NONLINEAR PLASMA ION OSCILLATIONS EXCITED BY A CURRENT

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The nonlinear phase of the development of current-excited ion Langmuir oscillations of a plasma is investigated just above the threshold of the system. The time evolution of the amplitude of a linearly unstable wave is studied up to the saturation stage. It is shown that under the conditions considered the effective mechanism that limits the growth of the amplitude is the nonlinear drift of the frequency of the excited wave. The investigation is carried out for a weak-collision plasma, in which oscillation buildup is due to the electron Cerenkov effect, as well as for the case of frequent collisions, when inverse conductivity and electron diffusion in the current-carrying plasma become the cause of the instability.

1. INTRODUCTION

THE present work is a continuation of the papers[1,2], in which we investigated the nonlinear phase of the development of oscillations in the acoustic part of the spectrum of the ion-sound instability of a dense nonisothermal ($T_e \gg T_i$), current-carrying plasma. Here, the results of[1,2] are supplemented by an investigation of the unstable, short-wavelength, ion Langmuir oscillations and are generalized to the case of a plasma in which collisions are infrequent and weak and the buildup of the oscillations are due to the electron Cerenkov effect. In consequence, the theory of the ion-sound instability of a nonisothermal, current-carrying plasma near the threshold assumes a final form.

Let us note a few distinctive features of the investigation. Having in mind a highly nonisothermal plasma, we shall, for simplicity, assume the ions to be cold and neglect effects due to the ion-oscillation interaction. Furthermore, as has already been noted, our investigation pertains to the case when the system is just above its threshold. We then have a narrow spectrum of oscillations excited in the plasma[1)], and we can restrict ourselves to the study of only one mode with the maximum increment, neglecting mixing of the phases of the excited oscillations. Such an analysis nevertheless allows us to elucidate a number of distinctive features characteristic of an unstable plasma (e.g., conditions of excitation, anomalous convection of the plasma, etc.[3]), and to investigate the time dependence of the amplitude of the unstable mode and the mechanism leading to its saturation. In the considered case of excitation of short-wavelength ion Langmuir oscillations, the mechanism limiting the growth of the amplitude is the nonlinear drift of the frequency of the excitable mode.

2. PLASMA-STABILITY BOUNDARY

According to the linear theory (see, for example,[4]), a current-carrying nonisothermal plasma in which the electrons drift relative to the ions with velocity $\mathbf{u}$ is unstable with respect to the excitation of ion-sound oscillations. Under these conditions ion Langmuir oscillations with

$$\omega^2 \approx \omega_{Li}^2 \tag{2.1}$$

and increment

$$\gamma = \begin{cases} \sqrt{\dfrac{\pi}{8}\dfrac{m}{M}}\dfrac{\omega_{Li}}{k^3 r_{De}^3}\left(\dfrac{\mathbf{ku}}{\omega}-1\right)-\dfrac{\nu_i}{2} & \text{for } \nu_e < kv_{Te}, \\ \dfrac{m}{2M}\dfrac{\nu_e}{k\,r_{De}^4}\left(\dfrac{\mathbf{ku}}{\omega}-1\right)-\dfrac{\nu_i}{2} & \text{for } \nu_e > kv_{Te}. \end{cases} \tag{2.2}$$

are excited in the short-wavelength part of the spectrum ($k^2r_{De}^2 \gg 1$). In (2.1) and (2.2) $\omega_{Li} = \sqrt{4\pi e^2N_0/M}$ is the ion Langmuir frequency, $r_{De} = v_{Te}/\omega_{Le}$ is the Debye radius of the electrons, $v_{Te} = \sqrt{T_e/m}$ is their thermal velocity, ν_e is the rate of collisions of electrons with ions or neutral atoms, and, finally, $\nu_i = \nu_{i0}$ is the rate of collisions between ions and neutral atoms in a weakly ionized plasma and $\nu_i = {}^8/_5\nu_{ii}k^2v_{Ti}^2/\omega_{Li}^2$, the rate in a completely ionized plasma. Notice that the collision limit ($\nu_e > kv_{Te}$) in the case of short wavelengths being considered is realized in a gas with a very low degree of ionization, when the mean free path of the electrons turns out to be smaller than their Debye radius and $\nu_e > \omega_{Le}$.

