GRAVITATIONAL SYNCHROTRON RADIATION

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The intensity, spectrum and angular distribution of the gravitational radiation emitted by a dumbbell rotating with relativistic velocity are calculated. For masses rotating with velocities close to that of light, it is important that the masses be of finite size. Conclusions are drawn regarding the emission of gravitational waves by a packet of light waves.

In recent years the various mechanisms of emission of gravitational waves under astrophysical and laboratory conditions have been the subject of lively discussions. This interest is related, in the first place, to the experiments on detection of gravitational waves.

In the present paper the problem of emission of gravitational waves by a rod and by a dumbbell, rotating with relativistic velocities around the respective center of mass, is discussed rigorously. The results obtained show up an essential difference between gravitational, electromagnetic[1] and scalar[2,3] synchrotron radiations. The results of the paper also allow us to draw some conclusions on the emission of gravitational waves by light (or by other massless particles).

In relation to this set of problems we note the paper of Peters[4] which considered the emission of gravitational waves by a particle passing through a Schwarzschild field at distances much larger than the gravitational radius, as well as the paper by Davis et al.[5], who considered the gravitational radiation for a particle falling along a straight line in the Schwarzschild field. We also recall the paper of Pustovoĭt and Gertsenshteĭn[6], which analyzes the gravitational radiation of a charged relativistic particle which moves in a magnetic field.

In the present paper we consider a periodic motion that gives rise to the characteristic peculiarities of synchrotron radiation, and, what is most important, the masses are not considered pointlike. The latter circumstance leads to important features of the radiation for ultrarelativistic velocities and allows one to reach conclusions about the radiation for the exact equality $v = c$.

Finally, we note also the paper by Misner et al.[3] which considers the emission of scalar waves for motion with relativistic velocities in a Schwarzschild field. Here also the source was assumed to be pointlike. The properties of scalar synchrotron radiation (its spectrum and angular distribution) differ substantially from those obtained by us for gravitational waves.

The solutions of the linearized Einstein equations for a weak gravitational field in the wave zone, far from the source, can be written in the form of retarded potentials[1]:

$$\psi_i^k = \frac{4G}{c^4R_0}\int_V (T_i^k)_{t-R/c}\, dV, \tag{1}$$

where R_0 is the distance from the source to the observation point. The solution is written in a harmonic coordinate system; $\psi_i^k = h_i^k - \frac{1}{2}\delta_i^k h$, where h_{ik} are the small corrections to the Galilean metric $g_{ik}^{(0)}: g_{ik} = g_{ik}^{(0)} + h_{ik}$. If the motion in the source of the gravitational waves is not determined gy gravitational forces then T_i^k are simply the components of the energy-momentum tensor[1].

For slow motions inside the source of gravitational waves the retardation for all the points of the source is the same and the integral of all components T_i^k can be reduced to the second time derivative of integrals containing only the component T_0^0 (cf.[1]). For velocities comparable to the velocity of light, one must carry out the calculations directly with Eq. (1).

The spectral density of the components ψ_α^β in the wave zone can be written in the form[1)]

$$\psi_\alpha^\beta(\omega) = \frac{2G}{\pi c^4 R_0}\int_{-\infty}^{\infty} dt \int_V T_\alpha^\beta(\mathbf{r}, t)\exp\left[i\omega\left(t - \frac{|\mathbf{R}-\mathbf{r}|}{c}\right)\right] dV. \tag{2}$$

For a periodic motion with period T and frequency $\Omega = 2\pi/T$ the radiation decomposes into a Fourier series, and the expression (2) can be rewritten for the harmonics $n = \omega/\Omega$:

$$\psi_\alpha^\beta(n) = \frac{4G}{c^4R_0}\exp\left(\frac{in\Omega R_0}{c}\right)\frac{\Omega}{2\pi}\int_V dV \int_{-\pi/\Omega}^{\pi/\Omega} dt \cdot \times T_\alpha^\beta(\mathbf{r}, t)\exp\left[in\Omega\left(t - \frac{\mathbf{nr}}{c}\right)\right]. \tag{3}$$

Here $\mathbf{n}$ is a unit vector in the direction of the observer (not to be confused with the label n of the harmonics).

