EFFECT OF COMBINATION COUPLING ON THE SPECTRAL AND STATISTICAL PROPERTIES OF MULTIMODE FLUCTUATIONS IN A TRAVELING-WAVE LASER

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A study is reported of the statistical and spectral properties of the electromagnetic field emitted by a two-level system in a traveling-wave cavity. It is assumed that the generation occurs at the atomic transition frequency, whereas for all the remaining modes the threshold condition is not satisfied (these are the subthreshold modes). The nature of the weak (subthreshold) emission depends on the presence or absence of the combination coupling between the subthreshold modes and on the direction of observation. A detailed analysis is given of the role of the combination coupling. All the required spectral characteristics are derived, including the shape of the spectral profiles, their position, the integrated power in the subthreshold mode, and so on.

INTRODUCTION

FLUCTUATIONS in laser radiation have been investigated by a number of workers (see, for example,[1-3]). However, no account has been taken of the effect of different modes on one another. It is known,[4] however, that a strong field in one mode has an important effect on the interaction between the resonance medium and other modes.[4] In this paper we consider generation in one mode and fluctuations in all other modes. The fluctuation problem will be solved in the linear approximation (such modes will be called subthreshold modes). Under certain definite conditions the main factor in the interaction between subthreshold modes is the combination coupling. The importance of this type of interaction between the modes in a laser was first pointed out by Lamb[5] (Malakhov and Sandler[6] have considered a similar problem but without taking into account the combination coupling). It will be shown below that combination coupling leads to a modification in the spectrum and in the statistical properties of fluctuations in the subthreshold modes.

Consider a ring laser in which one traveling wave is generated. The situation is quite different for subthreshold modes traveling the same direction as the generated wave (forward direction), and for waves traveling in the opposite direction (backward direction). Waves traveling in the backward direction are completely independent of one another even in the presence of a strong generation field. There is only a change in the probabilities of emission and absorption of a photon by atoms in the upper and lower working levels, respectively (all other parameters are expressed in terms of these characteristics). In this case, our results are identical with those reported by Rautian,[7] who did not take into account the combination coupling between the modes. On the other hand, for waves traveling in the forward direction, and having frequencies located symmetrically relative to the strong-field frequency, the combination coupling need not be taken into account. In the simplest case, when the generation frequency is equal to the atomic transition frequency, this leads to the splitting of the radiated line associated with each subthreshold mode into two components. (We note that similar splitting for a standing-wave laser is exhibited by the formulas derived by Kuznetsova, who investigated the stability of monochromatic generation.[4]) A pair of modes subject to combination coupling will thus give four lines (see Fig. 1), two of which (a1 and a2) can, in accordance with[8], be associated with amplitude modulation of the generating wave and the other two (ph1 and ph2) with phase modulation. It is important that lines a1 and a2 and, similarly, lines ph1 and ph2 are strongly correlated with one another. Below we shall obtain expressions for the frequencies and intensities of these lines, and the photon statistics for them.

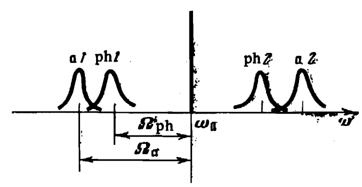

FIG. 1

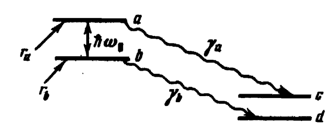

FIG. 2

The shape of the spectrum can also be obtained by classical analysis of the electromagnetic field, whereas problems involving the intensity of the fluctuations and photon statistics require a quantum-mechanical approach.

1. SEMICLASSICAL ANALYSIS

Consider a resonant medium placed in an optical traveling-wave cavity, and suppose that the medium consists of fixed atoms whose energy structure is as shown in Fig. 2. One traveling wave is generated at the frequency ω_0 of the atomic transition. As indicated in the introduction, the subthreshold modes will be analyzed in the linear approximation. Each of the subthreshold waves traveling in the backward direction and each pair of subthreshold waves traveling in the forward direction and having frequencies symmetrically located relative to the strong generation mode are completely independent of all the other subthreshold modes. The entire analysis can therefore be performed for any one subthreshold wave traveling in the backward direction and any pair with combination coupling. The field in the cavity is then of the form

$$E(x, t) = \mathscr{E}_0 \cos[\omega_0(t - x/c) + \varphi_0] + E'(x, t) + E''(x, t), \tag{1}$$

where $\mathscr{E}_0$, φ_0, and ω_0 are the amplitude, phase, and frequency of the strong field. Next,

$$E'(x, t) = \mathscr{E}_+' \exp[-i(\omega_0 + \Delta)(t - x/c)] + \mathscr{E}_-'^* \exp[i(\omega_0 - \Delta)(t - x/c)] + \text{c.c.}$$

is the field of the pair of waves with frequencies $\omega_0 \pm \Delta$ traveling in the forward direction. The quantity $\Delta = 2\pi cn/L$ is the mode separation in the cavity ($n = 1, 2, \ldots$), L is the cavity perimeter, and

$$E''(x, t) = \mathscr{E}'' \exp[-i(\omega_0 + \Delta)(t + x/c)] + \text{c.c.}$$

is the field in the subthreshold wave with frequency $\omega_0 + \Delta$, traveling in the backward direction.

