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SUPERFLUID COMPONENT DENSITY DISTRIBUTION NEAR THE HeI–HeII PHASE INTERFACE IN EXTERNAL FIELDS

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The distribution of the order parameter (the density of the superfluid component) near the HeI–HeII phase interface in external fields is investigated on the basis of two modified variants of the phenomenological Ginzburg-Pitaevskiĭ theory of superfluidity. The results of a numerical solution of the respective equations for three types of external fields: the gravitational field, the field of the van der Waals forces near a rigid wall, and the field of a point electric charge (ions in helium), are presented.

IN the experimental study of critical phenomena we are obliged, as a rule, to deal with "samples" along which the temperature or any other thermodynamic parameter (density, impurity concentration, etc.) varies slightly. A phase transition in the ordered state is then accomplished not simultaneously along the entire sample. In this connection, special interest attaches to the distribution of the order parameter near the interface between the ordered and disordered phases, where correlation effects are strong.

We investigate in the present paper the nature of the distribution of the order parameter near the interface, using as an example the λ-transition in helium located in an external field of the form $\rho U(\mathbf{r})$ acting on a unit mass of the liquid. To this type of fields pertain the gravitational field ($U = gx$), the field of the van der Waals forces near a rigid wall, the field of the electrostrictional forces near a charged body, etc. For the case of the gravitational field the corresponding problem has already been considered by a number of authors[1,2] on the basis of two improved versions[2,3] of the phenomenological Ginzburg-Pitaevskiĭ (GP) theory[4]. In the present paper we discuss in greater detail the form of the equation describing the equilibrium distribution of the order parameter in nonuniform external fields, and present the results of a numerical solution of this equation for the three types of fields listed above.

1. THE EQUILIBRIUM EQUATION FOR THE ORDER PARAMETER

In the phenomenological theory of superconductivity we introduce for the description of the state of the superfluid component the parameter $\Psi(\mathbf{r})$ whose equilibrium value in the spatially inhomogeneous case is determined (see[4]) from the condition for the minimum of the functional[1]:

$$\mathcal{F}(\rho, T; \Psi) = \int \left[F_0(\rho, T, \Psi) + \frac{\hbar^2}{2m} (\nabla\Psi)^2 \right] dV. \tag{1.1}$$

The use of the density ρ (or the pressure P, as has been done before (see, for example,[1,2])) as an independent variable is, however, not quite correct, since in the state of thermodynamic equilibrium the fixed thermodynamic variable is the chemical potential μ, and the density and pressure should, generally speaking, vary with the order parameter. In accordance with the foregoing, we should, in the presence of spatial variations of the order parameter, minimize the functional

$$O(\mu, T; \rho, \Psi) = \int \left[F_0(\rho, T, \Psi) + \frac{\hbar^2}{2m} (\nabla\Psi)^2 - \mu\rho \right], \tag{1.2}$$

simultaneously with respect to $\Psi(\mathbf{r})$ and $\rho(\mathbf{r})$.

When an external field is present we must add the term $\rho U(\mathbf{r})$ to the integrand in (1.2) and, furthermore, if the density ρ changes appreciably over distances of the order of the interatomic distances, also the term $\frac{1}{2}\delta(\nabla\rho)^2$ which takes the density correlation into account. To determine the equilibrium values of $\Psi(\mathbf{r})$ and $\rho(\mathbf{r})$, we derive the following system of equations:

$$\frac{\hbar^2}{m} \nabla^2\Psi = \left(\frac{\partial F_0}{\partial \Psi} \right)_{\rho, T}, \tag{1.3}$$

$$\delta(\nabla\rho)^2 = \left(\frac{\partial F_0}{\partial \rho} \right)_{\Psi, T} - \mu + U(\mathbf{r}). \tag{1.4}$$

If the total-density gradients appearing in the system are small, and this will henceforth be assumed, then it is convenient in the above expressions to go over from the free energy density, $F_0(\rho, T, \Psi)$ to the thermodynamic-potential density, $\Omega_0 = F_0 - \mu_0\rho$, in the variables (μ_0, T, Ψ), where

$$\mu_0 \equiv \left(\frac{\partial F_0}{\partial \rho} \right)_{T, \Psi} = \mu - U(\mathbf{r}). \tag{1.5}$$

The equations for $\Psi(\mathbf{r})$ and $\rho(\mathbf{r})$ are then separable and take the form:

$$\frac{\hbar^2}{m} \nabla^2\Psi = \left(\frac{\partial \Omega_0}{\partial \Psi} \right)_{\mu_0, T}, \tag{1.6}$$

$$\rho = -\left(\frac{\partial \Omega_0}{\partial \mu_0} \right)_{\Psi, T}. \tag{1.7}$$

To proceed we need to know the explicit form of the Ψ-dependent, singular part of the thermodynamic potentials. In[2,3], where the thermodynamic potential $\Phi_0(P, T, \Psi)$ is considered, they use for this purpose the expansion

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[1] There is a definite ambiguity in the problem of choosing a macroscopic complex order parameter $\Psi(\mathbf{r})$ in helium [5]. We retain the original meaning for Ψ: $\rho_s = m|\Psi|^2$ [4]. Furthermore, we shall, for simplicity, henceforth assume the liquid to be stationary. Under these conditions $\Psi(\mathbf{r})$ can be assumed to be a real positive quantity.