As a source of waves we consider a rigid rod, with arbitrary mass distribution along the rod, rotating around its own center of mass. We introduce the spherical coordinates r, θ, and φ centered in the rotation center and with the equatorial plane coinciding with the plane of rotation of the rod. Let the azimuth of the direction of the observer be zero. Then

$$\mathbf{rn} = r\sin\theta\sin\xi, \quad \xi = \pi/2 + \varphi.$$

In order to calculate the spatial components ψ_α^β (which are the only ones of importance) only the spatial components of the energy-momentum tensor of the source are important. This part of the energy-momentum tensor of the rotating rod consists of two components. The first is related to the kinetic energy of rotation of a given point of the body, and the second corre-

[1)]We shall be interested only in the space components. Greek indices take on the values 1, 2, 3; latin indices take the values 0, 1, 2, 3.

sponds to elastic stresses directed along the radius:

$$T_{\alpha\beta} = T_K p_\alpha p_\beta + T_H e_\alpha e_\beta.$$

The quantities T_K and T_H are defined below (cf. (6), (7)), **e** is a unit vector directed along the radius and **p** is a unit vector along the velocity.

For the calculation we select a Cartesian coordinate system with one of the axes (denoted by X^1) directed along **n**. Then, according to [1], the intensity of the gravitational radiation per unit solid angle in the direction **n** for the n-th harmonic, equals[2)]

$$\frac{dI_n(\theta)}{dO} = \frac{R_0^2 c^3 n^2 \Omega^2}{8\pi G}\left(|\psi_{23}(n)|^2 + \frac{1}{4}|\psi_{22}(n) - \psi_{33}(n)|^2\right). \tag{4}$$

In the selected Cartesian coordinate system the components of the unit vectors **p** and **e**, which are of interest to us, have the form

$$p_2 = \sin\xi, \quad p_3 = \cos\theta\cos\xi; \quad e_2 = \cos\xi, \quad e_3 = -\cos\theta\sin\xi.$$

If the rod is not infinitely thin, one can set for some arbitrary point of the body $\xi = \Omega t$, selecting an appropriate origin for the time variable. For any other point $\xi = \Omega t + \mu$, where μ is the difference of the azimuths of the two points.

We write the spatial components of the energy-momentum tensor in the laboratory system. We write T_K and T_H in the form of a product of quantities having the dimension of energy density per unit length of the rod, multiplied by functions taking into account the finite width and thickness of the rod. This is, of course, not the most general form, but is quite sufficient for our purposes. Thus,

$$T_K = \frac{\rho(\eta)\beta^2 c^2 \eta^2}{(1-\beta^2\eta^2)^{1/2}}\Delta_1(\mu)\Delta_2(\theta),$$

$$T_H = -\int_\eta^1 \frac{\rho(\eta)\beta^2 c^2 \eta\, d\eta}{(1-\beta^2\eta^2)^{1/2}}\Delta_1(\mu)\Delta_2(\theta). \tag{5}$$

Here $\eta = r/r_0$, where r_0 is the distance from the center of mass the remotest point of the body, βc is the velocity at that point and ρ is the linear mass density (i.e., mass per unit length of the rod). We note that ρ contains the energy related to stresses in the rod. The functions Δ_1 and Δ_2 are selected in such a manner that their integrals equal unity[3)]. Taking into consideration that

$$T_H = -\int_\eta^1 \frac{T_K\, d\eta}{\eta},$$

we obtain

$$|\psi_{22}(n) - \psi_{33}(n)| = \left|\frac{4Gr_0}{c^4 R_0}\int_{-\pi}^{\pi} d\mu\, e^{-in\mu}\int_0^1 d\eta\right. \tag{6}$$

$$\left.\times\left[T_K\left(\frac{n^2}{z^2}\frac{1+\cos^2\theta}{\eta^2} - 1\right) - T_H\right]J_n(z\eta)\right|, \tag{7}$$

$$|\psi_{23}(n)| = \left|\frac{4Gr_0}{c^4 R_0}\int_{-\pi}^{\pi} d\mu\, e^{-in\mu}\int_0^1 d\eta\, \frac{n}{z\eta}\left(J_n'(z\eta) - \frac{J_n(z\eta)}{z\eta}\right)(T_K - T_H)\right|,$$

where $z = n\beta\sin\theta$ and J_n is the Bessel function.