Using existing calculations[4,7,8] of the polarizability of an atom in the field given by Eq. (1), we can write down the truncated equations for the generation fields and the subthreshold modes as follows:

$$\dot{\mathscr{E}}_0 = \frac{\delta\omega}{2}\left(\frac{N}{N_{th}}\frac{1}{1+I} - 1\right)\mathscr{E}_0, \tag{2}$$

$$\dot{Z}_a = \lambda_a(I) Z_a, \quad \dot{Z}_{ph} = \lambda_{ph}(I) Z_{ph}, \quad \dot{\mathscr{E}}'' = \lambda(I)\mathscr{E}''. \tag{3}$$

In the equations for the subthreshold modes traveling in the forward direction, which are given by Eq. (3), we use, instead of the slow field amplitudes $\mathscr{E}_+'$, $\mathscr{E}_-'^*$, their linear combinations which are denoted by Z_a and Z_{ph}:

$$Z_a = \mathscr{E}_+' \exp(i\varphi_0) + \mathscr{E}_-'^* \exp(-i\varphi_0),$$
$$Z_{ph} = \mathscr{E}_+' \exp(i\varphi_0) - \mathscr{E}_-'^* \exp(-i\varphi_0). \tag{4}$$

The convenience of using these variables lies in the fact that instead of a set of coupled equations we obtain two independent equations. The quantity Z_a describes the amplitude modulation of the generation, whereas Z_{ph} describes the phase modulation.[8] The remaining designations are as follows: $\delta\omega$—cavity line width, $I = d^2\mathscr{E}_0^2/\hbar\gamma\gamma_{ab}$—dimensionless generation power, $N = N_a - N_b$—the difference between the populations of the working levels, N_{th}—threshold difference between the working level populations, and

$$\lambda_a = {}^1\!/_2\delta\omega(N(1 - 2\beta I)/N_{th}(1 + I)(1 - if) - 1),$$

$$\lambda_{ph} = {}^1\!/_2\delta\omega\left(\frac{N}{N_{th}(1+I)}\frac{1}{(1-if)} - 1\right)$$

$$\lambda = {}^1\!/_2\delta\omega(N(1 - \beta I)/N_{th}(1 + I)(1 - if) - 1);$$

$$\beta = \frac{1 - if/2}{(1 - if)(1 - if/\varkappa) + I}, \qquad f = \frac{\Delta}{\gamma_{ab}}, \qquad \varkappa = \frac{\gamma}{\gamma_{ab}} \leqslant 1.$$

Equations (2) and (3) are written on the assumption that the field relaxation time in the cavity is much longer than the time $1/\gamma_{ab}$ for the atomic coherence decay. Moreover, it is assumed that $\gamma_a = \gamma_b \equiv \gamma$, but γ_{ab} is not, in general, equal to γ ($\gamma_{ab} \geq \gamma$).

The importance of combination coupling can be elucidated by comparing λ (which does not include combination coupling) with the quantities λ_a and λ_{ph}. It is clear that in λ_a the term proportional to βI is double in comparison with the analogous term in λ, and is absent altogether from λ_{ph}.

In the case of time-independent generation ($\dot{\mathscr{E}}_0 = 0$), we find from Eq. (2) that $N/N_{th}(1 + I) = 1$. This substantially simplifies the expressions for λ_a, λ_{ph}, and λ. The solutions of Eq. (3) are, of course, obvious. They enable us to write down the field in the cavity in the following explicit form:

$$E'(x, t) = {}^1\!/_2 Z_a(0)\exp(-\Gamma_a t)\{A_+(x)\exp[-i(\omega_0 + \Omega_a)t] + A_-^*(x)\exp[i(\omega_0 - \Omega_a)t]\}$$
$$+ {}^1\!/_2 Z_{ph}(0)\exp(-\Gamma_{ph} t)\{A_+(x)\exp[-i(\omega_0 + \Omega_{ph})t] - A_-^*(x)\exp[i(\omega_0 - \Omega_{ph})t]\} + \text{c.c.}, \tag{5}$$

$$E''(x, t) = {}^1\!/_2\mathscr{E}''(0)\exp(-\Gamma t)\exp\left[i(\omega_0 + \Omega_0)t + i(\omega_0 + \Delta)\frac{x}{c}\right] + \text{c.c.}$$

where

$$A_\pm(x) = \exp[-i\varphi_0 + i(\omega_0 \pm \Delta)x/c],$$
$$\Omega_a = \Delta - \operatorname{Im}\lambda_a, \quad \Omega_{ph} = \Delta - \operatorname{Im}\lambda_{ph}, \quad \Omega_0 = \Delta - \operatorname{Im}\lambda = {}^1\!/_2(\Omega_a + \Omega_{ph}),$$
$$\Gamma_a = -\operatorname{Re}\lambda_a, \quad \Gamma_{ph} = -\operatorname{Re}\lambda \quad \Gamma = -\operatorname{Re}\lambda. \tag{6}$$

