ are determined from the conditions $p^2 = -m^2$, $p_- = p_0 - p_3$ so that

$$p_3 = \frac{m^2+p_\perp{}^2-p_-{}^2}{2p_-}, \qquad p_0 = \frac{m^2+p_\perp{}^2+p_-{}^2}{2p_-}, \qquad p_\perp{}^2 = p_1{}^2+p_2{}^2. \tag{7}$$

Accordingly, let us change in Eq. (5) from an integration over $p_3$ and $p_3'$ to an integration over $p_-$ and $p_-'$:

$$\int_{-\infty}^{\infty}\frac{dp_3}{p_0}\int_{-\infty}^{\infty}\frac{dp_3'}{p_0'}\ldots = \int_0^\infty\frac{dp_-}{p_-}\int_0^\infty\frac{dp_-'}{p_-'}\ldots \tag{8}$$

Now we note that the process under consideration is characterized by the conservation laws

$$q_1 = p_1+p_1', \qquad q_2 = p_2+p_2', \qquad q_- = -q_3 = p_-+p_-', \tag{9}$$

Furthermore, it is convenient to change from the integration variables $p_1$, $p_1'$ to the variables $q_1$, $q_1' = p_1 - p_1'$. The square of the matrix element does not depend on $q_1'$, and here

$$\int dq_1' = 2eBT, \tag{10}$$

where $T$ denotes the total time for observation of the process. In fact, the integral over the phase $\varphi' = k_0(x_3 - t)$

$$\int_{-\infty}^{\infty}d\varphi'\exp\left[-is\varphi'+i\frac{\alpha}{2}\varphi'^2-i\frac{4\beta}{3}\varphi'^3\right]$$

in the matrix element (compare with[7]) reduces to the Airy function by the replacement of $\varphi'$ by $z'$:

$$\varphi' = \frac{\alpha}{8\beta}-\frac{z'}{(4\beta)^{1/3}} = \frac{1}{2ea}\left[\frac{q_1}{q_-}(p_--p_-')-q_1'\right]-\frac{z'}{(4\beta)^{1/3}}. \tag{11}$$

Since the effective values of $z'$ and $q_1$ are finite, the "center of gravity" of the effective values of $\varphi'$ is determined by the value of $q_1'$, which also leads to expression (10).

Then changing from the variables $p_2$, $p_-$, $p_2'$, and $p_-'$ to the variables $q_2$, $q_-$, $\tau$, and $\rho = p_-'/q_-$, we obtain

$$T^{-1}W = \frac{\alpha^2Z^2m}{\pi^4\zeta}\int_0^\infty dq_\perp q_\perp\int_0^{2\pi}d\varphi\int_0^\infty dq_-q_-\int_0^1 d\rho\int_{-\infty}^{\infty}d\tau\,\frac{\sigma\{\ldots\}}{q^4yp_-p_-'}$$

$$q_1 = q_\perp\cos\varphi, \quad q_2 = q_\perp\sin\varphi, \quad q^2 = q_\perp{}^2+q_-{}^2,$$

$$\tau = \frac{eF_{\mu\nu}{}^*p_\mu p_\nu'}{m^4\varkappa} = \frac{p_-p_2'-p_-'p_2}{mq_-}. \tag{12}$$

The symbol $\{\ldots\}$ in Eq. (12) has the following meaning:

$$\{\ldots\} = \frac{2}{q_-{}^2}\left[\frac{m^2q_\perp{}^2(1+\tau^2)}{4\rho(1-\rho)}+\rho(1-\rho)q^4+m\tau q_\perp q^2(1-2\rho)\sin\varphi\right.$$
$$\left.-m^2\tau^2q_\perp{}^2\sin^2\varphi\right]\Phi^2(y)+\frac{2m^2\sigma}{yq_-{}^2}\left[\frac{q_\perp{}^2}{4\rho(1-\rho)}-q_\perp{}^2\cos^2\varphi\right]\Phi'^2(y),$$
$$y = \left(\frac{m}{2\zeta\rho(1-\rho)q_-}\right)^{2/3}\sigma.$$