In order to estimate the integrals one can utilize the asymptotic formulas for the Bessel functions (cf. [7]). As a result of the calculations we obtain for the flux density of radiation per unit solid angle the following expressions:

a) if $n(\cos^2\theta + 1 - \beta^2)^{3/2} \ll 1$, then

$$\frac{dI_n(\theta)}{dO} = \frac{G}{c^5}\frac{\Omega^2|\Phi(n)|^2}{4\pi^3}n^{2/3}\left\{6^{-1/3}\Gamma^2\left(\frac{1}{3}\right)n^{2/3}\left[1 - \beta^2 + 2\cos^2\theta + \frac{1}{2}\left(\frac{3}{n}\right)^{2/3}\right]^2 + 4\cos^2\theta\cdot 6^{1/3}\Gamma^2\left(\frac{2}{3}\right)\right\}, \tag{8}$$

where

$$\Phi(n) = r_0\int_{-\pi}^{\pi} d\mu\, e^{-in\mu}\int_{\tilde\eta}^1 \eta^{-2}T_K(\eta,\mu)\, d\eta = \Phi_\Delta(n)\Phi_\varkappa(n), \tag{9}$$

$$\Phi_\Delta(n) = \int_{-\pi}^{\pi} d\mu e^{-in\mu}\Delta_1(\mu), \tag{10}$$

$$\Phi_\varkappa(n) = r_0\beta^2\int_{\tilde\eta}^1 \frac{c^2\rho(\eta)\, d\eta}{(1-\beta^2\eta^2)^{1/2}} \tag{11}$$

and $\tilde\eta = 1 - \frac{1}{2}(3/n)^{2/3}$. It was assumed that

$$\int_{\tilde\eta}^1 \eta^{-1}T_K\, d\eta \approx \int_{\tilde\eta}^1 \eta^{-2}T_K\, d\eta;$$

b) if $n(\cos^2\theta + 1 - \beta^2)^{3/2} \gg 1$, then an exponentially small factor of the type

$$\exp\{-\tfrac{2}{3}n(\cos^2\theta + 1 - \beta^2)^{3/2}\}$$

appears in the expression for the energy flux density, and one may consider the energy flux density to be negligibly small in this case.

We note that for relativistic rotation velocities there are large stresses in the rod. According to the general principles the energy density in the proper reference frame at any point is not smaller than the stresses at that point.

In the sequel we shall consider rods in which a considerable fraction of the mass is concentrated at the very ends, whereas at all other points the mass density is taken to be the minimally necessary one, i.e., not to be smaller than the stresses $\rho_1 = -T_H c^{-2}$. We denote $\tilde\rho = \rho - \rho_1$. The total energy of the rod consists of two terms:

$$E = r_0 c^2\int_0^1 \frac{\rho(\eta)\, d\eta}{(1-\beta^2\eta^2)^{1/2}} = r_0 c^2\int_0^1 \frac{\tilde\rho(\eta)\, d\eta}{(1-\beta^2\eta^2)^{1/2}} + r_0 c^2\int_0^1 \frac{\rho_1(\eta)\, d\eta}{(1-\beta^2\eta^2)^{1/2}},$$

which can be shown to be approximately equal. Therefore we have approximately

$$r_0 c^2\int_0^1 \frac{\tilde\rho(\eta)\, d\eta}{(1-\beta^2\eta^2)^{1/2}} \approx \frac{1}{2}E. \tag{12}$$