$$\Phi_{0\,sing}(P,\ T,\ \Psi) = \varphi_0 t^2 \ln|t| + At|t|^{1/3}\Psi^2 + {}^1\!/_2 B|t|^{2/3}\Psi^4 + {}^1\!/_3 C\Psi^6. \tag{1.8}$$

Here, t stands for the distance to the λ-curve:

$$t = T - T_\lambda(P) = T - T_\lambda - \frac{dT_\lambda}{dP}(P - P_\lambda).$$

The expansion (1.8) is similar to the original classical expansion in the GP theory[4], but differs from it in the coefficients, whose dependence on temperature has been modified to reflect a more accurate behavior of the heat capacity C_p and superfluid-component density near the λ-transition. These two sets of experimental data are, however, insufficient for a unique determination of all the three phenomenological coefficients A, B, and C. In this connection one of these coefficients is usually assumed to be small. In Mamaladze's paper[3] he, by analogy with the original classical version of the GP theory and also using data on the critical thickness of films and capillaries, assumes the coefficient C to be small, whereas Slyusarev and Strzhemechnyĭ[2] assume the coefficient B to be small.

In the scaling theories one assumes for $\Phi_{0\,\mathrm{sing}}(P,\ T,\ \Psi)$ the more general expression

$$\Phi_{0\,sing}(t,\ \Psi) = \varphi_0 t^2 \ln|t| + t^2 f(t/\Psi^3), \tag{1.9}$$

which reduces to (1.8) only when $\Psi/|t|^{1/3} \ll \Psi_0^\pm$, where Ψ_0^+ and Ψ_0^- are certain constants that are, in order of magnitude, close but generally speaking not equal to the coefficient Ψ_{0e} defined by the temperature dependence of the equilibrium value Ψ_e of the order parameter in the low-temperature phase: $\Psi_e = \Psi_{0e}(-t)^{1/3}$. In the opposite case when $\Psi/|t|^{1/3} \gg \Psi_0^\pm$ (exactly such a situation obtains near the interface between the ordered and disordered phases, where $t \to 0$, and, on account of correlation effects, Ψ is a finite quantity), the expansion of the function f should rather be directly in powers of t/Ψ^3. Then (see also[6]),

$$\Phi_{0\,sing}(t,\ \Psi) = 3\varphi_0 t^2 \ln\Psi + \tilde{A}\Psi^6 + \tilde{B}t\Psi^3 + \tilde{C}t^2. \tag{1.10}$$

We could try to obtain for $\Phi_{0\,\mathrm{sing}}$ a good interpolation expression that is suitable for any value of the ratio $\Psi/|t|^{1/3}$ by smoothly matching the expansions (1.8) and (1.10) at $\Psi/|t|^{1/3} = \Psi_0^\pm$. Note that such a procedure yields very good results in the case of magnetic phase transitions[7]. As applied to the λ-transition in helium, however, it is difficult to carry it out consistently, in view of the absence at present of a sufficient amount of experimental data. For this reason, we shall use the Slyusarev-Strzhemechnyĭ[2] and Mamaladze[3] forms of the expansion (1.8) right up to the interface t = 0 itself, in the hope that in practice the vicinity of the interface ($\Psi/|t|^{1/3} \gg \Psi_0^\pm \sim \Psi_{0e}$) where (1.8) becomes inapplicable is small and the order parameter Ψ(r) does not change by any appreciable amount there. The nature of the curves shown in the following sections justifies, to a certain extent, such a viewpoint.

We have thus far been talking about the thermodynamic potential $\Phi_0(P,\ T,\ \Psi)$. On changing to the variables (ρ, T, Ψ) and (μ_0, T, Ψ) the form of the singular part of thermodynamic potential, generally speaking, changes. However, as shown in the Appendix, owing to the small anomaly of the compressibility in helium, this change is negligible. In final analysis, the difference lies only in the fact that for each selected pair of variables (X, T) we should take t in the formula (1.8) to mean the distance to the λ-curve in the same variables. In particular, for the case of the thermodynamic potential $\Omega_0(\mu_0,\ T,\ \Psi)$ of interest to us

$$t = T - T_\lambda - \frac{dT_\lambda}{d\mu}(\mu - \mu_\lambda - U(\mathbf{r})) = t_0 + \frac{dT_\lambda}{d\mu}U(\mathbf{r}). \tag{1.11}$$

Here t_0 is the initial distance to the λ-curve in the absence of an external field.

With all the above observations taken into account, the equation for the order parameter can finally be written in the form

$$\frac{\hbar^2}{2m}\nabla^2\Psi = At|t|^{1/3}\Psi + B|t|^{2/3}\Psi^3 + C\Psi^5, \tag{1.12}$$

where, we recall, t depends on the coordinates in the presence of an external field and is given by the formula (1.11)[2)].

