RADIATION EMITTED BY RELATIVISTIC PARTICLES IN PERIODIC STRUCTURES

V. N. BAĬER, V. M. KATKOV, and V. M. STRAKHOVENKO

Novosibirsk State University

Submitted July 19, 1972

Zh. Eksp. Teor. Phys. 63, 2121–2131 (December, 1972)

The process of emission of radiation from a moving relativistic electron in a periodic external electromagnetic field is considered. The polarization, spectral, and angular characteristics of the radiation are obtained. The region in which quantum effects are important is considered. It is shown that the quasiclassical operation method suggested earlier by the authors is applicable also to strongly inhomogeneous fields. Possible experiments are discussed.

1. INTRODUCTION

THE possible applications of magnetic bremsstrahlung radiation have stimulated interest in the radiation emitted from relativistic particles in periodic electromagnetic structures which are frequently called undulators. The properties of the radiation emitted in such structures are basically the same as those of the radiation of relativistic particles in the field of a plane electromagnetic wave, which may be regarded as a special type of undulator.

High-energy particles $[\gamma = 1/(1 - v^2)^{1/2} \gg 1]$[1)] emit mainly forward into a small angle $\sim 1/\gamma$, and the characteristics of the radiation depend strongly on the relationship between the angle of deflection (rotation) ψ of the particle in the field and the angle $1/\gamma$. In case I, defined by $\psi \gg 1/\gamma$, the radiation is formed in the coherence length l_m, defined by[2)]

$$l_m \approx \frac{R}{\gamma} = \frac{m}{eH} = \frac{1{,}7 \cdot 10^3}{H[\mathrm{Oe}]}\,\mathrm{cm},$$

and is of universal nature (see, for example,[1,2]) with the maximum of the spectral distribution of the radiation intensity located in the region $\omega \sim |\dot{\mathbf{v}}|\gamma^3$. In this case, the radiation is formed within each element of the periodicity of an undulator and the total intensity of the radiation is the incoherent sum of the intensities in each element. In the opposite case II, defined by $\psi \ll 1/\gamma$, the radiation depends strongly on the details of the structure of the external field. However, even in this case the properties of the radiation formed in periodic structures are basically universal, a considerable coherence is exhibited by the radiation emitted from different parts of the structure, and the maximum of the spectral distribution of the intensity is shifted (in a given field) by a factor l_m/l in the direction of higher frequencies (l is the length of the undulator period). In this situation only one or several of the lowest harmonics are emitted at the frequency of the periodic structure. This makes case II particularly interesting.

Some aspects of the use of periodic electromagnetic structures (undulators) have been discussed many times in the literature: they include microwave generation,[3] scattering of light on light,[4] and measurements of the energy of relativistic particles (see, for example,[5] and the literature cited there).

We shall determine the polarization and spectral characteristics of the radiation emitted by relativistic particles in an arbitrary undulator[3)] in case II (when $\psi \ll 1/\gamma$, i.e., when the length of the period is much less than the coherence length in the appropriate field). We shall discuss the classical problem of the emission of radiation in the case when the quantum corrections are small as well as the emission in the quantum region where experimental realization is easier than in case I.

2. RADIATION IN THE CLASSICAL REGION

We shall consider the radiation emitted from an infinite undulator in which motion is quasiperiodic (we shall understand this to be the motion which is periodic in the system in which the average displacement of the particle is zero, as shown in Fig. 1). The problem can be analyzed conveniently using the Fourier component of the velocity[4)]

$$\mathbf{v}_{\tilde{\omega}} = \int_{-\infty}^{\infty} \mathbf{v}(t)\, e^{i(\omega t - \mathbf{kr})}\, dt = \sum_{n=-\infty}^{\infty} \int_0^T \mathbf{v}(t)\, e^{ikx} e^{i\varphi_0 n} dt = \mathbf{v}_{\tilde{\omega}}^T \sum_{n=-\infty}^{\infty} e^{in\varphi_0}, \quad (1)$$