Consider the simplest case of a dumbbell, consisting of two identical pointlike masses, equal to M in their own reference frames, and connected by a thin rod with the density $\rho = \rho_1 = -T_H/c^2$. In this case

$$\Delta_1 = \tfrac{1}{2}[\delta(\mu) + \delta(\mu - \pi)], \quad \tilde\rho = (2M/r_0)\delta(\eta - 1), \quad \Phi(n) \approx \tfrac{1}{4}\beta^2 E[1 + (-1)^n]. \tag{13}$$

Integrating (8) over the angles we obtain the spectral density of the total radiation. For estimates we make use of the fact that Eq. (8) is valid for $|\cos\theta| \lesssim n^{-1/2}$, and outside this angle interval we shall consider the radiation to be negligible, owing to the exponential decay of the Bessel function. As a result we obtain for

[2)]Concretely, for the validity of Eq. (4) it is necessary that at the observation point only ψ_{23} and $\psi_{22}-\psi_{33}$ be nonzero among all the components of ψ_{ik}; as was shown in [1] this can always be realized at a given point by means of coordinate transformations which leave the values of ψ_{23} and $\psi_{22}-\psi_{33}$ unchanged. The latter fact allows one to use for calculations the values of the components of the gravitational wave in the coordinate system considered in the text.

[3)]Here and in the sequel we set $\Delta_2 = \delta(\theta - \pi/2)$, since the extension of the source in the radial direction is not of interest.

$n(1-\beta^2)^{3/2} \ll 1$ for the spectral density of radiation, integrated over all the directions,

$$I_n \approx \frac{G\Omega^2E^2}{c^5}\left[\frac{1+(-1)^n}{2}\right]^2\frac{1}{n^{1/3}}[C+(1-\beta^2)Dn^{2/3}+(1-\beta^2)^2Fn^{4/3}]. \quad (14)$$

The numerical coefficients C, D, F are of the order of unity. Here we have retained the terms $n^{2/3}(1-\beta^2)$ and $n^{4/3}(1-\beta^2)^2$, which are small for $n(1-\beta^2)^{3/2} \ll 1$, however they do not give a small contribution to the integral over the spectrum. For $n(1-\beta^2)^{3/2} \gg 1$ the function I_n vanishes exponentially and can be omitted.

With the help of (8) one can estimate also the angular distribution of the total intensity W, i.e., the intensity summed over the harmonics. For this purpose we take into account that for $\cos^2\theta \ll 1-\beta^2$ the integration[4)] over n goes up to $n \approx n_0 = (1-\beta^2)^{-3/2}$, and for $\cos^2\theta \ll 1-\beta^2$, up to $n \approx |\cos^3\theta|^{-1}$. Thus, for $\cos^2\theta \ll 1-\beta^2$

$$\frac{dW}{dO} \approx \frac{G}{c^5}\frac{\Omega^2E^2}{(1-\beta^2)^{3/2}}, \quad (15)$$

and for $1-\beta^2 \ll \cos^2\theta \ll 1$

$$\frac{dW}{dO} = \frac{G}{c^5}\frac{\Omega^2E^2}{|\cos^3\theta|}. \quad (16)$$

The total intensity is obtained either by integrating (14) over n from 2 to $n_0 = (1-\beta^2)^{-3/2}$ (for $n > n_0$ the quantity I_n is exponentially small), or by integrating (15) with respect to $\cos\theta$ from $-(1-\beta^2)^{1/2}$ to $(1-\beta^2)^{1/2}$. Outside this angle interval the radiation is small. The results of both calculations coincide, of course, and the total power of the gravitational radiation is

$$W \approx \frac{G}{c^5}\frac{\Omega^2E^2}{1-\beta^2}. \quad (17)$$

We recall that E is the total energy of the rod in the coordinate system of the observer:

$$E = r_0\int_0^1 c^2\rho(\eta)(1-\beta^2\eta^2)^{-1/2}\,d\eta.$$

The spectral characteristics of gravitational synchrotron radiation differ from the corresponding characteristics of electromagnetic synchrotron radiation. According to Eq. (14), for not too large $n < n_0$ we have $I_n \propto n^{-1/3}$, whereas for the electromagnetic case $I_n \propto n^{1/3}$. However, although in the gravitational case I_n decreases with the growth of n, the decrease is slow, and the integrated radiation is essentially accumulated on high harmonics, near $n_0 = (1-\beta^2)^{-3/2}$, after which I_n becomes exponentially small.