Here the integration over $\varphi$ is elementary to perform. Then integrating over $\tau$ (compare with[8,9]) we obtain

$$T^{-1}W = \frac{\alpha^2Z^2m}{\pi^2}\int_0^\infty du\int_1^\infty\frac{dx}{x}\int_4^\infty\frac{dv}{v^2[v(v-4)]^{1/2}}\left\{\left[4+v\left(\frac{2}{xu^2}-1\right)(x-1)\right.\right.$$
$$\left.\left.+6(x-1)\right]\Phi_1(z)-2\left(1-\frac{1}{x}\right)\frac{v(v-2)}{u^2}\left(\frac{\zeta u}{v}\right)^{2/3}\Phi'(z)\right\};$$
$$z = \left(\frac{v}{\zeta u}\right)^{2/3}\left(1+\frac{xu^2}{v}\right), \qquad x = \frac{q^2}{q_-{}^2}, \qquad u = \frac{q_-}{m}, \tag{13}$$
$$v = \frac{1}{\rho(1-\rho)} = \frac{q_-{}^2}{p_-p_-'}, \qquad \zeta = \frac{B}{B_0}, \qquad \Phi_1(z) = \int_z^\infty dt\,\Phi(t).$$

By making the appropriate change of variables, one can carry out one more integration in (13). Assuming, for example, that

$$u = w/x, \qquad v = 4s/x$$

and then integrating with respect to $x$, we obtain

$$T^{-1}W = \frac{\alpha^2Z^2m}{\pi^2}\int_0^\infty dw\int_1^\infty\frac{ds}{s^{5/2}}\left\{\frac{2s}{3w^2}(s-1)^{3/2}+\frac{1}{2}s^{1/2}\,\mathrm{Arth}\left(1-\frac{1}{s}\right)^{1/2}\Phi_1(z')\right.$$
$$\left.-\left[\frac{2s+1}{3}(s-1)^{1/2}-s^{1/2}\,\mathrm{Arth}\left(1-\frac{1}{s}\right)^{1/2}\right]\frac{4s}{w^2}\left(\frac{\zeta w}{4s}\right)^{2/3}\Phi'(z')\right\},$$
$$z' = (4s/\zeta w)^{2/3}(1+w^2/4s). \tag{14}$$

The integrands in expressions (12)—(14) have the meaning of differential probabilities. However, if one is not interested in differential distributions, then one can simplify formulas (13) and (14) by integrating by parts. Thus, integrating the terms of the integral containing $(x - 1)\Phi_1(z)$ twice by parts with respect to $v$ (and using the relation $\Phi(z) = z^{-1}\Phi''(z)$), from expression (13) we obtain the formula first found by Ritus:[6]

$$T^{-1}W = \frac{4\alpha^2Z^2m}{\pi^2}\int_0^\infty du\int_1^\infty\frac{dx}{x}\int_4^\infty\frac{dv}{v^2[v(v-4)]^{1/2}}\left\{\Phi_1(z)\right.$$
$$\left.-\frac{v(2v+1)}{6u^2}\left(\frac{\zeta u}{v}\right)^{2/3}\left(1-\frac{1}{x}\right)\Phi'(z)\right\}.$$

In similar fashion one can easily find the following expression instead of (14):

$$T^{-1}W = \alpha^2Z^2mJ(\zeta),$$
$$J(\zeta) = \frac{1}{2\pi^2}\zeta^{2/3}\int_{z_0}^\infty\frac{dz'}{z'}[-\Phi'(z')]\int_{\theta_1}^{\theta_2}\frac{d\theta}{\theta^{1/2}}\left\{\left[\frac{7+2s}{6s}+\frac{(1-s)\theta}{sz'\zeta^{2/3}}\right.\right.$$
$$\left.\left.+\frac{\theta^2z'\zeta^{2/3}(13s-1)}{24s^2}\right]\left(1-\frac{1}{s}\right)^{1/2}-\frac{\theta^2z'\zeta^{2/3}}{2s^2}\mathrm{Arth}\left(1-\frac{1}{s}\right)^{1/2}\right\}, \tag{15}$$
$$z_0 = 3/\zeta^{2/3}, \quad s = {}^1/_4\theta^2[z'\zeta^{2/3}-\theta].$$

Here $\theta_1$ and $\theta_2$ are the positive roots of the equation

$$z'\zeta^{2/3} = \theta + 4/\theta^2. \tag{16}$$

The results of a numerical calculation of the function $J(\zeta)$ in formula (15) are shown in the accompanying figure.