where $\varphi_0 = \omega T - \mathbf{k}\cdot\mathbf{l} = \omega T(1 - \mathbf{n}\cdot\mathbf{v}) = \tilde{\omega}T$, $\mathbf{v}$ is the average velocity of a particle, $\mathbf{l} = \mathbf{v}T$ is the length of the period of the structure, $\tilde{\omega} = \omega(1 - \mathbf{n}\cdot\mathbf{v})$. Bearing in mind that

$$\sum_{n=-\infty}^{\infty} e^{in\varphi_0} = 2\pi \sum_{k=-\infty}^{\infty} \delta(\varphi_0 - 2\pi k), \quad (2)$$

we find that

$$\mathbf{v}_{\tilde{\omega}} = 2\pi \mathbf{v}_{\tilde{\omega}}^T \sum_{k=-\infty}^{\infty} \delta(\varphi_0 - 2\pi k); \quad (3)$$

the transition from (1) to (3) corresponds to the transition from the Fourier integral to the Fourier series.

We have already mentioned that the properties of

---

1) We shall use the system of units with c = 1 and the metric $ab = a_0b_0 - \mathbf{a}\cdot\mathbf{b}$.

2) The coherence length is a function of the frequency; here, it is taken at the frequency of the maximum in the spectral distribution.

3) When the present paper was completed we learned of the preprint [6] in which the spectral distribution of the intensity was found in the classical region for some periodic structure models in case II.

4) The same approach is used in the quantum region.

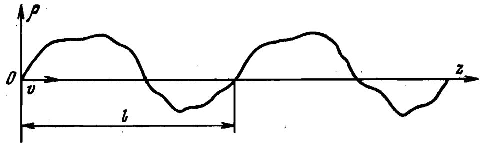

FIG. 1. Dependence of the deviation of a particle from its equilibrium position (ρ) on z.

the radiation depend strongly on the relationship between the angle of deflection of the particle in one period and the angle $1/\gamma$. In case I, when $\psi = |\dot{\mathbf{v}}|T = \Delta v \gg 1/\gamma$, it follows from Eq. (1) that mainly the frequencies $\omega \sim |\dot{\mathbf{v}}|\gamma/(1-\mathbf{n}\cdot\mathbf{v})$ are emitted and $\varphi_0 \gg 1$. In case II, when $\psi = |\dot{\mathbf{v}}|T = \Delta v \ll 1/\gamma$, the frequencies $\omega \sim 1/T(1-\mathbf{n}\cdot\mathbf{v})$ are mainly emitted and $\varphi_0 \lesssim 1$.

In case I, the phase φ_0 is large in the fundamental frequency region and then

$$2\pi\sum_k \delta(\varphi_0 - 2\pi k) \to 2\pi\int dk\,\delta(\varphi_0 - 2\pi k) = 1, \tag{4}$$

so that the radiation emitted in each element of the periodic structure is independent and the intensity of the radiation along a given direction can be found by adding the intensities at each wavelength (see[1,2]). However, at frequencies $\omega \lesssim \gamma^2/T$ in case I, the coherence length is comparable with the structure period so that the coherence must be allowed for in the emission of these frequencies.

In case II ($\Delta v \ll 1/\gamma$), the radiation is formed in many elements of the periodic structure and the coherence is always observed in the fundamental frequency region. In this case, it is convenient to transform $v^T_{\mu\tilde{\omega}}$ to the form

$$v^T_{\mu\tilde{\omega}} = \frac{i}{kv}\left(\dot{v}^T_{\mu\tilde{\omega}}(kv) - v_\mu(k\dot{v}^T_{\tilde{\omega}})\right), \tag{5}$$

where $kv = \tilde{\omega} = \omega(1-\mathbf{n}\cdot\mathbf{v})$,

$$\dot{v}^T_{\mu\tilde{\omega}} = \int_0^T \dot{v}_\mu(t)\,e^{i\tilde{\omega}t}dt, \tag{6}$$