In the expression for the total power of the gravitational radiation, (17), the denominator contains $1-\beta^2$. We fix the quantity E and will take β to be closer to unity, i.e., we let the velocity of rotation of the extremity of the rod tend to the speed of light. The expression (17) then becomes divergent. It would seem that for $v \to c$ and finite energy E that the radiation becomes infinite in the frame of the observer. However, the expression (17) stops being valid (as well as the expressions (13)–(16)) for sufficiently small $1-\beta^2$, owing to the necessity of taking into account the finite size of the emitting masses[5)].

[4)]The summation for large n can obviously be replaced by an integration.

[5)]A similar remark is valid not only for the gravitational, but also for the electromagnetic and scalar radiations.

Let us now take into account the finite dimensions of the masses. We start with taking into account a finite extension of the masses along the radius and neglect, for the time being, the finite width of the rod. Let the mass of the rod be concentrated in the interval $1-\kappa < \eta < 1$. For simplicity we shall assume that the linear density does not vary in this interval. Then

$$\Delta_1 = {}^1/_2[\delta(\mu)+\delta(\mu-\pi)],$$

$$\bar\rho(\eta) = \begin{cases} 2M/r_0\kappa, & 1-\kappa<\eta<1 \\ 0, & \eta<1-\kappa \end{cases}.$$

Let

$$\kappa \ll 1-\beta^2. \quad (18)$$

We consider the harmonics satisfying the inequality

$$n^{-2/3} \gg \kappa. \quad (19)$$

Then (11) yields

$$\Phi_\kappa(n) \approx {}^1/_2E\beta^2. \quad (20)$$

The condition (18) leads to automatic validity of (19) for all harmonics up to $n_0 = (1-\beta^2)^{-3/2}$ after which the radiation becomes exponentially small and negligible. Therefore if the inequality (18) holds, all formulas for the synchrotron radiation of point masses hold, and the distribution of the density along the rod can be disregarded completely.

Assume now that the opposite inequality is satisfied

$$\kappa \gg 1-\beta^2. \quad (21)$$

We consider again the harmonics satisfying the inequality (19). Expression (2) holds as before, and the formulas for the spectral resolution of the radiation do not differ in any way from the formulas for the point masses. However, now in the transition to higher harmonics the inequality (19) is violated before the value n_0 is reached. We denote $n_* = \kappa^{-3/2}$. Then, for n situated in the interval

$$n_* < n < n_0, \quad (22)$$

we obtain

$$\Phi_\kappa(n) \approx {}^1/_2E\beta^2(\kappa^{3/2}n)^{-1/3}. \quad (23)$$

Consequently, we have in the interval (22) for the spectral density of the radiation

$$I_n \propto n^{-1}. \quad (24)$$

This power-law decay of I_n with the increase of n is still slow, the integral with respect to n of I_n accumulates (logarithmically) near n_0, as before, and we can write down the following expression for the total power of the radiation, if (21) is true:

$$W \approx \frac{G}{c^5}\frac{\Omega^2E^2}{\kappa}\ln\frac{\kappa}{1-\beta^2}. \quad (25)$$

For the interval of angles where most of the radiation is emitted we obtain, nevertheless, the same expression as for point masses:

$$|\cos\theta| \lesssim (1-\beta^2)^{1/2}. \quad (26)$$

From a physical point of view taking into account the radial distribution of masses can be reduced to the following. The higher the number labeling the harmonic, the larger must the ratio $1/(1-v^2/c^2)$ be for the moving masses which contribute to the radiation in this harmonic. But such high velocities will be exhibited only by masses situated closer and closer to the end points of the rod. For large n, when $n \gg \kappa^{-3/2}$, only a small part of the mass $\sim (\kappa n^{2/3})^{-1/2}$ will contribute to the radiation on this harmonic.