It follows from the expressions (2) that the oscillations propagating along the direction of drift of the electrons have the maximum growth constant, the stability boundary for a weakly ionized plasma being defined by the relations

$$\begin{aligned} &\frac{u_{\text{lim}}}{v_s}=\frac{3}{2}\left(\sqrt{\frac{8}{\pi}\frac{M}{m}}\frac{\nu_{i0}}{\omega_{Li}}\right)^{1/2}, k_{\text{lim}}r_{De}=\frac{3}{2}\frac{v_c}{u_{\text{lim}}} \text{ for } \nu_e < kv_{Te}, \\ &\frac{u_{\text{lim}}}{v_s}=\frac{4}{3}\left(\frac{3M}{m}\frac{\nu_{i0}}{\nu_e}\right)^{1/4}, k_{\text{lim}}r_{De}=\frac{4}{3}\frac{v_s}{u_{\text{lim}}} \text{ for } \nu_e > kv_{Te}, \end{aligned} \tag{2.3)}$$

and for a completely ionized plasma by

$$\frac{u_{\text{lim}}}{v_s}=\frac{5}{4}\left(4\sqrt{\frac{2}{\pi}\frac{M}{m}}\frac{T_i}{T_e}\frac{\nu_{ii}}{\omega_{Li}}\right)^{1/5}, k_{\text{lim}}r_{De}=\frac{5}{4}\frac{v_s}{u_{\text{lim}}}. \tag{2.4}$$

Here, $v_s = \sqrt{T_e/M}$ is the velocity of isothermal sound in the plasma. It follows from the formulas (2.3) that in contrast to long wavelength sound vibrations, short

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1) The relative width of the spectrum of the excitable oscillations $\Delta k/k \sim \sqrt{\epsilon}$, where ϵ is a parameter indicating how far the system is above the threshold (see below).

wavelength ion Langmuir oscillations can be excited at relatively small electron velocities $u < v_s$: but for this to happen, the following conditions must be fulfilled: $\omega_{Li} > \nu_{ie}\sqrt{M/m}$ in the case of a weakly ionized plasma[2)] and $\omega_{Li} > \nu_{ii}(T_i/T_e)\sqrt{M/m}$ for a completely ionized plasma. These conditions are assumed to be fulfilled below.

3. EQUATION FOR THE AMPLITUDE OF A NONLINEAR ION LANGMUIR WAVE

Sagdeev[5] has shown that stationary nonlinear ion-sound waves can exist in a conservative system (in the absence of dissipation) and has studied the properties of these waves. Allowance for dissipation leads, in the presence of a current in the plasma, to the appearance of an ion-sound instability. Near the instability threshold, when $\gamma \ll \omega$, dissipation may be allowed for in the linear approximation. Furthermore, as follows from the linear treatment, in the region of short-wavelength ion Langmuir oscillations, the perturbations of the electron density and velocity are negligibly small:

$$\rho_e/\rho_i \sim v_e/v_i \sim 1/k^2r_{De}^2 \ll 1. \tag{3.1}$$

This permits us to neglect the nonlinearities due to the perturbations of the electron component and write the basic system of equations describing the development of short-wavelength ion Langmuir oscillations in a nonisothermal, current-carrying plasma in the following form:

$$\frac{\partial f_e}{\partial t} + \mathbf{v}\frac{\partial f_e}{\partial \mathbf{r}} - \frac{e}{m}\mathbf{E}\frac{\partial f_{0e}}{\partial \mathbf{v}} = \frac{\nu_e}{N_0}\frac{\partial f_{0e}}{\partial \mathbf{v}}\int f_e \mathbf{v}\,d\mathbf{v}, \tag{3.2}$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} + \nu_i\mathbf{V} = \frac{e}{M}\mathbf{E}, \tag{3.3}$$

$$\frac{\partial N}{\partial t} + \operatorname{div} N\mathbf{V} = 0, \tag{3.4}$$

$$\frac{e}{M}\operatorname{div}\mathbf{E} = \omega_{Li}^2\frac{N - N_e}{N_0}. \tag{3.5}$$

Here, f_e is the perturbation of the electron distribution function $f_{0e}(v)$ (a Maxwellian distribution with drift[3)]), $\mathbf{E}$ is the perturbation of the electric field, N and $\mathbf{V}$ are the density and hydrodynamic velocity of the ions and, finally, $N_e = N_0 + \int f_e dv$ is the electron density. Collisions are taken into account in the electron kinetic equation (3.2) with the aid of the model collision integral[6] describing the isotropization of the electron distribution function when the electrons collide with heavy particles.