Before proceeding to the solution of some concrete problems on the basis of Eq. (1.12), let us simplify somewhat the notation. For this purpose let us impose on the coefficients A, B, and C two restrictions by requiring, as usual, that they yield the correct experimental values of the "jump" in the heat capacity ΔC_μ and the coefficient ρ_{s0} connected with the temperature dependence of the equilibrium value of the superfluid-component density, $\rho_{se} = \rho_{s0}(-t)^{2/3}$, and let us change to the new variables

$$\psi = \Psi/\Psi_{0e}, \quad \bar{\mathbf{r}} = \mathbf{r}/\xi_{0M},$$

where $\Psi_{0e} = (\rho_{s0}/m)^{1/2} = (1.43\rho_\lambda/m)^{1/2}$ and $\xi_{0M} = T_\lambda\hbar^2\Psi_{0e}^2/2m\Delta C_\mu = 2.73$ Å$\times$(deg.)$^{2/3}$ is the coefficient associated with the temperature dependence of the coherence length in the version of the phenomenological theory[3] with C = 0. In terms of the indicated variables Eq. (1.12) takes the form

$$\nabla^2\psi = \frac{3}{3+M}\{t|t|^{1/3}\psi + (1-M)|t|^{2/3}\psi^3 + M\psi^5\}, \tag{1.13}$$

where M is a dimensionless parameter whose value is equal to 0 and 1 in the versions of the phenomenological theory expounded respectively in[3] and[2].

2. SOLUTION OF CERTAIN PROBLEMS

A. The HeI—HeII Phase Boundary in a Gravitational Field

In the case of the gravitational field

$$U(x) = gx. \tag{2.1}$$

It is convenient to choose as the origin of the x coordinate the location of the classical interface determined from the condition

$$t(x) = t_0 + \frac{dT_\lambda}{d\mu}gx = 0, \tag{2.2}$$

[2)] In the equations used in [1] and [2] the derivative $\rho_\lambda dT_\lambda/dP$ stands in place of the derivative $dT_\lambda/d\mu$. The implicit identification of these two derivatives also occurred in Ahlers' experimental work [8]. Note that although the difference between $dT_\lambda/d\mu$ and $\rho_\lambda dT_\lambda/dP$ is quite small (~2%), the allowance for it brings the value obtained in [8] for the slope of the λ-curve, $dP_\lambda/dT = -113.9$ atm/°K, closer to the best experimental value $dP_\lambda/dT = -111.05$ atm/°K [9] (the corrected value for dP_λ/dT following from [8] is equal to -111.6 atm/°K).

and to go over to the new dimensionless variables[1]:

$$u = x/l_g, \quad \eta = \Psi(\rho_{sg}/m)^{-1/2}, \tag{2.3}$$

where the characteristic scales l_g and ρ_{sg} do not depend on temperature and are equal (at the saturated-vapor pressure) to

$$l_g = \xi_{0M}^{3/5}(|dT_\lambda/d\mu|g)^{-2/5} = 6{,}8\cdot 10^{-3}\,\mathrm{cm}, \quad \rho_{sg} = 1{,}43\rho_\lambda\xi_{0M}/l_g = 5.85\cdot 10^{-6}\rho_\lambda. \tag{2.4}$$

In terms of the indicated variables the equation for the order parameter assumes the form

$$\frac{d^2\eta}{du^2} = \frac{3}{3+M}\{M\eta^5 + (1-M)|u|^{2/3}\eta^3 - u|u|^{1/3}\eta\}. \tag{2.5}$$

The solutions of Eq. (2.5) corresponding to M = 0 and M = 1 are shown in Fig. 1. The dashed curve is the plot of the distribution $\eta(u)$ which one would get in the absence of correlation effects: $\eta = u^{1/3}$ $(u > 0)$ and $\eta = 0$ $(u < 0)$. In the region $u < 0$ the solutions of the equation have the asymptotic form

$$\eta(u) \sim (-u)^{-1/3}\exp\{-0{,}6(3/(3+M))^{1/2}(-u)^{5/3}\}.$$

As the interface approaches the free surface of the liquid the distribution $\eta(u)$ changes, acquiring the shape of a "bell" (or one half of a bell in the case of the boundary condition $\eta'|_S = 0$ at the free surface) deformed by the field. As the free surface is approached, the height of this bell decreases gradually and the solution vanishes at a certain $u = u_c$. The corresponding values of u_c and $x_c = u_c l_g$ for two values of the parameter M and the two types of boundary conditions $\eta|_S = 0$ and $\eta'|_S = 0$, found by a numerical solution of the linear equation

$$\frac{d^2\eta}{du^2} = -u|u|^{1/3}\frac{3}{3+M}\eta,$$

are presented in Table I. The fact that the curves with M = 0 and M = 1, shown in Fig. 1, are very close, in spite of the essentially different form in these cases of the nonlinear term in Eq. (2.5), is noteworthy. The difference in the nonlinearity is most important in the region where $\eta > |u|^{1/3}$, i.e., where the expansion (1.8) ceases to be applicable and the expansion (1.10) ought to be used (the corresponding region is marked out in Fig. 1 by a circle). The proximity of the curves with M = 0 and M = 1 points, however, to the fact that the exact form of the equilibrium equation in this region is not of great importance, and Eq. (1.12) is, in all probability, a satisfactory approximation for the description of the behavior of $\rho_s(x)$ in the transition layer.