and the velocity $\mathbf{v}$ is regarded as constant. If we use Eq. (5), we find that the general expression for the intensity of the radiation emitted by a relativistic particle in an infinite undulator is of the form[5)]

$$dI = \frac{e^2}{4\pi}\frac{d^3k}{2\pi T}\frac{\omega^2}{\tilde{\omega}^4}\left|(\mathbf{e}^*\dot{\mathbf{v}}^T_{\tilde{\omega}})(1-\mathbf{nv}) + (\mathbf{e}^*\mathbf{v})(\mathbf{n}\dot{\mathbf{v}}^T_{\tilde{\omega}})\right|^2\sum_{k=1}^{\infty}\delta(\varphi_0 - 2\pi k). \tag{7}$$

If the vector $\mathbf{n}$ is fixed, Eq. (7) is a sum of harmonics.

Summing over the polarizations of the emitted photons, we find that (ignoring terms $\sim 1/\gamma$)

$$\langle dI\rangle_{\mathbf{n}\omega} = \frac{e^2}{4\pi}\frac{d^3k}{2\pi T}\frac{\omega^2}{\tilde{\omega}^4}|\dot{v}^T_{\tilde{\omega}}|^2\frac{1}{2\gamma^4}\Big\{2y^2 + (1-2y)\Big[1+\cos 2(\beta-\varphi_1) - 2\sin^2\frac{\varphi}{2}\sin 2\beta\sin 2\varphi_1\Big]\Big\}\sum_{k=1}^{\infty}\delta(\varphi_0 - 2\pi k), \tag{8}$$

where $y = \gamma^2(1-\mathbf{n}\cdot\mathbf{v}) = \gamma^2\tilde{\omega}/\omega$. The Fourier component of the acceleration $\dot{\mathbf{v}}^T_{\tilde{\omega}}$ is expanded along two mutually perpendicular unit vectors $\mathbf{l}_1$ and $\mathbf{l}_2$ in a plane orthogonal to the vector $\mathbf{v}$:

$$\dot{\mathbf{v}}^T_{\tilde{\omega}} = |\dot{\mathbf{v}}^T_{\tilde{\omega}}|e^{i\delta}(\mathbf{l}_1\cos\beta + \mathbf{l}_2\sin\beta e^{i\varphi}), \tag{9}$$

$\mathbf{n}\cdot\mathbf{v} = v\cos\vartheta$, $\mathbf{n}\cdot\mathbf{l}_1 = \sin\vartheta\cos\varphi_1$. After integration over the azimuthal angle of emergence of the photons, we find that

$$\langle dI_{y\omega}\rangle = \frac{e^2}{4\pi}\frac{2}{T}\frac{d\omega}{\omega^2}\frac{dy}{y^6}|\dot{\mathbf{v}}|^2\gamma^6 F(1-2y+2y^2)\sum_{k=1}^{\infty}\delta\left(\frac{\omega yT}{\gamma^2} - 2\pi k\right), \tag{10}$$

where we have introduced the quantity

$$F = |\dot{\mathbf{v}}^T_{\tilde{\omega}}|^2\tilde{\omega}^2/4|\dot{\mathbf{v}}|^2, \tag{11}$$

which depends only on the structure of the field. Equation (10) follows from a formula given in[1] for the case of quasiperiodic motion defined by Eq. (3). Integrating Eq. (10) over the frequency, we obtain the angular distribution of the radiation

$$\langle dI_y\rangle = \frac{e^2}{4\pi}|\dot{\mathbf{v}}|^2\gamma^4\frac{dy}{2y^3}(1-2y+2y^2)\frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{F(\varphi_0 = 2\pi k)}{k^2}. \tag{12}$$

Finally, integrating over y, we find that the total intensity is

$$\langle I\rangle = \frac{4}{3}\frac{e^2}{4\pi}|\dot{\mathbf{v}}|^2\gamma^4\frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{F(\varphi_0 = 2\pi k)}{k^2}. \tag{13}$$

For any structure of the field we find that

$$\frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{F(\varphi_0 = 2\pi k)}{k^2} = \frac{1}{2}, \tag{14}$$

so that the total radiation intensity is given by the well-known expression $\langle I\rangle = \frac{2}{3}|\dot{\mathbf{v}}|^2\gamma^4 e^2/4\pi$.