Eliminating the quantities N and $\mathbf{E}$ from Eqs. (3.3)–(3.5), we reduce them to one nonlinear equation

$$\operatorname{div}\left[\frac{\partial^2 \mathbf{V}}{\partial t^2} + \omega_{Li}^2\mathbf{V} + \nu_i\frac{\partial \mathbf{V}}{\partial t} + \frac{\partial}{\partial t}(\mathbf{V}\nabla)\mathbf{V} + \mathbf{V}\operatorname{div}\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V}\right)\right] + \omega_{Li}^2\frac{\partial \rho_e}{\partial t} = 0, \tag{3.6}$$

where

$$\rho_e = \frac{1}{N_0}\int f_e dv$$

is the relative perturbation of the electron density. It is easy to obtain from the kinetic equation (3.2) the expression for the Fourier transform $\rho_e(\mathbf{k}, \omega)$:

$$\rho_e(\mathbf{k},\omega) = \frac{e}{M}\frac{i\mathbf{kE}}{k^2v_s^2}\left[1 + i\left(\sqrt{\frac{\pi}{2}} + \frac{\nu_e}{kv_{Te}}\right)\frac{\omega - \mathbf{uk}}{kv_{Te}}\right]. \tag{3.7}$$

We obtain from the relation (3.7):

$$v_s^2\nabla\rho_e = -\frac{e}{M}\mathbf{E} + \frac{e}{M}\int d\mathbf{r}'Q(\mathbf{r}-\mathbf{r}')\left(\frac{\partial}{\partial t} + \mathbf{u}\nabla_\parallel\right)\mathbf{E}(\mathbf{r}', t), \tag{3.8}$$

where

$$Q(\mathbf{R}) = (2\pi)^{-3}\int d\mathbf{k}Q(k)e^{i\mathbf{kR}}, \qquad Q(\mathbf{k}) = \frac{1}{kv_{Te}}\left(\sqrt{\frac{\pi}{2}} + \frac{\nu_e}{kv_{Te}}\right). \tag{3.9}$$

Taking into account (3.8) and the smallness of the dissipative terms, we can write Eq. (3.6) in the short-wavelength region of the spectrum in the form

$$\operatorname{grad}\operatorname{div}\left[\omega_{Li}^2\mathbf{V} + \frac{\partial^2\mathbf{V}}{\partial t^2} + \nu_i\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\operatorname{div}\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V}\right) + \frac{\partial}{\partial t}(\mathbf{V}\nabla)\mathbf{V}\right] + \frac{\omega_{Li}^2}{v_s^2}\int d\mathbf{r}'Q(\mathbf{r}-\mathbf{r}')\left(\frac{\partial}{\partial t} + (\mathbf{u}\nabla)\right)\frac{\partial^2\mathbf{V}}{\partial t^2} = 0. \tag{3.10}$$

It is easy to show that Eq. (3.10) corresponds, in the limit $\nu_e > kv_{Te}$, to the standard hydrodynamic (inertialess) description of the motion of electrons, and, in the limit $\nu_e < kv_{Te}$, to the collisionless hydrodynamics of a nonisothermal plasma[7,8]. It is this equation that determines the development in the short-wavelength region of the spectrum of an ion-sound instability in a nonisothermal, current-carrying plasma.

4. TIME EVOLUTION AND SATURATION OF A NONLINEAR ION LANGMUIR WAVE

We shall henceforth limit ourselves to the study of the development of a one-dimensional ion-sound instability for waves propagating along the drift of the electrons, i.e., $\nabla \parallel \mathbf{u} \parallel \mathbf{V}$. The investigation of such waves is important, since, in the first place, according to the linear theory, they possess the maximum increment and, secondly, they are weakly stabilized by induced scattering on the ions[3,9]. In the one-dimensional case Eq. (3.10) can be represented in the form

$$\frac{\partial^2 V}{\partial t^2} + \omega_{Li}^2V + \nu_i\frac{\partial V}{\partial t} + V\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial t}V\frac{\partial V}{\partial x} - \frac{\omega_{Li}^2}{v_s^2}\int dx'Q_1(x-x')\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x'}\right)\frac{\partial^2 V}{\partial t^2} = 0, \tag{4.1}$$

where

$$Q_1(x) = \frac{1}{2\pi}\int dk\,e^{ikx}Q_1(k), \quad Q_1(k) = \frac{1}{k^2}Q(k). \tag{4.2}$$

In the linear approximation, it follows from Eq. (4.1) that the amplitude of the ion Langmuir oscillations with the spectrum (2.1) increases exponentially in time with the growth constant (2.2). It is convenient in the analysis of the nonlinear phase of the development of the instability to introduce a parameter ϵ which indicates how far the system is above the threshold:

$$\varepsilon = u/u_{\lim} - 1, \tag{4.3}$$

where $u_{\lim}$ is given by the expressions (2.3) and (2.4). Near the instability threshold, when $0 < \epsilon \ll 1$, it is

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[2)]Notice that the indicated inequality pertains to the weak collision ($\nu_e < kv_{Te}$) limit. In the strong-collision case ($\nu_e > kv_{Te}$), it is necessary, according to (2.3), that $mT_e > MT_i$.