For the frequencies Ω_a and Ω_{ph} (see Fig. 1) we have

$$\Omega_a = \Delta - \Delta\frac{\delta\omega}{2\gamma_{ab}}\frac{\varkappa^2 - \varkappa(\varkappa + 2)I + f^2}{(\varkappa + \varkappa I - f^2)^2 + (\varkappa + 1)^2 f^2},$$

$$\Omega_{ph} = \Delta - \Delta\frac{\delta\omega}{2\gamma_{ab}}\frac{1}{1 + f^2}.$$

It will be shown in Sec. 2 that these frequencies are present in the beat spectrum. For practical purposes, it is convenient to have the frequency splitting ($\Omega = \Omega_a - \Omega_{ph}$) of the amplitude and phase modulation:

$$\Omega = \Delta\frac{\delta\omega}{2\gamma_{ab}}\frac{I}{1 + f^2}\frac{2 + \varkappa(3 + I) + \varkappa f^2}{\varkappa(1 + I - f^2/\varkappa)^2 + f^2(1 + \varkappa)^2/\varkappa}. \tag{7}$$

Figure 3 shows Ω as a function of the strong field. As the power increases, the splitting at first smoothly rises but, having reached a maximum at the point $I_{max} = I_0 + (I_0^2 = d)^{1/2}$, where

$$I_0 = \frac{f^2 + \varkappa^2}{\varkappa(\varkappa + 2)},$$

$$d = \frac{f^4 + f^2(3 + \varkappa^2 + 2/\varkappa) + \varkappa(2 + 3\varkappa)}{\varkappa(\varkappa + 2)}$$

it decreases to a value $\operatorname{Im}\lambda_{ph}$ that is independent of I. The point I_0 is characterized by the fact that, for this value of the strong-field power, the frequency of the

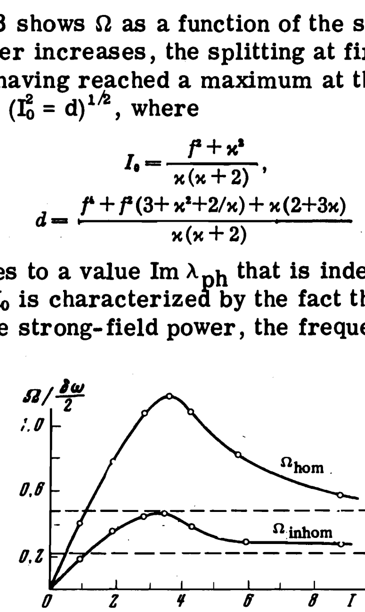

FIG. 3

amplitude modulation is equal to the cavity frequency of the subthreshold mode. The frequency of the subthreshold mode traveling in the backward direction is shown by Eqs. (5) and (6) to be equal to the mean frequency of the wave doublet traveling in the forward direction.

We shall now consider some numerical estimates for the splitting of the amplitude and phase modulation frequencies. For the ruby laser ($\kappa \sim 10^{-7}$) we find from Eq. (7) that $\Omega \approx f\delta\omega \sim 10^4$ Hz for the subthreshold modes for which $f^2 \approx \kappa I$. For the gas laser with homogeneous amplification $\kappa \leq 1$, and for the subthreshold modes for which $f^2 \approx \kappa I \approx 1$ we have $\Omega \sim \delta\omega \sim 10^6$ Hz.

In point of fact, the emission lines are not monochromatic. They all have the Lorentz shape. The width of the lines a1 and a2 (Fig. 1) is Γ_a, whereas that of the lines ph1 and ph2 is Γ_{ph}. The resolution condition for the two lines in the doublet can be written in the form

$$|\Gamma_a|+|\Gamma_{ph}|<|\Omega|.$$

This condition is satisfied for the above examples.

The amplitude and phase modulation frequency splitting occurs also in the case of inhomogeneous amplification. In the limiting case of inhomogeneous broadening $\gamma(1+I)^{1/2} \ll ku$ when the longitudinal and transverse relaxation constants are equal, $\gamma = \gamma_{ab}$, the splitting is characterized by

$$\Omega=\frac{\delta\omega}{2^{3/2}}\sqrt{1+I}\left[\left(\frac{1}{B}+\frac{1}{1+f^2}\right)(f\sqrt{B+a}-\sqrt{B-a})-\frac{I}{fB}\sqrt{B+a}\right]+\frac{\delta\omega}{2}\frac{I}{f(1+f^2)},$$

where

$$B=(a^2+4f^2)^{1/2},\quad a=1+I-f^2,\quad I=(N/N_{\rm th})^2-1.$$

Figure 3 shows the Ω_{inhom} corresponding to the last formula with f = 1.