Equation (7) can be used to find also the polarization characteristics of the radiation. We shall introduce the unit vectors

$$\mathbf{e}_1 = [\mathbf{nv}]/|[\mathbf{nv}]|, \quad \mathbf{e}_2 = [\mathbf{n}[\mathbf{nv}]]/|[\mathbf{n}[\mathbf{nv}]]|, \tag{15}^*$$

so that the polarization properties of the radiation can be described by the matrix

$$dI_{ik} = \frac{\langle dI\rangle}{2}\left(\delta_{ik} + \sum_{n=1}^{3}\xi_n\sigma_{ik}{}^n\right), \tag{16}$$

where σ^n are the Pauli matrices and ξ_n are the Stokes parameters:

$$\xi_n = a_n/c; \quad n = 1, 2, 3, \tag{17}$$

where

$$\begin{aligned} a_1 &= 2[\sin 2(\varphi_1-\beta) + 2\sin 2\beta\cos 2\varphi_1\sin^2(\varphi/2)]y(1-y),\\ a_2 &= 2y(1-y)\sin 2\beta\sin\varphi\\ a_3 &= -[\cos 2(\varphi_1-\beta) - 2\sin 2\beta\sin 2\varphi_1\sin^2(\varphi/2)]\cdot\\ &\quad\times(1-2y+2y^2) - (1-2y),\\ c &= 2y^2 + (1-2y)[1+\cos 2(\varphi_1-\beta) - 2\sin 2\beta\sin 2\varphi_1\sin^2(\varphi/2)]. \end{aligned} \tag{18}$$

For given $\mathbf{n}$ and ω, the radiation is completely polarized (usually elliptically), i.e., the Stokes parameters satisfy the relationship $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$, which can be proved by simple calculations.

The polarization properties of the radiation are also interesting in the case when integration is performed over a series of variables. After integration over the frequency and azimuthal angle of the emergence of photons, we obtain

$$\xi_1 = 0, \quad \xi_2 = \frac{2y(1-y)S}{1-2y+2y^2}, \quad \xi_3 = \frac{2y-1}{1-2y+2y^2}, \tag{19}$$

where Eq. (14) is taken into account and

$$S = \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{F\sin 2\beta\sin\varphi}{k^2}\Big|_{\varphi_0=2\pi k}. \tag{20}$$

[5)] We shall use the Heaviside system of units.

*$[\mathbf{nv}] \equiv \mathbf{n}\times\mathbf{v}$

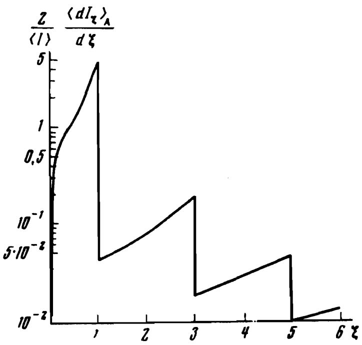

FIG. 2

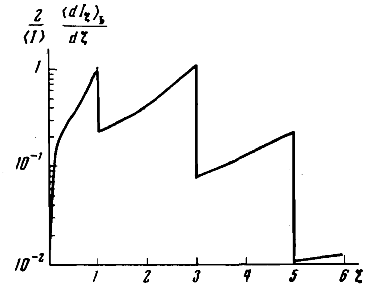

FIG. 3

Finally, after integration over the polar angle of the radiation, we find the polarization properties of the radiation as a whole:

$$\xi_1 = 0, \quad \xi_2 = S/2, \quad \xi_3 = 1/2. \tag{21}$$

Thus, the radiation emitted from an undulator in case II is partly polarized and the degree of linear polarization (ξ_3) is universal and independent of the structure of the field, whereas the degree of circular polarization depends strongly on this structure.