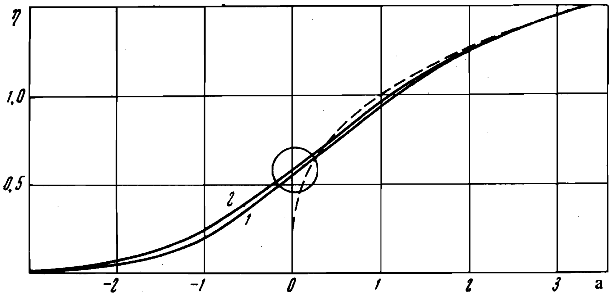

FIG. 1. Order-parameter distribution $\eta(u)$ near the HeI-HeII phase interface in the gravitational field; 1—according to the version in [3] of the phenomenological theory of superfluidity; 2—according to the version in [2]; the dashed curve is the plot obtained without allowance for correlation effects.

Table I

	M=0		M=1	
	$\eta=0$	$\eta'=0$	$\eta=0$	$\eta'=0$
u_c	2,290249	1.058224	2,496685	1.15357
x_c, cm	$1{,}55\cdot 10^{-2}$	$0{,}72\cdot 10^{-2}$	$1{,}70\cdot 10^{-2}$	$0.784\cdot 10^{-2}$

B. BOUNDARY CONDITION FOR THE ORDER PARAMETER NEAR A RIGID WALL

Near a rigid wall, on liquid helium acts an attractive potential which owes its origin to the van der Waals dispersion forces[10]. At small distances to the wall $(x \ll x_0)$ this potential has the form

$$U(x) = -\Theta/x^3, \tag{2.6}$$

while at large distances $(x \gg x_0)$ the exponent approaches four. The constant Θ, which characterizes the range of the potential, and the characteristic distance x_0, over which the change in the power law occurs, depend on the material of the wall. A typical value of $\Theta = 16 - 100$°K[11] (x is in this case measured in atomic layers; one layer a = 3.6 Å), and $x_0 \approx 50 - 200$ Å.

From the classical point of view the action of the potential U(x) ought to lead to the formation near the solid surface of a film of normal He I[12] 3), whose thickness, $d_0 = (|dT_\lambda/d\mu|\Theta/|t_0|)^{-n}$, ought to increase as the λ-point is approached first as $|t_0|^{-1/3}$ and, then as $|t_0|^{-1/4}$. On the face of it, it seems that the corresponding functions should be added as corrections to the formulas for the shift of the λ-point in narrow gaps and capillaries $d_c \sim (-t_0)^{-2/3}$, the superfluid-mass defect near a rigid surface

$$\Delta = \int_0^\infty [\rho_{s\infty} - \rho_s(x)]dx = \mathrm{const}$$

etc., figuring in the GP theory which assumes the vanishing of Ψ directly on a rigid surface. In reality, however, as the computations carried out below show, the distribution $\rho_s(x)$ for $(T_\lambda - T)/T_\lambda \ll 1$ is only slightly sensitive to the form of the external potential, and formulas of the type $d_c = d_0 + \pi\xi(T)$, where d_0 is of the order of a few interatomic distances, remain valid right up to the λ-point.

The equilibrium equation for the order parameter near a rigid wall has in the corresponding dimensionless variables $(\eta = \psi/(-t_0)^{1/3},\ u = \bar{x}/(-t_0)^{2/3})$ the form4)

$$\frac{d^2\eta}{du^2} = \frac{3}{3+M}\left\{M\eta^5 + (1-M)\left|1-\left(\frac{u_0}{u}\right)^3\right|^{2/3}\eta^3 + \left[\left(\frac{u_0}{u}\right)^3 - 1\right]\left|\left(\frac{u_0}{u}\right)^3 - 1\right|^{1/3}\eta\right\}, \tag{2.7}$$

$$u_0 = \frac{a}{\xi_{0M}}\left|\frac{dT_\lambda}{d\mu}\Theta t_0\right|^{1/3} = u_{00}(-t_0)^{1/3},$$

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3) When $T_\lambda - T < 0.4$ °K the normal film is not formed, since the helium near the wall goes over sooner into the solid state.

4) Taking into account the slight sensitivity of $\eta(u)$ to the concrete form of the potential U(x) at large distances, we taken the exponent in the expression for U(x) in the following computations to be equal to three.

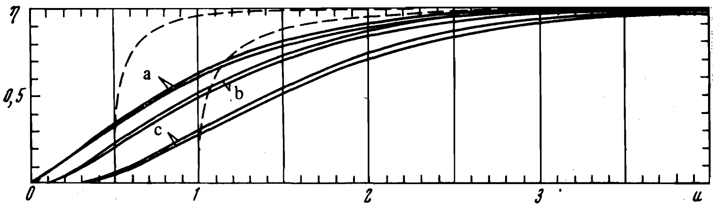

FIG. 2. Order-parameter distribution near a rigid wall for a few values of the parameter u_0: a–u_0 = 0, b–¼, c–½. The upper curves correspond to the version of the phenomenological theory with M = 1 [2], the lower curves–to the version with M = 0 [3]. The dashed curves are the plots of the order-parameter distribution which one gets when correlation effects are not taken into account. The curves (a) correspond to the usual GP theory in which van der Waals' forces are not taken in account.