Zeiger[9] has shown that, in the case of three-mode generation in a gas traveling-wave laser with symmetric location of the frequencies, it is possible to have different generation regimes, depending on the pump and the mode spacing. In addition to regions in which the amplitude or phase modulation is stable, there is also a region in which stable three-frequency regime does not exist. In our view, this fact can be explained by the coexistence of amplitude and phase modulation regimes occurring at different frequencies. In the standing-wave laser this was recently confirmed experimentally.[10]

2. QUANTUM-MECHANICAL ANALYSIS

We shall now suppose that the field radiated by the a ↔ b transition (working transition) is quantized. It will be described by the density matrix $\rho(t)$. In spite of the fact that we are considering the multimode problem, we can use some of the results of Lamb and Scully.[1] For example, the time derivative of the field density matrix is

$$\dot\rho=\sum_{\alpha=a,b}\langle N_\alpha[F_{cc}^{(\alpha\alpha)}(\infty,t)+F_{dd}^{(\alpha\alpha)}(\infty,t)-\rho(t)]\rangle = \sum_{\alpha,\alpha'=a,b}\left\langle N_\alpha\gamma_{\alpha'}\int_t^\infty F_{\alpha'\alpha'}^{(\alpha\alpha)}(t',t)\,dt'-\frac{1}{2}\rho(t)\right\rangle. \quad (8)$$

In these expressions N_a and N_b are the working level populations in the absence of the radiated field at the working transition, and γ_a and γ_b are the decay constants of the working levels (Fig. 2). The angular brackets represent averaging over the volume of the medium (and, in general, over the motion of the atoms). F is the density matrix for one atom and the radiation field in the interaction representation. The subscripts on this matrix represent the corresponding matrix elements between the levels a, b, c, or d (see Fig. 2), and the superscripts represent the initial conditions for which the solution of the equation

$$\dot F_s=-i\hbar^{-1}[H,F_s]-{}^1\!/_2[\Gamma,F_s]^+. \quad (9)$$

is sought. The matrices F_s and F are related by the unitary transformation

$$F_s(t)=\exp[i\hbar^{-1}H_0(t-t_0)]F(t)\exp[-i\hbar^{-1}H_0(t-t_0)], \quad (10)$$

where H_0 is the interaction Hamiltonian for the atom and the field. The complete Hamiltonian is $H = H_0 + H_1$, where H_1 is the interaction Hamiltonian

$$H_1=\sum_{\mathbf k}(\hbar g_{\mathbf k}^+a_{\mathbf k}+\hbar g_{\mathbf k}a_{\mathbf k}^+)$$

with the coupling constant

$$(g_{\mathbf k})_{ab}=\left(\frac{2\pi}{\hbar\omega_k V}\right)^{1/2}\mathbf d_{ab}\mathbf e\, e^{-i\mathbf k\mathbf r},$$

a_k^+ and a_k are the photon creation and annihilation operators in the mode with wave vector **k**, and Γ is the relaxation operator, whose matrix elements are $\Gamma_{\alpha\alpha'} = \gamma_\alpha\delta_{\alpha\alpha'}I$ (I is the unity operator).

Equation (9) is taken from the work of Scully and Lamb.[1] It presupposes that the transverse relaxation constant is related to the longitudinal relaxation constant by the formula $\gamma_{ab} = 1/2(\gamma_a + \gamma_b)$. The necessary condition for a pure field equation (i.e., a closed equation for the field density matrix ρ) is the same as for the one-mode problem,[1] i.e., the change in the field must be slow in comparison with the change in the state of the atom. This condition is consistent with the requirements of Sec. 1.

The formal solution of Eq. (9) can be written in the form

$$F_s(t)=U(t,t_0)F(t_0)U^+(t,t_0) = U(t,t_0)\rho(t_0)U^+(t,t_0), \quad (11)$$

where the operator describing the development of the system is

$$U(t,t_0)=\exp[-i\hbar^{-1}H(t-t_0)-{}^1\!/_2\Gamma(t-t_0)].$$

The fact that this is a solution can be verified by direct substitution. In writing down Eq. (11), we took into account the fact that the total density matrix F coincides with the field matrix ρ at the initial time. Using the resolvent operator

$$G(x)=(x+{}^1\!/_2 i\Gamma-\hbar^{-1}H)^{-1}$$

we can rewrite the operator $U(t, t_0)$ in the form

$$U(t,t_0)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}dx\,G(x)\exp[-ix(t-t_0)]. \quad (12)$$

As before, we shall be interested in the case where one traveling wave with wave vector **k** is strong and all the others are weak. Accordingly, we shall write down the interaction Hamiltonian in the form of two components $H_1 = \hbar v + \hbar v_1$, where $\hbar v$ determines the interaction with the strong field and $\hbar v_1$ with the weak field. We shall write the resolvent operator in the form of a series in powers of v_1, and will restrict our attention to terms up to the second order:

$$G = G^{(1)} + G^{(1)}v_1G^{(1)} + G^{(1)}v_1G^{(1)}v_1G^{(1)}, \qquad (13)$$

where $G^{(1)}$ represents the operator

$$G^{(1)}(x) = (x + {}^1/_2 i\Gamma - \hbar^{-1}H_0 - v)^{-1},$$

the equation for which is

$$G^{(1)} = G^{(0)} + G^{(0)}vG^{(1)}, \qquad (14)$$

and $G^{(0)}$ represents the free motion of the field and atom

$$G^{(0)} = (x + {}^1/_2 i\Gamma - \hbar^{-1}H_0)^{-1}.$$

Equation (14) can readily be solved in the quasi-resonance approximation, i.e., by taking into account only those processes in which the energy is conserved to within atomic line widths:

$$G^{(1)}_{an,an'} = \delta_{nn'}\frac{x - E_b/\hbar + i\gamma_b/2 - (n+1)\omega}{(x - x_1)(x - x_2)}$$

$$G^{(1)}_{bn+1,an'} = \delta_{nn'}\frac{v_{bn+1,an}}{(x - x_1)(x - x_2)},$$

where

$$x_{1,2} = \frac{1}{2}\left(\frac{E_a + E_b}{\hbar} - i\gamma_{ab} + (2n+1)\omega\right) \pm\left[\frac{1}{4}\left(\omega - \omega_0 - \frac{\gamma_a - \gamma_b}{2}\right)^2 + |v_{an,bn+1}|^2\right]^{1/2}.$$

By determining $G^{(1)}$ in explicit form, we have also determined G as given by Eq. (13). If we then evaluate the integral in Eq. (12), we obtain the matrix elements of the operator $U(t, t_0)$, which immediately enables us to construct the matrix elements of the density matrix F_S in which we are interested. All that remains is to carry out the transformation which is the inverse of Eq. (10), i.e., transform from the density matrix F_S in the Schrödinger representation to the matrix F in the interaction representation, and substitute the result in Eq. (8). This yields the kinetic equation for the field density matrix $\rho(t)$, which describes the electromagnetic field radiated by the working transition. This field includes both strong generation and the subthreshold emission. If in Eq. (13) we restrict our attention to the first term and reject all the others (zero-order approximation in v_1) we obtain the equation for only the strong generation, which is identical with the corresponding equation given by Scully and Lamb.[1]

The kinetic field equation which we have obtained has a complicated structure and is not very convenient for analysis. It is substantially simplified, however, if it is assumed that the generation field is classical. Formally, this means that we replace the corresponding operators by the complex strong-field amplitudes. This can be done with adequate accuracy because neglecting fluctuations in the generation field is equivalent to neglecting quantities of the order of $(\langle n\rangle)^{-1/2}$ in comparison with unity, where $\langle n\rangle$ is the mean number of photons in the generated mode (even for a small helium-neon laser with an output of $I \sim 10^{-1}$ we have $\langle n\rangle \sim 10^9$).

Because of the classical nature of the strong field, we shall consider the kinetic equation only for the subthreshold radiation. Its structure is such that it is clear that each of the backward waves and each pair of forward waves with frequencies symmetrically located relative to the generation field frequency are completely independent of all the other subthreshold modes. It is therefore possible to write down the equation for any particular chosen pair of waves. It will be of the form

$$\begin{aligned}\dot{\rho} = &-A_1(a_1a_1^+\rho - a_1^+\rho a_1 + \rho a_2a_2^+ - a_2^+\rho a_2)\\ &- (A_2 + {}^1/_2\delta\omega)(\rho a_1^+a_1 - a_1\rho a_1^+ + a_2^+a_2\rho - a_2\rho a_2^+)\\ &+ A_3(a_1a_2\rho - a_2\rho a_1 + \rho a_1^+a_2^+ - a_2^+\rho a_1^+)\\ &+ A_4(\rho a_1a_2 - a_1\rho a_2 + a_1^+a_2^+\rho - a_1^+\rho a_2^+) + \text{h.c.}\end{aligned} \qquad (15)$$

In these expressions, the subscript 1 represents the mode with frequency $\omega_1 = \omega - \Delta$ and subscript 2 corresponds to the frequency $\omega_2 = \omega + \Delta$. The coefficients A_1, A_2, A_3, and A_4 for waves traveling in the forward direction are given by

$$A_1 = \frac{\delta\omega_0}{2}\frac{1 + I/2}{1 - if} + A_4, \quad A_2 = \frac{\delta\omega_0}{2}\frac{I/2}{1 - if} + A_3,$$

$$A_3 = \frac{\delta\omega_0}{16}\frac{I^2}{(1 - if)^2}\left(\frac{2}{1 - if} - \frac{2}{I}\right)\left(\frac{I}{(1 - if)^2 + I} - 1\right),$$

$$A_4 = \frac{\delta\omega_0}{8}\frac{I(1 + I/2)}{(1 - if)^2}\left(\frac{2}{1 - if} + \frac{1}{1 + I/2}\right)\left(\frac{I}{(1 - if)^2 + I} - 1\right). \qquad (16)$$

These coefficients are obtained on the assumption that the strong-field frequency ω is equal to the atomic transition frequency ω_0, that pumping occurs only to the upper working level, and that all the relaxation constants are identical, i.e., $\gamma_a = \gamma_b = \gamma_{ab} \equiv \gamma$.

For the backward traveling waves, one simply sets the coefficients A_3 and A_4 equal to zero and leaves A_1 and A_2 as they stand.