## 3. LIMITING CASES AND CONCLUSIONS

In the limiting case of a weak field, $\zeta \ll 1$, the values $z_{\mathrm{eff}} \gg 1$ are effective, and the Airy function can be replaced by its asymptotic expression.[10] Then we obtain[1]

$$T^{-1}W = \frac{\alpha^2 Z^2 m}{2\sqrt{\pi}}\left(\frac{\zeta}{2\sqrt{3}}\right)^{5/2} \exp\left(-\frac{2\sqrt{3}}{\zeta}\right). \tag{17}$$

Let us cite the effective values of the integration variables appearing in expressions (12)–(14):

$$\begin{gathered}(v-4)_{\mathrm{eff}},\quad (x-1)_{\mathrm{eff}},\quad (u-2^{1/2})^2_{\mathrm{eff}} \equiv (q_-/m - 2^{1/2})^2_{\mathrm{eff}} \sim \zeta,\\ \tau^2_{\mathrm{eff}} \approx (p_2'/m)^2_{\mathrm{eff}} \sim \zeta, \quad (q_\perp/m)^2_{\mathrm{eff}}, \quad (p_-'/m - 2^{-1/2})^2_{\mathrm{eff}} \sim \zeta.\end{gathered} \tag{18}$$

It follows from (18) that in the effective region $q_-$ (which is equal to $p_- + p_-'$) $\sim m2^{1/2}$. This means that even in a weak field the particles of the pair are created relativistically (otherwise one would have $q_- \approx 2m$ and $p_-, p_-' \approx m$). However, the relative velocity of a pair's components is not large: $v_r \sim \zeta^{1/2} \ll 1$, so that for $\alpha/v_r \approx \alpha/\zeta^{1/2} \gtrsim 1$ it would also be necessary to take their Coulomb interaction into consideration. In the opposite limiting case of a strong field we have[2)]

$$\begin{gathered}T^{-1}W = \alpha^2 Z^2 m J(\zeta),\\ J(\zeta) = \frac{13}{6}\frac{\zeta}{3^{1/2}\pi}\left\{\ln\frac{\zeta}{2\cdot 3^{1/2}} - C - \frac{58}{39}\right\}, \qquad \zeta^{1/3} \gg 1,\end{gathered} \tag{19}$$

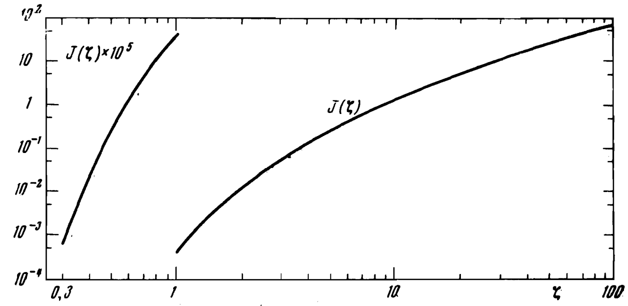
where C = 0.577 is Euler's constant. The effective values of the integration variables are as follows:

$$\left(\frac{p_-'}{m}\right)^2_{\mathrm{eff}}, \left(\frac{q_-}{m}\right)^2_{\mathrm{eff}} \sim \zeta^{-2}, \left(\frac{p_2'}{m}\right)^2_{\mathrm{eff}} \sim 1, \left(\frac{q}{m}\right)^2_{\mathrm{eff}} \sim \zeta^{-2} \div 1. \tag{20}$$

Formula (19) agrees qualitatively with the results obtained in[2,3], where only the logarithmic term was considered. The coefficient associated with the logarithm agrees relatively well with the approximate calculation by Erber.[3]

We note that with respect to the method of calculation, Erber's result refers to pair production by a Coulomb center although it is applied in order to estimate the probability for pair production by an electron. The case of pair production by a Coulomb center was not considered in the article by Baĭer et al.,[4] but one can obtain the result from the formula for $W_{\mathrm{OF}}$ (containing a fourfold integral) if the mass of the initial particle tends to infinity. Here the contributions to the probability coming from $W_{\mathrm{FF}}$ and $W_{\mathrm{FO}}$ must vanish (as to the meaning of the notation $W_{\mathrm{FF}}$, $W_{\mathrm{OF}}$, and $W_{\mathrm{FO}}$, see [4]). By making such a change in the second formula (31) of[4] and taking account of the factor ½ which was omitted in the initial formula, we obtain the logarithmic term of our formula (19). We note that the conditions for the applicability of the second formula (31) in[4] turn out to be less stringent than the conditions cited in[4], and they allow the limiting transition to the case of infinite mass.