Table II

u_0	$M=0$		$M=1$		$M=0$		$M=1$	
	$\Delta(u_0)$	$[\Delta-\Delta(0)]\cdot u^2_0$	$\Delta(u_0)$	$[\Delta-\Delta(0)]\cdot u^2_0$	u_c	$[u_c-u_c(0)]\cdot u^2_0$	u_c	$[u_c-u_c(0)]\cdot u^2_0$
0	1,414	1	1.32	0.866	3.1416	1	3.6276	0,866
$^1/_8$	1,432	1,15	1.333	1,02	3.1605	1,21	3.6439	1.043
$^1/_4$	1,486	1,15	1.380	1.01	3.2167	1.20	3.6932	1,050
$^1/_2$	1,665	1.00	1.547	0.92	3,4078	1.06	3.8671	0.958
1	2,166	0,752	2.031	0,714	3.9338	0,792	4.3683	0.741
2	3,35	0,485	3.21	0.472	5.1250	0,496	5.5437	0.479

where t_0 is the distance to the λ-curve at infinity where there is no field.

Examples of the η(u) curves obtained by a numerical solution of Eq. (2.7) for some values of the parameter u_0 are shown in Fig. 2. We see that at temperatures not too close to the λ-point ($u_0 \sim 1$), the η(u) distributions differ essentially from the distributions predicted by the usual GP theory, which does not take the van der Waals forces into account (the curves (a)), as well as from the distributions which one obtains when correlation effects are not taken into account (the dashed curves). As $T \rightarrow T_\lambda$, however, the corresponding curves increasingly approach the curves (a). In particular, in the range of values of u where η(u) appreciably differs from zero, the solution η(u) is even better approximated by the formula

$$\eta_>(u) = \nu\,\mathrm{th}\frac{u-b}{\nu\sqrt{2}}\left\{1-(1-\nu^2)\,\mathrm{th}^2\left[\frac{u-b}{\nu\sqrt{2}}\right]\right\}^{-1/2}, \tag{2.8}$$

where $\nu^2 = (3 + M)/(3 + 3M)$. The formula (2.8) corresponds to the solution of Eq. (2.7) in the absence of an external field, with the boundary condition η = 0 obtaining, however, not at the rigid wall itself, but at some distance b from it.

On the basis of numerical computations alone it is difficult to follow how the distance b at which η effectively vanishes varies with temperature. Let us, therefore, use an approximate analytic representation of the solutions, a representation which is easily obtained by smoothly matching at u = u_0 the expression (2.10) with the asymptotic form of η(u) at small u:

$$\eta_<(u) = cu\exp\{-u_0^2/\xi u\}, \quad \xi^2 = {}^1/_3(M+3). \tag{2.9}$$

We then obtain for the parameters b and c the following values:

$$b = \frac{u_0^2}{\xi+u_0}, \quad c = \frac{\xi\exp(u_0/\xi)}{(\xi+u_0)\sqrt{2}}. \tag{2.10}$$

It follows from the formula (2.10) for b that if the range of the field is long compared to the coherence length ($u_0 \gg \xi$), then the variations in η(u) follow the variations in the external field, and η vanishes practically at the classical interface. If, however, $u_0 \ll \xi$, then $b \approx u_0^2/\xi$, or, in terms of dimensional variables,

$$d_0 = b\xi_M(T) = u_{00}^2\xi_{0M}[3/(M+3)]^{1/2}. \tag{2.11}$$

Thus, at temperatures close to the λ-point, the characteristic distance d_0 over which the influence of the van der Waals forces is effectively felt is a finite quantity and does not depend on the proximity to the λ-point. This circumstance is confirmed also by numerical computations the results of which are partially presented in Table II. In the first and third columns of this table are given the values of the superfluid-mass defect

$$\Delta = \int_0^\infty [1-\eta^2(u)]du,$$

computed in the absence (u_0 = 0) and presence of a field, and in the fifth and seventh columns, the values of the critical thickness of the film with the boundary condition η = 0 at the free surface. The values of the critical thickness were determined by a numerical solution of the linear equation

$$\frac{d^2\eta}{du^2} = \frac{3}{3+M}\left[\left(\frac{u_0}{u}\right)^3-1\right]\left|\left(\frac{u_0}{u}\right)^3-1\right|^{1/6}\eta. \tag{2.12}$$

The quantities in the even columns are analogs of the parameter d_0(T) measured in this case in units of

$$d_0(T_\lambda) = \frac{a^2}{\xi_{0M}}\left(\left|\frac{dT_\lambda}{d\mu}\right|\Theta\right)^{2/3}.$$

As can be seen from the table and from the approximate expression (2.12), d_0(T) decreases, generally speaking, with distance from the λ-point.