The notation used in Eq. (16) was introduced in Sec. 1. In contrast to Sec. 1, the cavity width $\delta\omega$ for the subthreshold emission is not, in general, equal to the cavity width $\delta\omega_0$ for the strong field. This enables us to use the results when we are concerned not with the generation field but with an external signal. In this situation, we need only replace $\delta\omega$ in accordance with the formula

$$\frac{2|g|^2N}{\gamma}\frac{1}{1 + I} = \delta\omega_0, \qquad (17)$$

where N is the population of the upper working level.

Equation (15) is the quantum-mechanical analog of the truncated semiclassical equations. The latter can be obtained from Eq. (15) by the following procedure: multiply the equation by the operator a_1 and take the trace over both modes; then multiply the equation again by the operator a_2 and take the trace over both modes. The result is a set of two equations with two unknowns (both of which will, of course, depend on the strong-field amplitude which has its own equation).

There is one further point which we must note. In the derivation of Eq. (15) we have to take an average over the volume of the medium [see Eq. (8)]. We assume that the medium occupies the entire volume of the cavity since, otherwise, the equation would contain terms responsible for the so-called linear coupling and describing scattering by the boundaries of the medium from one mode into another.

It will now be convenient to transform to the diagonal representation of the density matrix. In accordance with

Eq. (10), the density matrix $\rho(t)$ can be associated with the quantity $P(\alpha_1\alpha_2 t)$ such that[11]

$$\rho(t)=\iint P(\alpha_1\alpha_2 t)\,|\alpha_1\alpha_2\rangle\langle\alpha_1\alpha_2|\,d^2\alpha_1\,d^2\alpha_2, \qquad (18)$$

where $d^2\alpha_1 = d\alpha_i' d\alpha_i''$, $\alpha_i' = \mathrm{Re}\,\alpha_i$, $\alpha_i'' = \mathrm{Im}\,\alpha_i$, $i = 1, 2$. In these expressions α_i and $|\alpha_i\rangle$ are the eigenvalue and eigenvector of the photon annihilation operator in the i-th mode, i.e., we have $\alpha_i|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$. The vector $|\alpha_1\alpha_2\rangle$ represents the product $|\alpha_1\rangle|\alpha_2\rangle$. A description of the transformation from the equation for ρ to the equation for P can be found, for example, in Gordon's paper.[12]

The equation for the density matrix in the diagonal representation $P(\alpha_1\alpha_2 t)$, written in terms of the variables

$$Z_a = {}^1/_2(\alpha_1+\alpha_2^*)\exp[i(\Delta-\Omega_a)t],$$
$$Z_{ph} = {}^1/_2(\alpha_1^*-\alpha_2)\exp[-i(\Delta-\Omega_{ph})t], \qquad (19)$$

permits the separation of variables in the form

$$P(\alpha_1\alpha_2 t) = P_a(Z_a t)P_{ph}(Z_{ph} t). \qquad (20)$$

For each of the functions $P_k(Z_k t)$ ($k = \alpha$, ph) the equation has the same form, i.e.,

$$\frac{\partial P_k}{\partial t}=\frac{\Gamma_k}{2}\frac{\partial}{\partial Z_k}(Z_kP_k)+\frac{\Gamma_k\sigma_k}{2}\frac{\partial^2 P_k}{\partial Z_k\,\partial Z_k^*}+\mathrm{c.c.} \qquad (21)$$

These two equations are completely analogous to the semiclassical equations given by Eq. (3) but, of course, have a more general character.

The constants Γ_a, Γ_{ph} and Ω_a, Ω_{ph} are defined in Sec. 1 [Eq. (6)]. They are given by

$$\Gamma_a=\delta\omega-{}^1/_2\,\mathrm{Re}(A_1-A_2-A_3+A_4);\ \Gamma_{ph}=\delta\omega-{}^1/_2\,\mathrm{Re}(A_1-A_2+A_3-A_4),$$
$$\Omega_a=\Delta-{}^1/_2\,\mathrm{Im}(A_1-A_2-A_3+A_4);\ \Omega_{ph}=\Delta-{}^1/_2\,\mathrm{Im}(A_1-A_2+A_3-A_4).$$

The constants σ_a and σ_{ph} are new and appear only in the quantum-mechanical theory. The fact that they are not zero is governed by the presence in Eq. (21) of terms involving the second derivative with respect to z. They in fact determine the correlation properties of the subthreshold radiation. They are given by the following formulas:

$$\sigma_a=\frac{\mathrm{Re}(A_1+A_4)}{\Gamma_a}=(1+I)(1+f^2)\Big/2\left\{\frac{\delta\omega}{\delta\omega_0}[(1+I-f^2)^2+4f^2]-(1+I)(1-I+f^2)\right\},$$
$$\sigma_{ph}=\frac{\mathrm{Re}(A_1-A_4)}{\Gamma_{ph}}=(1+{}^1/_2 I)\Big/2\left[\frac{\delta\omega}{\delta\omega_0}(1+f^2)-1\right]. \qquad (22)$$

As in Sec. 1, the quantities Z_a and Z_{ph} describe, respectively, the amplitude and phase modulation of the strong signal. It will be seen from the ensuing formulas [it is, of course, also clear from Eq. (21) which is well known in the theory of Markov processes] that σ_a and σ_{ph} determine the mean values of $|Z_a|^2$ and $|Z_{ph}|^2$ (i.e., they determine the amplitude and phase modulation power) and their dispersion.