By way of illustration, we shall consider two specific cases when the periodic structure consists of regions with a homogeneous magnetic field of different directions, the angle of rotation of a particle in a field of given direction is $\psi = |\dot{\mathbf{v}}|T \ll 1/\gamma$, the z axis is directed along $\mathbf{v}$, the x axis along $\mathbf{l}_1$, and the y axis along $\mathbf{l}_2$.

Case A:

$$\begin{gathered} H_y = H,\ 0 < z < l/2, \\ H_y = -H,\ l/2 < z < l, \\ H_x = H_z = 0. \end{gathered} \tag{22}$$

In this case, we have

$$F_A(\varphi_0 = 2\pi k) = 4\sin^4(\pi k/2), \qquad S = 0. \tag{23}$$

Substituting F_A in Eqs. (10)—(13), we obtain the spectrum of the emitted photons. Figure 2 shows the graph of the function $2\langle I\rangle^{-1}\langle dI\rangle_A/d\xi$, where $\xi = \omega/\omega_c$ and $\omega_c = 4\pi\gamma^2/T$.

The polarization of the radiation is partial and linear ($|\xi| = 1/2$), making an angle $\pi/4$ with the direction of the acceleration [see Eq. (21)] and there is no circular polarization. However, the circular polarization can be obtained in a magnetic field of a more complex configuration, discussed below.

Case B:

$$\begin{gathered} H_y = -H,\quad 0 < z < {}^1\!/\!_8 l,\quad {}^5\!/\!_8 l < z < {}^3\!/\!_4 l, \\ H_y = H,\quad {}^1\!/\!_8 l < z < {}^1\!/\!_4 l,\quad {}^1\!/\!_2 l < z < {}^5\!/\!_8 l, \end{gathered} \tag{24}$$

$$\begin{gathered} H_x = -H,\quad {}^1\!/\!_4 l < z < {}^3\!/\!_8 l,\quad {}^7\!/\!_8 l < z < l, \\ H_x = H,\quad {}^3\!/\!_8 l < z < {}^1\!/\!_2 l,\quad {}^3\!/\!_4 l < z < {}^7\!/\!_8 l. \end{gathered}$$

In this case, we find that $\beta = \pi/4$, $\varphi = -\varphi_0/4$, and

$$F_B(\varphi_0 = 2\pi k) = 32\sin^2\frac{\pi k}{2}\sin^4\frac{\pi k}{8}, \qquad S = -\frac{8}{\pi^2}[3G - 4G_1]. \tag{25}$$

Here,

$$G = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = 0.916, \quad G_1 = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\cos\left[\frac{(2k+1)\pi}{4}\right] = 0.753.$$

If we substitute F_B in Eqs. (10)—(13), we find the spectrum of the emitted photons, which is plotted in Fig. 3 (the units are the same as in case A). The radiation is now partly polarized in the elliptical sense [see Eq. (21)].

We have considered an undulator of infinite length but it can be shown that the expressions obtained are also applicable to an undulator with N periods to an accuracy within $\sim 1/N$.