Of particular interest is the consideration of the finite sizes of the emitting masses in the direction of the velocity, since in this case one can pass to the limit when the whole emitting mass moves with the velocity of light. We consider a finite extension of the masses along the direction of motion, neglecting the extension in the radial direction. In this case

$$\tilde{\rho}(\eta) = \frac{2M}{r_0}\delta(\eta - 1), \quad 2\pi\Delta_1 = f(\mu).$$

Here the function $f(\mu)$ which describes the finite dimensions of the masses in the direction of motion is selected to be different from zero only in the intervals $|\mu| < \Delta\pi$ and $|\mu - \pi| < \Delta\pi$; its integral over each of these intervals equals π; otherwise the function $f(\mu)$ is arbitrary.

According to (10)

$$\Phi_\Delta(n) = \frac{1+(-1)^n}{2}\frac{1}{\pi}\int_{-\Delta\pi}^{\Delta\pi} d\mu\, e^{in\mu}f(\mu); \tag{27}$$

with

$$\frac{1}{\pi}\int_{-\Delta\pi}^{\Delta\pi} f(\mu)\,d\mu = 1$$

according to the normalization we have selected. It can be seen from (27) that for $n\Delta \ll 1$ the formulas for the radiation are the same as for $\Delta = 0$, i.e., pointlike masses. Thus, if $\Delta \ll n_0^{-1} = (1-\beta^2)^{3/2}$, the radiation on all harmonics, up to n_0 coincides with the radiation of point masses and one can set $\Delta = 0$.

For $n\Delta \gg 1$ the integral (27) is small, since the integrand contains a product of the slowly varying function $f(\mu)$ with a rapidly oscillating exponential function. The Parseval identity yields

$$\sum_{n=2}^{\infty}|\Phi_\Delta(n)|^2 \leqslant \frac{1}{\pi}\int_{-\Delta\pi}^{\Delta\pi} f^2(\mu)\,d\mu \sim \Delta^{-1},$$

therefore for $n \gg n_{**} \equiv \Delta^{-1}$

$$|\Phi_\Delta(n)|^2 \propto n^{-1-q}, \quad I_n \propto n^{-4/3-q} \quad (q > 0) \tag{28}$$

and the integral over the spectrum accumulates near $n \approx n_{**}$. The concrete value of q depends on the form of the function $f(\mu)$. Thus, e.g., for $f(\mu)$ in the form of a step, q = 1.

Assuming continuity and vanishing at the endpoints of the segment $[-\pi\Delta, \pi\Delta]$ of all the derivatives for harmonics with index $n > n_{**}$, one obtains an exponential fall-off. In addition, the spectrum for $n > n_{**}$ may be nonmonotonic: the expression (28) may be modulated by some bounded function. Thus, for the same step function, such a function may be $\sin^2 n\pi\Delta$.

Thus, if the condition

$$\Delta > (1-\beta^2)^{3/2} \tag{29}$$

is satisfied, harmonics with $n \approx n_{**}$ give the main contribution to the integral synchrotron radiation and the radiated power becomes

$$W \approx \frac{G}{c^5}\frac{\Omega^2E^2}{\Delta^{2/3}} \tag{30}$$

remains finite for $\beta \to 1$. The radiation is essentially concentrated in the equatorial plane, within the angle interval

$$|\cos\theta| \leqslant \Delta^{1/3}. \tag{31}$$

The change of the spectrum for $n > n_{**}$ owing to taking into account the finite size of the masses is related to interference of the radiation from different portions of the rod.