The quantities Γ_a and Γ_{ph}, which form the denominators of the expressions for σ_a and σ_{ph}, represent the difference between the cavity width and the gain (see Sec. 1). When the denominators of Eqs. (22) and (23) vanish, this corresponds to the instability boundary of the single-mode generation for the growth of the amplitude or phase modulation.[8,13,14]

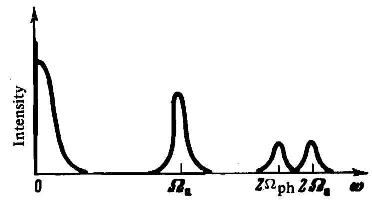

FIG. 4

The solutions of Eq. (21) for arbitrary initial conditions but subject to normalizability are[15,16]

$$P_k(Z_k,t)=\int d^2Z\,P(Z,0)K(Z,0|Z_k t),$$
$$K(Z,t_0|Z_k t)=\frac{1}{2\pi q_k\sigma_k}\exp\left(-\frac{|Z_k-Zp_k^*|^2}{q_k\sigma_k}\right); \qquad (23)$$
$$q_k=1-p_k,\ p_k=\exp[-\Gamma_k(t-t_0)].$$

In principle, these equations contain all the information about the subthreshold radiation. From the experimental standpoint, the most interesting are the Fourier transforms of correlators $\langle\mathscr{E}(t)\mathscr{E}^+(t-\tau)\rangle$ and $\langle\mathscr{E}(t)\mathscr{E}^+(t)\mathscr{E}(t-\tau)\mathscr{E}^+(t-\tau)\rangle$. The first of these determines the radiated frequency spectrum and the second determines the beat spectrum. $\mathscr{E}(t)$ is the positive-frequency electric-field operator in the Heisenberg representation.

It is possible to construct the above correlators and any other correlators directly with the aid of the density matrix in the interaction representation, i.e., in our case, with the aid of Eq. (13). For example, in the case of the frequency spectrum, this yields

$$J(\omega)=\int_{-\infty}^{\infty}\langle\mathscr{E}(t)\mathscr{E}^+(t-\tau)\rangle_{t\to\infty}e^{i\omega\tau}d\tau=\mathscr{E}_0^2\delta(\omega-\omega_0)$$
$$+8\pi^2\hbar\omega\sigma_a\left[\frac{\Gamma_a}{2\pi}\frac{1}{(\omega-\omega_0-\Omega_a)^2+\Gamma_a^2/4}+\frac{\Gamma_a}{2\pi}\frac{1}{(\omega-\omega_0+\Omega_a)^2+\Gamma_a^2/4}\right]$$
$$+8\pi^2\hbar\omega\sigma_{ph}\left[\frac{\Gamma_{ph}}{2\pi}\frac{1}{(\omega-\omega_0-\Omega_{ph})^2+\Gamma_{ph}^2/4}+\frac{\Gamma_{ph}}{2\pi}\frac{1}{(\omega-\omega_0+\Omega_{ph})^2+\Gamma_{ph}^2/4}\right].$$

Therefore, each of the two modes with combination coupling is the sum of two Lorentz profiles, one of which is centered on $\omega_0 \pm \Omega_a$ and has an integrated power $8\pi^2\hbar\omega\sigma_a$ and the other is centered on $\omega_0 \pm \Omega_{ph}$ and has an integrated power $8\pi^2\hbar\omega\sigma_{ph}$ (see Fig. 1). We shall not write out explicitly the shape of the beat spectrum. Figure 4 illustrates it in a general form. It is clear that not all the possible beats are present. For example, there are no beats at the frequency of Ω_{ph}. This is a consequence of the fact that the lines ph1 and ph2 in Fig. 1 form the phase modulation of the strong signal. In this case, beats between the central line and ph1, and between the central line and ph2, occur in antiphase and, therefore, cancel one another. For the same reason, there are no beats at the frequency $\Omega_a - \Omega_{ph}$. The beats between a1 and ph1 are in antiphase with the beats between a2 and ph2. Although beats at the difference frequency $\Omega = \Omega_a - \Omega_{ph}$ are absent from the spectrum, nevertheless, this frequency can be determined by measuring the separation between the lines $2\Omega_a$ and $2\Omega_{ph}$. This method of measuring the splitting of Ω is not, however, very convenient. The point is that the intensities of these lines are low (beats between two low-power signals). A more promising approach would seem to be an experiment in which the receiver intercepts only a portion of the radiation. For example, suppose that, in some way, the lines a2 and ph2 are excluded, i.e., the receiver intercepts only the weak signals a1 and ph1 and the strong signal at the center of the amplification line. In this experiment the beat spectrum will contain lines at frequencies Ω_a, Ω_{ph}, and $\Omega_a - \Omega_{ph}$. The lines at the frequencies Ω_a and Ω_{ph} are strong since they are proportional to the generation power. They would appear to be the most convenient for analysis.