[3)]It is assumed that the external electric field is small compared with the Dreicer field and, therefore, the distortion of the distribution function in the collisionless limit due to the appearance of "run-away" electrons can be neglected.

natural to suppose that the main contribution to the profile of the excited—in the plasma—wave with the maximum growth constant is made by the first few harmonics of the mode $k = k_{lim}$. Limiting ourselves to the first two harmonics, we shall seek the solution of Eq. (4.1) in the form (see also the Appendix)

$$V = V_1 e^{i(kx-\omega t)} + V_2 e^{2i(kx-\omega t)} + \text{c.c.}, \qquad (4.4)$$

where $v_{1,2}(t)$ are slowly varying amplitudes and ω is an unknown real frequency. Substituting (4.4) into (4.1) and equating the coefficients of the same exponential functions, we obtain

$$\begin{gathered}\frac{dV_1}{dt} - \left(i\frac{\omega^2-\omega_{Li}^2}{2\omega}+\gamma\right)V_1 + 3ikV_1^*V_2 - ik^2V_1^*V_1^2/\omega = 0,\\ \frac{dV_2}{dt} + \beta\frac{\nu_i}{2}V_2 + \frac{3}{4}ikV_1^2 - \frac{3}{4}i\omega_{Li}V_2 = 0.\end{gathered} \qquad (4.5)$$

Here, γ is the linear growth constant near the instability threshold and $\beta \equiv (Q(k) - Q(2k))/Q(k)$ (i.e., $\beta = 0.5$ when $\nu_e < kv_{Te}$, and $\beta = 0.75$ when $\nu_e > kv_{Te}$).

Introducing the dimensionless quantities $A_1 = V_1V_1^*/v_s^2$ and $\tau = \gamma t$, and taking into account the fact that near the instability threshold $\gamma \sim \nu_i\epsilon \ll \nu_i$, we find from the system (4.5) the equation determining the time evolution of the amplitude of the fundamental mode of the ion oscillations that can be excited by a current:

$$\frac{1}{2}\frac{dA_1}{d\tau} = A_1 - \frac{2\beta\nu_i}{\gamma}k^2r_{De}^2A_1^2. \qquad (4.6)$$

The solution of this equation has the form

$$A_1(\tau) = \left\{\frac{2\beta\nu_i}{\gamma}k^2r_{De}^2 + \left(\frac{1}{A_1(0)} - \frac{2\beta\nu_i}{\gamma}k^2r_{De}^2\right)e^{-2\tau}\right\}^{-1}, \qquad (4.7)$$

where $A_1(0)$ is the initial value determined by thermal noise in the plasma. We find from (4.7), in particular, the amplitude of the stationary, short-wavelength ion Langmuir oscillations[4)]

$$A_1(\infty) = \frac{\gamma}{2\beta\nu_i}\frac{\omega_{Li}^2}{k^2v_s^2} = \frac{\alpha}{2\beta}\frac{1}{k^2r_{De}^2}\varepsilon, \qquad (4.8)$$

where $\alpha = 3/2$ for a weakly ionized plasma in the limit $\nu_e < kv_{Te}$ and $\alpha = 2$ when $\nu_e > kv_{Te}$; for a completely ionized plasma, however, $\alpha = 7/2$. The quantity $A_2 = V_2V_2^*/v_s^2$ is then of the order of $\sim\epsilon^2$.

Knowing the nature of the variation of the ion-velocity oscillation amplitude, we can easily determine the corresponding amplitude of the electric field of the wave. We have from the linearized Eq. (3.3)

$$A_1 = |E_1|^2/4\pi N_0 T_e, \qquad (4.9)$$

where E_1 is the amplitude of the fundamental harmonic of the field.