Turning to the real situation, we note that the value of d_0 at the λ-point is the maximum value. At the same time for certain substances $d_0(T_\lambda)$ turns out to be 3–13 Å. In this connection it is clear that the results obtained concerning the temperature dependence d_0(T), even if we put aside the question of validity at such distances of the macroscopic treatment, can, generally, only be of qualitative importance. We recall that we derived Eq. (1.13) only for the case of weak fields when terms containing density derivatives may not be taken into account. Allowance for the density gradients ought to have led to a smoother growth of t(x) in the region of small x than follows from formula (2.6) and, consequently, to smaller values of d_0. On the other hand, the slope of the λ-curve $|dT_\lambda/d\mu|$ (or $|dT_\lambda/dP|$) increases with the density. This effect acts in the opposite direction. Finally, the magnitude of the parameter b could also be essentially influenced (especially at low temperatures) by the postulated formation near a rigid surface of a film of solid He^4, which also ought to have led to the growth of d_0.[5]

C. The Superfluid Component Density Distribution Near Charged Bodies

In the presence of an electric field the extra chemical potential of the liquid is equal to

$$U(r)=-\left(\frac{\partial\varepsilon}{\partial\rho}\right)_T\frac{E^2(r)}{8\pi},\qquad(2.13)$$

where E is the strength of the electric field and ϵ is the permittivity.

Using the relation

$$\varepsilon-1=4\pi\alpha\rho/G\ll1,$$

which is fulfilled with a high degree of accuracy in liquid helium, we can rewrite (2.13) in the form

$$U(r)=-\frac{\alpha}{2G}E^2(r),\qquad(2.14)$$

where $\alpha = 0.1233$ cm³/mole is the molar polarizability and G = 4.002 is the molecular weight of helium.

The threshold value of the intensity for the electrical breakdown in helium $E_{thr} \sim 2$ MV. To this value corresponds the shift of the λ-point, $\Delta T_\lambda \sim 10^{-3}$ °K. Thus, in the case of charged bodies of macroscopic dimensions, the observation of the proximity effects connected with the coexistence of the superfluid and normal phases of helium is possible only in a narrow neighborhood of the λ-point: $T_\lambda - T \lesssim 10^{-3}$ °K.

Let us continue the discussion, using as an example bodies of the simplest geometric shape: a cylinder and a sphere. In this case the λ-transition temperature varies with distance to the center of the cylinder (sphere) according to the law

$$t(r)=t_0+(r_{00}/r)^n=t_0[1-(r_0(T)/r)^n],\qquad(2.15)$$

where n is equal to 2 or 4 respectively for the cylinder or sphere.

If the radius

$$r_0(T)=\left(\left|\frac{dT_\lambda}{d\mu}\right|\alpha Q^2/2G\varepsilon^2|t_0|\right)^{1/n}=r_{00}(-t_0)^{1/n},$$

(Q is the charge of the body) is sufficiently large, then t(r) can be expanded near the location of the classical interface $r = r_0(T)$. We then obtain for an interface thickness $l(T)$

$$l(T)=\xi_{0M}^{3/5}(r_{00}/|t_0|n)^{2/5}=\xi_M(T)[r_0(T)/n\xi_M(T)]^{2/5},\qquad(2.16)$$

while the condition for the applicability of this formula is

$$[l(T)/r_0(T)]n\ll1\qquad(2.17)$$

or

$$[\eta\xi_M(T)/r_0(T)]^{3/5}\ll1.\qquad(2.17a)$$

The condition (2.17) is fulfilled practically for any macroscopic body. The superfluid component density distribution near the boundary will then be the same as in the case of the gravitational field (see section A). The only additional factor here is that the location and, hence (according to (2.16)), the thickness of the interface can be varied by varying the charge of the body.

Let us now consider another limiting case when the charged body is of microscopic dimensions. Such objects in helium are positive and negative ions. In the case of an ion of charge Ze the distance $r_0(T)$ at which classically (in the absence of correlation effects) a transition to the normal phase should occur is equal to

$$r_0(T)=\left(\left|\frac{dT_\lambda}{d\mu}\right|\alpha Z^2e^2/2G\varepsilon^2|t_0|\right)^{1/4}=r_{00}(-t_0)^{1/4},\qquad(2.18)$$

where at the saturated-vapor pressure $r_{00} = 4.52Z^{1/2} \times 10^{-8}$ cm × (deg)$^{1/4}$.

As in the case of the van der Waals forces, the ratio $r_0(T)/\xi(T) = 1.65(-t_0)^{5/12}$ quite rapidly tends to zero as $T \to T_\lambda$. For this reason, we can expect that in reality the influence of the electric polarization forces on the order-parameter distribution near an ion when $T \to T_\lambda$ will be felt not over a distance $r_0(T)$ from the center of the ion, but over a certain distance R_9 which practically does not depend on temperature and does not exceed a few interatomic distances. A concrete estimate yields for R_0 the value[6]

$$R_0(T\to T_\lambda)=0{,}5(1{,}5)^{3/5}r_{00}(r_{00}\sqrt{3/(3+M)}/\xi_{0M})^{3/5}=4.4Z^{1/2}\cdot10^{-8}\text{ cm}.$$

This value is somewhat smaller than the assumed radius $R_+ = 5.35$ Å of the solid core in the case of positive ions, and considerably smaller than the radius $R_- \approx 16$ Å of the cavity in the case of an electron "bubble." Thus, the influence of the electric field can be neglected in considering the question of the structure of an ion near the λ-point. The dominant effect is the vanishing of Ψ in the vicinity of the solid core, or at the free surface of the "bubble."