Let us now consider the time-independent solution ($t \to \infty$) for our problem. Using Eqs. (20) and (23), we obtain the following expression for two subthreshold modes with combination coupling:

$$P(\alpha_1\alpha_2)=\frac{1}{4\pi^2\sigma_a\sigma_{ph}}\exp\left(-\frac{|\alpha_1+\alpha_2^*|^2}{4\sigma_a}-\frac{|\alpha_1-\alpha_2^*|^2}{4\sigma_{ph}}\right). \quad (24)$$

From this we obtain $\langle\alpha_1\rangle = \langle\alpha_2\rangle = 0$ while $\langle\alpha_1\alpha_2\rangle$ is not zero but is proportional to the difference $\sigma_a - \sigma_{ph}$. In the absence of combination coupling (and only in this case) the quantities σ_a and σ_{ph} are equal and, therefore, there is no correlation between α_1 and α_2, i.e., $(\langle\alpha_1\alpha_2\rangle = 0)$.

We require to determine the statistics of photons in the subthreshold modes in the presence of the strong field. This means that we must evaluate the quantity

$$\rho_{n_1n_1}=\sum_{n_2}\rho_{n_1n_2,n_1n_2}.$$

It is readily seen that

$$\rho_{n_1n_1}=\int P(\alpha_1)\langle n_1|\alpha_1\rangle\langle\alpha_1|n_1\rangle\, d^2\alpha_1,$$

$$P(\alpha_1)=\int P(\alpha_1\alpha_2)\, d^2\alpha_2.$$

Thus, to establish the statistics of photons in one mode we must integrate Eq. (24) with respect to the variable of one of the modes. As a result of this operation we obtain

$$P(\alpha_1)=\frac{1}{\pi(\sigma_a+\sigma_{ph})}\exp\left(-\frac{|\alpha_1|^2}{\sigma_a+\sigma_{ph}}\right).$$

It is well known that the Gaussian function for the diagonal representation of the density matrix corresponds to the Bose-Einstein statistics. This is conveniently characterized by a temperature T which, in our case, is given by

$$1/(\sigma_a+\sigma_{ph}) = \exp(\hbar\omega/kT) - 1.$$

It is clear that the strong field at the atomic transition frequencies does not lead to a change in the photon statistics. It merely changes the temperature of the equilibrium photon gas. Moreover, this change is different for the forward and backward subthreshold radiation.

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[1] M. O. Scully and W. E. Lamb, Phys. Rev. 159, 203 (1967).
[2] A. P. Kazantsev and G. I. Surdutovich, Zh. Eksp. Teor. Fiz. 56, 2001 (1969) [Sov. Phys.-JETP 29, 1075 (1969)].
[3] B. L. Zhelnov and G. I. Smirnov, Opt. Spektrosk. 32, 388 (1972).
[4] T. I. Kuznetsova and S. G. Rautian, Zh. Eksp. Teor. Fiz. 49, 1605 (1965) [Sov. Phys.-JETP 22, 1098 (1966)].
[5] W. E. Lamb, in: Quantum Optics and Quantum Electronics, ed. by O. V. Bogdankevich and O. N. Krokhina (Russ. Transl., Mir, 1966, p. 283).
[6] A. N. Malakhov and M. S. Sandler, Izv. Vyssh. Ucheb. Zaved., Radiofizika 14, 845 (1971).
[7] S. G. Rautian, in: Nelineĭnaya optika (Nonlinear Optics), FIAN, 1968, Vol. 43, p. 3.
[8] É. E. Fradkin and Z. K. Yankauskas, Zh. Tekh. Fiz. 38, 356 (1968) [Sov. Phys.-Tech. Phys. 13, 260 (1968)].
[9] S. G. Zeiger, Opt. Spektrosk. 34, 1 (1972).
[10] S. A. Gonchukov, O. E. Parodinkov, E. D. Protsenko, and V. A. Semchishen, ZhETF Pis. Red. 14, 235 (1971) [JETP Lett. 14, 155 (1971)].
[11] R. Glauber, transl. in Kvantovaya optika i kvantovaya radiofizika, (Quantum Optics and Quantum Electronics) ed. by O. V. Bogdankevich and O. N. Krokhin, Mir, 1966, p. 93.
[12] J. P. Gordon, Phys. Rev. 161, 367 (1967).
[13] H. Risken and K. Nummedal, J. Appl. Phys. 39, 4662 (1968).
[14] M. I. Kliot-Dashinskiĭ, P. P. Filatov, and É. E. Fradkin, sb. Voprosy prikladnoĭ matematiki i geometricheskogo modelirovaniya (Problems in Applied Mathematics and Geometric Simulation), ed. by S. N. Numerova, LISI, 1971, p. 23.
[15] H. Risken, Z. Phys. 191, 302 (1966).
[16] T. I. Kuznetsova, in: Nelineĭnaya optika (in: Nonlinear Optics), FIAN, 1968, Vol. 43, p. 116.

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