It should be noted that, in the short-wavelength limit, in contrast to the long-wavelength limit, the stabilization of the amplitude of the growing wave is determined not only by energy transfer to the higher damped harmonics, but also depends essentially on the nonlinear shift of the frequency of the oscillations. This frequency shift is easily determined from the system (4.5):

$$\frac{\Delta\omega}{\omega_{Li}} = \frac{\omega-\omega_{Li}}{\omega_{Li}} = 2\frac{k^2v_s^2}{\omega_{Li}^2}A_1(\infty) = \frac{\alpha}{\beta}\varepsilon.$$

It is due to the oscillations of the ions in the field of the ion Langmuir wave. The possibility of stabilization of the instability by the nonlinear frequency shift can already be seen from the relations (2.2). Indeed, since the increment $\gamma \sim (\mathbf{k}\cdot\mathbf{u} - \omega)$, a nonlinear increase in the frequency decreases the increment, and can make it vanish when the system is just above the threshold.

Notice also that for a fixed level above the threshold, i.e., for the same value of ϵ, the amplitude of the stationary ion-sound waves in the short-wavelength part of the spectrum is considerably larger than in the long-wavelength part. This is connected with the resonance nature of the energy transfer to the higher harmonics in the latter case. In the short-wavelength limit the higher harmonics of the ion Langmuir oscillations are not natural oscillations of the system and, therefore, interact weakly with the fundamental harmonic.

4)Nonlinear dissipative effects limit the amplitude of the ion Langmuir oscillations (according to the lowest estimate) to [9]

$$\frac{E_1^2}{4\pi N_0 T_e} \approx \sqrt{\frac{m}{M}}\frac{T_e}{T_i}\left(\frac{\omega_{Li}}{kv_s}\right)^8.$$

This limit is higher than (4.8) provided

$$\text{D} \quad \varepsilon < \frac{T_e}{T_i}\sqrt{\frac{m}{M}}\left(\frac{\omega_{Li}}{kv_s}\right)^6$$

APPENDIX

Going over to a coordinate Fourier representation, we write Eq. (4.1) in the form

$$\begin{gathered}\frac{\partial^2V_k}{\partial t^2} + \omega_{Li}^2V_k + \nu_i\frac{\partial V_k}{\partial t} - \frac{\omega_{Li}^2}{v_s^2}Q_1(k)\left(\frac{\partial}{\partial t}+iku\right)\frac{\partial^2V_k}{\partial t^2}\\ + i\int dk_1\,dk_2k_1\delta(k_1+k_2-k)\left[V_{k2}\frac{\partial V_{k1}}{\partial t} + \frac{\partial}{\partial t}(V_{k1}V_{k2})\right]\\ - \int dk_1\,dk_2\,dk_3\,k_3(k-k_1)\delta(k_1+k_2+k_3-k)V_{k1}V_{k2}V_{k3} = 0.\end{gathered} \qquad (A.1)$$

Setting $V_k = \mathscr{V}_k(t)e^{-i\omega_k t}$, where $\mathscr{V}_k(t)$ is a slowly varying amplitude, we find

$$\begin{gathered}\frac{d\mathscr{V}_k}{dt} - (\gamma_k + i\Delta\omega_k)\mathscr{V}_k + \frac{i}{2\omega_k}\int dk_1\,k_1\exp\{i(\omega_k-\omega_{k_1}-\omega_{k-k_1})t\}\\ \times(\omega_{k_1}+\omega_k)\mathscr{V}_{k_1}\mathscr{V}_{k-k_1} - \frac{i}{2\omega_k}\int dk_1\,dk_2k_2(k-k_1)\\ \times\mathscr{V}_{k_1}\mathscr{V}_{k_2}\mathscr{V}_{k-k_1-k_2}\exp\{i(\omega_k-\omega_{k_1}-\omega_{k_2}-\omega_{k-k_1-k_2})t\} = 0,\end{gathered} \qquad (A.2)$$

where γ_k is the linear growth constant and $\Delta\omega_k = (\omega_k^2 - \omega_{Li}^2)/2\omega_k$. Near the instability threshold we can limit ourselves to the consideration of the first two harmonics of the fundamental instability mode of the oscillations with $k = k_{lim}$ and $\omega_k \approx \omega_{Li}$. Then $\mathscr{V}_k = V_1\delta(k - k_{lim}) + V_2\delta(k - 2k_{lim})$, $\omega_{2k} \approx 2\omega_{Li}$, and Eq. (A.2) reduces to the system (4.5). On the other hand, Eq. (A.2) is the more general equation and enables us to also investigate the weakly turbulent state of the system, when a large number of modes of the oscillations is excited, and phase mixing occurs.

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137