We shall now discuss the classical range in more general terms. Periodic electromagnetic structures (undulators) can be used to generate directed (into an angle $\gtrsim 1/\gamma$) intense radiation beams of wavelengths ranging from infrared to x rays. The intensity of such radiation is proportional to the square of the external field. Therefore, it is desirable to use strong fields if high intensities are needed. However, in a field $H = 10^4$ Oe the wavelength of the radiation (the maximum of the spectral distribution) is ~ 0.2 cm. In fields of this kind only case I (or, at most, the intermediate case $\psi \sim 1/\gamma$) can be realized by forming a stationary magnetic "lattice." However, the distinguishing characteristics of case II—the shift of the maximum of the spectral distribution toward higher energies, the high monochromaticity, and the generation of circularly polarized radiation—are pronounced only in the region $\psi \ll 1/\gamma$. One way of reaching this region would be to employ lasers. It should then be possible to generate x rays with the aid of electrons of moderate energy. One possible scheme runs as follows.[6)] Electrons of energy ϵ travel in a storage ring of radius R. In the linear part of the ring the electrons enter a laser resonator where a focusing region, several centimeters long, is established. This region acts as an undulator. For $\epsilon = 25$ MeV and $R = 25$ cm the driving field in the storage ring is $H_S = 3\times10^3$ G, so that the maximum of the magnetic bremsstrahlung spectrum in the ring lies in the infrared ($\hbar\omega \sim 0.5$ eV). If we use a laser emitting at $\lambda = 10\ \mu$ (CO_2) we can generate photons of energy ~ 300 eV. If the field of the laser wave at the focus is $H = 2\times10^4$ Oe, the energy lost as a result of the emission of radiation will be comparable with the losses due to the magnetic bremsstrahlung in the storage ring.

3. RADIATION IN THE QUANTUM REGION

It is shown in[7,8] that the best approach to the problem of radiation emitted by a high-energy particle in an external electromagnetic field is the formalism employing the quasiclassical approximation even in the initial expressions. This approach is based on the fact

6)The authors are grateful to A. N. Skrinskiĭ for discussing this program.

that we can ignore the noncommutative nature of the dynamic variables (terms of the order of $\hbar\omega_0/\epsilon$, where ω_0 is the frequency of revolution) and only include the commutators of these variables with the field of the emitted photons, i.e., the radiative recoil (terms of the order of $\hbar\omega/\epsilon$). The operator method developed in[7,8] makes it possible, after the necessary commutations and expansions of exponential functions, to go over to quantities describing the classical trajectory of a particle. The exponential functions are expanded on the assumption that the length of formation of the radiation is much shorter than the characteristic length of the field inhomogeneities. An analysis made in[2] shows that this restriction is not essential and all that we have to do is to postulate that the motion is quasiclassical, $\hbar\omega_0/\epsilon \ll 1$, so that the expansion of the exponential functions yields (with the same precision) the following expression:

$$e^{-ikx(t_2)}e^{ikx(t_1)} = T\exp\left\{-\frac{i}{\hbar}\int_{t_1}^{t_2} dt[\mathscr{H} - \hbar\omega - \sqrt{(\mathbf{P}(t)-\hbar\mathbf{k})^2+m^2}]\right\}, \qquad (26)$$

where $\mathscr{H}$ and **P** are, respectively, the energy and momentum operators; T is the chronological product operator.

Expanding the right-hand side of the above equation in terms of $(1 - \mathrm{nv}) \sim 1/\gamma^2$ and retaining only the larger terms in the expansion, we obtain an expression which is formally identical with Eq. (16) in[7]:

$$e^{-ikx(t_2)}e^{ikx(t_1)} = \exp\left\{-i\frac{\mathscr{H}}{\mathscr{H}-\hbar\omega}[kx(t_2) - kx(t_1)]\right\}, \qquad (27)$$

where $k^2 = 0$.

Since the other commutators are also unrelated to the nature of the field, it follows that Eqs. (16)—(18) of[7] are valid with the quasiclassical precision in an arbitrary external field, provided this field is much smaller than the critical value ($H \ll H_0 = m^2/e\hbar$). Thus, these formulas also hold in the case when the length of the inhomogeneity of the field is much shorter than the characteristic length of the formation of magnetic bremsstrahlung. The quasiclassical criterion is now $\hbar\Omega/\epsilon \equiv 2\pi\hbar/T\epsilon$, i.e., the frequency of quanta in the external field should be much less than the electron energy.