The results obtained so far are valid for pair production by a Coulomb center (provided $\alpha Z \ll 1$). However, under condition (23) one can also use these results to obtain the probability for pair production by nuclei. If a nucleus of mass M has 4-momentum $P_\mu$ in the laboratory reference frame, then taking account of the motion of the nucleus amounts to making the following substitutions in formulas (12)–(15) for the total probability:




Plot of the function $J(\zeta)$ determining the probability for pair production (see formula (15)).

$$\zeta \to [(eF_{\mu\nu}P_\nu)^2]^{1/2}/Mm^2, \qquad T \to TM/P_0. \tag{21}$$

If the quantities $q_-/m$ and v are also written in invariant form

$$\begin{gathered}q_-/m = \varkappa/\zeta, \quad \varkappa = [(eF_{\mu\nu}q_\nu)^2]^{1/2}/m^3,\\ v = (F_{\mu\nu}q_\nu)^2/[(F_{\mu\nu}p_\nu)^2(F_{\mu\nu}p_\nu')^2]^{1/2}\end{gathered} \tag{22}$$

and if the variables $\zeta$, $q_-/m$, $q^2$, $\tau$, and v are understood as the corresponding invariants, then the integrands appearing in formulas (12)–(14) will represent the differential probabilities for pair production by a heavy particle in the field $F_{\mu\nu}$, in particular, in the presence of a magnetic field.

Finally, the last remark, which we owe to Ritus, concerns the range of validity of the derived results. For $\zeta \gg 1$ the effective value of $q_-/m$ is $\sim\zeta^{-1}$, so that for $\zeta \sim (M/m)^{1/2}$ the work of the field over the length $(1/q_-)_{\mathrm{eff}}$ is of the order of M, that is, during the evolution time of the process the heavy particle's state of motion is substantially changed by the field, and it is necessary to take this change into consideration. Therefore, our results are applicable for nuclei provided that

$$\zeta \ll (M/m)^{1/2}. \tag{23}$$

In this connection, formula (19) does not have a region of validity for pair production by nuclei. In fact, the function $J(\zeta)$ in formula (15) reaches the asymptotic regime (19) for values of $\zeta$ ($\zeta^{1/3} \gg 1$, that is, $\zeta \gtrsim 10^3$) at which the approximation of an infinitely massive nucleus is no longer valid.

In conclusion we thank V. I. Ritus for fruitful discussions. We also thank A. T. Matachun for programming the numerical calculations.

[2)]Some of the details of the derivation of formula (19) are given in our preprint [11].


[1]N. B. Narozhnyĭ, Thesis, MIFI (1964).

[2]V. P. Yakovlev, Zh. Eksp. Teor. Fiz. 49, 318 (1965) [Sov. Phys.-JETP 22, 223 (1966)].

[3]Thomas Erber, Rev. Mod. Phys. 38, 626 (1966).

[4]V. N. Baĭer, V. M. Katkov, and V. M. Strakhovenko, Yad. Fiz. **14**, 1020 (1971) [Sov. J. Nucl.-Phys. **14**, 572 (1972)].
[5]O. P. Prylutsky and I. L. Rosental, 12th International Conference on Cosmic Rays, Conference Papers, Vol. 2, Tasmania (1971), p. 1801.
[6]V. I. Ritus, Nucl. Phys. **B44**, 236 (1972).
[7]A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **46**, 776, 1768 (1964) [Sov. Phys.-JETP **19**, 529, 1191 (1964)].
[8]David E. Aspnes, Phys. Rev. **147**, 554 (1966).
[9]A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **52**, 1707 (1967) [Sov. Phys.-JETP **25**, 1135 (1967)].
[10]L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika (Quantum Mechanics), Fizmatgiz, 1963, Sec. C. (English Transl., Pergamon Press, 1965 pp. 596–597).
[11]N. B. Narozhny and A. I. Nikishov, FIAN Preprint N-148 (1971).

Translated by H. H. Nickle
121