Taking into account the finite size of both dimensions of the rod leads to the following: if $\kappa > \Delta^{2/3} > 1-\beta^2$ first there appears a kink for $n \approx n_* = \kappa^{-3/2}$, and for $n_* < n < n_{**}$ we will have $I_n \propto n^{-1}$. The total intensity is determined by

$$W \approx \frac{G}{c^5}\frac{\Omega^2E^2}{\varkappa}\ln\frac{\varkappa}{\Delta^{2/3}}. \tag{32}$$

For $1-\beta^2 < \kappa < \Delta^{2/3}$ there appears a kink first for $n \approx n_{**} = \Delta^{-1}$ and for $n_{**} < n < n_*$ we have $I_n \propto n^{-4/3-q}$; the total intensity is determined by Eq. (30). For the case when $\Delta > (1-\beta^2)^{3/2}$ the difference between the velocities of the ends of the rod and c becomes insignificant. The angle interval in which the fundamental radiation is concentrated is determined by Eq. (31).

Thus, the fundamental results of this paper are the following. The spectrum of gravitational synchrotron radiation is decreasing and proportional to $n^{-1/3}$ up to a critical harmonic number, equal to

$$n_{\text{кр}} = \min\{(1-\beta^2)^{-3/2},\ \varkappa^{-3/2},\ \Delta^{-1}\}, \tag{33}$$

where κ and Δ determine the size of the sources in the radial and angular directions. For $n > n_{cr}$ the spectral density decreases faster. The integrated intensity of radiation is

$$W \approx \frac{G}{c^5}\Omega^2E^2 n_{\text{cr}}^{2/3}\cdot\begin{cases} 1 & \text{for } n_{\text{cr}} < \varkappa^{-3/2} \\ \ln(\varkappa\tilde{n}^{2/3}) & \text{for } n_{\text{cr}} = \varkappa^{-3/2} \end{cases}, \tag{34}$$

where $\tilde{n} = \min\{(1-\beta^2)^{-3/2}, \Delta^{-1}\}$. The radiation is concentrated near the equator in an angle interval

$$|\cos\theta| \leqslant \tilde{n}^{-1/3}. \tag{35}$$

We have not carried out a rigorous analysis of the emission of gravitational waves by a wave packet moving along a circle in a waveguide. However, the equations for the case $\Delta \neq 0$ allow one to carry out the limit $v \to c$; in this case the total radiation must be determined by Eq. (30). At the same time, as stressed by Ya. B. Zel'dovich, the detailed picture of emission of gravitational waves, with a detailed determination of the influence of this radiation on the intensity and frequency of waves which make up the emitting packet, can be obtained only by explicitly taking into account the wave nature of the emitter and the eigenfunction expansion of the motion of the packet in the waveguide.

In order to estimate the emission by a wave packet in a waveguide having the shape of a torus, we rewrite the Equation (30) in the form

$$W \approx \frac{G}{c^5}\frac{c^2E^2}{R^{4/3}(c\tau)^{2/3}}, \tag{30'}$$

where R is the radius of the torus and $c\tau$ is the width of the wave packet. For $R \sim 100$ cm and $c\tau \sim 10$ cm we obtain

$$W \approx 10^{-44}E^2 \text{ [erg/sec]}.$$

In order to compare the electromagnetic and gravitational radiations of an electron we note, e.g., that

$$W_{\text{el}} \approx \frac{e^2}{m^2}\frac{E^2\Omega^2}{1-\beta^2},$$

so that[6]

$$W_{cr} / W_{el} \approx Gm^2/e^2 \approx 10^{-43},$$

The listed examples show that under laboratory conditions the gravitational synchrotron radiation is small.

During the preparation of the manuscript of the present paper for the press there appeared the preprint of M. Davis, R. Ruffini, J. Tiomno and F. Zerili[8], where it is asserted that no gravitational synchrotron radiation appears in a periodic motion of a free probe particle in a Schwarzschild potential with velocity close to c, on unstable orbits close to $r = (3/2)r_g$. The absence of synchrotron radiation is in this case related to the strong curvature of the null geodesics in the Schwarzschild field for $r \approx (3/2)r_g$.