Since the energy of the emitted quanta is $\hbar\omega \sim \hbar\Omega\gamma^2$ and the classical theory is applicable in the range $\hbar\omega \ll \epsilon$, the quantum effects in radiation can be identified with the aid of the parameter $\eta = 2\hbar\Omega\gamma^2/\epsilon$. If $\eta \ll 1$, we are dealing with the classical case, whereas if $\eta \gtrsim 1$ the quantum effects are important in the radiation process. The intensity of the emission of photons with the four-momentum k_μ is [see Eqs. (9), (17), and (18) of[7]][7)]

$$dI = \frac{e^2}{4\pi}\frac{d^3k}{(2\pi)^2}\left|\int_{-\infty}^{\infty} dt\, R(t)\exp\left\{\frac{i\varepsilon}{\varepsilon-\omega}kx(t)\right\}\right|^2, \qquad (28)$$

where R(t) is formally identical with the matrix element of the transition involving free particles subject to the laws of conservation and to the time dependence of the momentum $p = p(t)$. For scalar particles we have

$$R(t) = (\mathbf{e}^*\mathbf{p}(t))\,/\,(\varepsilon(\varepsilon-\omega))^{1/2}. \qquad (29)$$

For spinor particles the corresponding relationship is $R(t) = \varphi_f^+[A(t) + i\boldsymbol{\sigma}\mathbf{B}(t)]\varphi$, where

$$A(t) = \frac{\mathbf{e}^*\mathbf{p}(t)}{2\sqrt{\varepsilon(\varepsilon-\omega)}}\left(\sqrt{\frac{\varepsilon-\omega+m}{\varepsilon+m}} + \sqrt{\frac{\varepsilon+m}{\varepsilon-\omega+m}}\right),$$

$$\mathbf{B}(t) = \frac{1}{2\sqrt{\varepsilon(\varepsilon-\omega)}}\left(\sqrt{\frac{\varepsilon-\omega+m}{\varepsilon+m}}[\mathbf{e}^*\mathbf{p}(t)] - \sqrt{\frac{\varepsilon+m}{\varepsilon-\omega+m}}[\mathbf{e}^*(\mathbf{p}(t)-\mathbf{k})]\right). \qquad (30)$$

We have selected here the transverse gauge for the photon polarization.

Here, as in the classical region, we shall be interested in the case-II motion in periodic structures, where the length of the field inhomogeneity (structure period) is much less than the characteristic length of the magnetic bremsstrahlung and $\Omega = 2\pi/T \gg \gamma/R = Hm/H_0$. In this case, the intensity of the radiation emitted by scalar particles is obtained if Eq. (7), deduced in the classical theory, is multiplied by the factor $(\epsilon - \omega)^3/\epsilon^3$ and the phase $\varphi_0(\omega)$ is replaced with the phase $\varphi_0' = \varphi_0(\omega')$, where $\omega' = \omega\epsilon/(\epsilon - \omega)$. For this reason the Stokes parameters for the radiation emitted by scalar particles are identical in the classical and quantum cases [see Eq. (18)]; this is true before integration over the frequency.

In the case of spinor particles, the expression for the intensity, summed over the spin of the final electrons and averaged over the spin of the initial electrons, has the form

$$dI = \frac{e^2}{4\pi}\frac{d^3k}{2\pi T}\left(\frac{\varepsilon-\omega}{\varepsilon}\right)^3\frac{\omega^2}{\tilde{\omega}^4}\left\{\frac{(2\varepsilon-\omega)^2}{4\varepsilon(\varepsilon-\omega)}|(\mathbf{e}^*\dot{\mathbf{v}}_{\tilde{\omega}})(1-\mathbf{nv}) + (\mathbf{e}^*\mathbf{v})(\mathbf{n}\dot{\mathbf{v}}_{\tilde{\omega}})|^2\right.$$
$$\left. + \frac{\omega^2}{4\varepsilon(\varepsilon-\omega)}[|\dot{\mathbf{v}}_{\tilde{\omega}}|^2(1-\mathbf{nv})^2(\mathbf{e}^*\mathbf{e}) - |(\mathbf{e}\dot{\mathbf{v}}_{\tilde{\omega}})(1-\mathbf{nv}) + (\mathbf{ev})(\mathbf{n}\dot{\mathbf{v}}_{\tilde{\omega}})|^2]\right\}$$
$$\times\sum_{\kappa=1}^{\infty}\delta(\varphi_0' - 2\pi k). \qquad (31)$$