At lower temperatures the influence of the electric field may, however, be of greater importance (especially in the case of positive ions for which $R_0 \approx R_+$). As an illustration we show in Fig. 3 a few examples of the curves $\eta(u)$ determined by a numerical solution of Eq. (1.13) written in spherical coordinates:

$$\frac{d^2\eta}{du^2}+\frac{2}{u}\frac{d\eta}{du}=\frac{3}{3+M}\left\{\left[\left(\frac{u_0}{u}\right)^4-1\right]\left|\left(\frac{u_0}{u}\right)^4-1\right|^{1/3}\eta\right.$$
$$\left.+\left|\left(\frac{u_0}{u}\right)-1\right|^{2/3}\eta^3(1-M)+M\eta^5\right\}.$$

Here, as in the case of the van der Waals forces,

$$\eta=\psi/(-t_0)^{1/3},\quad u=\bar r/(-t_0)^{2/3},\quad u_0=r_0(T)/\xi_M(T).$$

---

[5] We could have tried to take the influence of the formation of the solid film into account with the aid of the boundary condition $\eta = 0$ at u = a. The formula (2.10) would have then changed as follows:

$$b=u_0^2[u_0+\xi\,\mathrm{th}\,(u_0^2/\xi a-u_0/\xi)]^{-1},$$

where $\xi^2 = (3 + M)/3$. When $a \to 0$ we obtain the former expression, while when $a \gg u_0^2/\xi$ we have $d_0(T) = a$, i.e., the effect connected with the formation of the solid film is the dominant effect.

[6] With allowance for the boundary condition $\Psi(R_\pm) = 0$ on the surface of the ion,

$$R_0(T\to T_\lambda)\approx0{,}5R_m\frac{2+\mathrm{th}\,\{0{,}4[(R_m/R_+)^{5/3}-1]\}}{1+2\,\mathrm{th}\,\{0{,}4[(R_m/R_+)^{5/3}-1]\}},$$

where $R_m = r_{00}(1.5r_{00})\sqrt{3/(3+M)}/\xi_{0M})^{3/5}$, and it is assumed that $R_+ < R_m$.

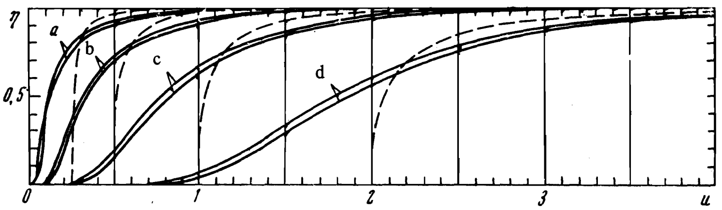

FIG. 3. Order-parameter distribution near an electric point charge (an ion in helium) for a few values of the parameter u_0: a) $u_0 = 1/4$, b) 1/2, c) 1, and d) 2. The upper curves correspond to the version of the phenomenological theory with M = 1 [2], the lower curves, with M = 0 [3]. The dashed curves show the order-parameter distribution which would be obtained in the absence of correlation effects.

3. CONCLUSIONS

We have considered above in detail the question of the influence of nonuniform external fields on the superfluid component density distribution. As it turned out, for a sufficiently slowly varying field—e.g., for the gravitational field—the corresponding effects (the penetration of the superfluidity into the normal-phase region) can appear at macroscopic distances.

In the case of thin films and capillaries it is shown that allowance for the van der Waals forces leads naturally to the boundary condition $\Psi = 0$ near a rigid wall, the point where Ψ vanishes being somewhat displaced from the rigid surface. This displacement does not however, exceed (3—13) Å and is comparable in magnitude with the shift which the postulated formation near the solid surface of a film of solid He^4 gives rise to.

In the case of charged particles (ions in helium) the influence of electric fields at $T \to T_\lambda$ is completely masked by the boundary condition $\Psi = 0$ at the surface of the solid core in the case of positive ions, or at the surface of the solid core in the case of positive ions, or at the surface of an electron "bubble." This condition itself leads, however, to the formation near an ion of a normal film the effective thickness of which increases as the λ-point is approached. This should clearly affect the mobility of the ions.

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APPENDIX

TRANSFORMATION OF INCOMPLETE THERMODYNAMIC POTENTIALS NEAR THE λ POINT

Let us choose as the basic thermodynamic potential, the potential $\Phi(P, T, \Psi)$ in the variables (P, T, Ψ). Let us suppose that near the λ-curve this potential has the form

$$\Phi(P, T, \Psi) = \Phi_0(P, T) + \varphi_0 t_P^2 \ln|t_P| + t_P^2 f(t_P/\Psi^3), \qquad (A.1)$$

$$t_P = T - T_\lambda(P) = T - T_\lambda - \frac{dT_\lambda}{dP}(P - P_\lambda), \qquad (A.2)$$

where $\Phi_0(P, T)$ is the regular part of the potential, and T_λ and P_λ are the coordinates of some fixed point on the λ-curve.