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APPENDIX

In the main text we have considered the case of a symmetric dumbbell. What is the character of the radiation for an asymmetric mass distribution?

For slow rotation, when the velocities are small, the radiation is emitted in the second harmonic, independently of the ratio of the masses. As was shown in the present paper, for equal masses and relativistic rotation with v of the order of c, the emission occurs in high, but even-order, harmonics.

We consider the case of a dumbbell consisting of two pointlike but unequal masses $M_1 < M_2$ connected by a rigid rod, and rotating around the center-of-mass. We denote by E_1 and E_2 the kinetic energies (including rest energies) of these two masses. We introduce $\beta_1 = v_1/c$ and $\beta_2 = v_2/c$, where v_1 and v_2 are the velocities of the masses. Obviously, the larger mass will have the smaller velocity. Therefore, for high enough harmonics, for

$$n \gg (1-\beta_2^2)^{-3/2}, \text{ but } n < (1-\beta_1^2)^{-3/2}, \tag{A.1}$$

the contribution to the radiation from the larger mass is exponentially small, of the order of $\exp(-(2/3)n(1-\beta_2^2)^{3/2})$, whereas the contribution from the smaller mass is quite finite. This has the effect that if (A.1) holds, the emission occurs not only in even harmonics, as is true for $M_1 = M_2$, but also equally on odd harmonics; moreover the radiation is caused totally by the smaller of the masses. We consider the lower harmonics, where

$$n \ll (1-\beta_2^2)^{-3/2}. \tag{A.2}$$

In this case the ratio of the radiation intensities in two neighboring odd and even harmonics is

$$I_{\text{odd}} / I_{\text{even}} \approx (1-\alpha)/(1+\alpha), \tag{A.3}$$

where $\alpha = E_2/E_1$. The quantity α is expressed in terms of the mass ratio and β:

$$\alpha = \frac{E_2}{E_1} = \frac{M_1}{M_2}\left[1-\beta_1^2+\left(\frac{M_1}{M_2}\right)^2\right]^{-1/2}. \tag{A.4}$$

For $M_1/M_2 \gg (1-\beta_1^2)^{1/2}$ (but possibly $M_1/M_2 \ll 1$) we obtain $\alpha \approx 1$ and the emission in the harmonics (A.2) is indistinguishable from the case $M_1 \approx M_2$. Finally, for $M_1/M_2 \ll (1-\beta_1^2)^{1/2}$ we have $\alpha \ll 1$ and $\beta_2 \ll 1$, i.e., the motion of the larger mass is nonrelativistic. The emission in the high n is determined only by the smaller mass and even and odd harmonics are present in equal amounts.

[1] L. D. Landau and E. M. Lifshitz, Teoriya polya (Classical Field Theory), Nauka, 1967 (Eng. Transl. Pergamon-Addison Wesley, 1968).

[2] V. L. Ginzburg and G. F. Zharkov, Zh. Eksp. Teor. Fiz. 47, 2279 (1964) [Sov. Phys.-JETP 20, 1525 (1965)].

[3] C. W. Misner, R. A. Brener, D. R. Brill, P. L. Chrzanowski, H. G. Hughes III, and C. M. Pereira, Phys. Rev. Lett. 28, 998 (1972).

[4] P. C. Peters, Phys. Rev. D1, 1559 (1970).

[5] M. Davis, R. Ruffini, W. H. Press and R. H. Proce, Phys. Rev. Lett. 27, 1466 (1971).

[6] V. I. Pustovoĭt and M. E. Gertsenshteĭn, Zh. Eksp. Teor. Fiz. 42, 163 (1962) [Sov. Phys.-JETP 15, 116 (1962)].

[7] I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov, summ, ryadov i proizvedeniĭ (Tables of Integrals, Sums, Series and Products), Fizmatgiz, 1962; Engl. Transl. Acad. Press, 1964.

[8] M. Davis, R. Ruffini, J. Tiomno and F. Zerilli, Preprint, Princeton, 1972.

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