Let us expand $\Phi_0(P, T)$ near the point (T_λ, P_λ) in powers of $\Delta T = T - T_\lambda$ and $\Delta P = P - P_\lambda$ up to quadratic terms and express ΔP in the expression obtained in terms of ΔT and t_P with the aid of (A.2). After some transformations we obtain:

$$\begin{aligned}\Phi_0(P, T) = \Phi_0(P_\lambda, T_\lambda) - \left(S_\lambda{}^0 - V_\lambda{}^0 \frac{dP_\lambda}{dT}\right)\Delta T \\ - \frac{1}{2}\left(\frac{dS_\lambda}{dT} - \frac{dP_\lambda}{dT}\frac{dV_\lambda}{dT}\right)(\Delta T)^2 - \frac{dP_\lambda}{dT}\frac{dV_\lambda}{dT} t_P \Delta T \\ - V_\lambda \frac{dP_\lambda}{dT} t_P + \frac{1}{2}\left(\frac{\partial V^0}{\partial P}\right)_{T,\lambda}\left(\frac{dP_\lambda}{dT}\right)^2 t_P^2.\end{aligned} \qquad (A.3)$$

The transition to the incomplete free energy $F(V, T, \Psi)$ is accomplished by adding to $\Phi(P, T, \Psi)$ the term $-PV$ and subsequently minimizing $F(V, T, \Psi; P)$ with respect to t_P. As a preliminary, it is convenient by analogy with (A.2), to introduce in place of V another variable:

$$t_V = T - T_\lambda(V) = \Delta T - \frac{dT_\lambda}{dV}\Delta V. \qquad (A.4)$$

For $F(V, T, \Psi; t_P)$ we have

$$\begin{aligned}F(V, T, \Psi; t_P) = \Phi(T, t_P, \Psi) - PV \\ = \Phi(t_P, T, \Psi) - \left[P_\lambda + \frac{dP_\lambda}{dT}(\Delta T - t_P)\right]\left[V_\lambda + \frac{dV_\lambda}{dT}(\Delta T - t_V)\right] \\ = F_0(V_\lambda, T_\lambda) - \left(S_\lambda + P_\lambda \frac{dV_\lambda}{dT}\right)\Delta T - \frac{1}{2}\left(\frac{dS_\lambda}{dT} + \frac{dP_\lambda}{dT}\frac{dV_\lambda}{dT}\right)(\Delta T)^2 \\ + \frac{dV_\lambda}{dT}\frac{dP_\lambda}{dT} t_V \Delta T + P_\lambda \frac{dV_\lambda}{dT} t_V + \left[\frac{1}{2}\left(\frac{\partial V^0}{\partial P}\right)_{T,\lambda}\left(\frac{dP_\lambda}{dT}\right)^2 t_P^2\right. \\ \left. + \varphi_0 t_P^2 \ln|t_P| + t_P^2 f\left(\frac{\Psi}{|t_P|^{1/3}}\right)\right] - \frac{dP_\lambda}{dT}\frac{dV_\lambda}{dT} t_V t_P.\end{aligned} \qquad (A.5)$$

It can be seen from (A.5) that the nature of the relation connecting t_P with t_V will depend on the relative magnitude of one or another term in the last square brackets in (A.5). The ratio of the second to the first term is equal in order of magnitude to $(k_T^1/k_T^0)\ln|t_P|$, where k_T^1 is the coefficient attached to the logarithmic part of the isothermal compressibility and k_T^0—to its regular part. In helium, at the saturated-vapour pressure, the ratio $k_T^1/k_T^0 \sim 1/160$. This means that the temperature range in which the logarithmic renormalization of the critical exponents becomes important is limited in helium by a practically inaccessible, $\sim 10^{-70}$ °K(!) wide neighborhood of the phase-transition point. Outside this region the logarithmic term can be neglected in determining the relation between t_V and t_P.

When $\Psi \lesssim \Psi_e(t_P)$ the relative magnitude of the third term does not exceed $\Delta k_T/k_T^0$, where Δk_T is the "jump" in the compressibility, which is $\sim 1/70$ at the saturated-vapor pressure. The relation between t_V and t_P in this case is, to a high degree of accuracy, linear:

$$t_P = \frac{dV_\lambda/dT}{(dP_\lambda/dT)(\partial V^0/\partial P)_{T,\lambda}} t_V \equiv \gamma t_V. \qquad (A.6)$$

The parameter γ determining the renormalization of the coefficients in the expansion of the function f in powers of $\Psi/|t|^{1/3}$ is close to unity: at the saturated-vapor pressure $\gamma \approx 1.05$—1.25. The expansion coefficients change just as insignificantly in the transformation of the thermodynamic potential from the variables (P, T, Ψ) to (μ, T, Ψ). In this case γ differs from unity by the amount

$$\rho_\lambda S_\lambda \Big/ \frac{dP_\lambda}{dT} \approx 2\%.$$

When $\Psi \gg \Psi_e(t)$ the function $t^2 f(t/\Psi^3)$ has the asymptotic form $C\Psi^6$. The third term in the square brackets does not then depend on t, and does not change when the thermodynamic potentials are transformed.

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