If we use the same axes and notation as before [see Eqs. (15)—(17)], we find that the Stokes parameters of spinor particles are

$$\xi_1^{(1/2)} = \frac{a_1}{c+\omega^2y^2/\varepsilon(\varepsilon-\omega)}, \quad \xi_2^{(1/2)} = \frac{(1+\omega^2/2\varepsilon(\varepsilon-\omega))a_2}{c+\omega^2y^2/\varepsilon(\varepsilon-\omega)},$$
$$\xi_3^{(1/2)} = \frac{a_3}{c+\omega^2y^2/\varepsilon(\varepsilon-\omega)}. \qquad (32)$$

Summing over the polarization of photons and integrating over the azimuthal angle of their emergence (factor 2π), we obtain (the index 0 refers to scalar particles and the index $1/2$ to spinor particles)

$$dI^{(0,1/2)} = \sum_{k=1}^{\infty}F\frac{e^2m^2\eta^2\chi^2\,du\,dy}{16\pi^3y^6u^2(1+u)^3}(y^2f^{(0,1/2)}(u) + 1/2 - y)\delta(uy - \eta k/2); \qquad (33)$$

here, we have used the following notation:

$$u = \frac{\omega}{\varepsilon-\omega};\ \eta = \frac{\omega_c}{\varepsilon} = \frac{4\pi\gamma}{Tm};\quad f^{(0)} = 1;\ f^{(1/2)}(u) = 1 + \frac{u^2}{2(1+u)};$$
$$\chi^2 = \frac{|\dot{\mathbf{v}}|^2\gamma^4}{m^2},$$

and F has been introduced earlier [see Eqs. (11). (23), and (25)].

Integration of Eq. (33) over the polar angles of the emergence of photons gives the spectral distribution

$$dI^{(0,1/2)} = \sum_{k=1}^{\infty}F\frac{e^2m^2\chi^2u\,du}{\pi^3\eta^2k^4(1+u)^3}\left(f^{(0,1/2)}(u) + 2\left(\frac{u}{\eta k}\right)^2 - \frac{2u}{\eta k}\right)\theta(\eta k - u). \qquad (34)$$

---

7)We shall henceforth assume that $\hbar = 1$.

The classical expressions for the intensity can be obtained from Eq. (34) by the substitution $(1+u)^3 \to 1$, $f \to 1$.

The total intensity of the radiation can be found by integrating first over u so as to obtain the angular distribution of the radiation:

$$dI^{(0,\frac{1}{2})} = \sum_{k=1}^{\infty} F \frac{e^2m^2\chi^2}{k^2\pi^3} \frac{x\,dx}{(1+\eta kx)^3} (f^{(0,\frac{1}{2})}(\eta kx) - 2x + 2x^2), \tag{35}$$

where $x = 1/2y$ $(0 \le x \le 1)$.

We shall now calculate the total intensity of the radiation in the asymptotic cases $\eta \ll 1$ and $\eta \gg 1$. If $\eta \ll 1$, we obviously obtain the classical expression for the intensity. If $\eta \gg 1$ and case A of Eq. (22) applies, we obtain (to the nearest power)

$$I_A^{(0)} = \frac{2e^2m^2\chi^2}{\pi^3\eta^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96} \frac{2e^2m^2\chi^2}{\pi^3\eta^2} = \frac{\pi}{48}\left(em\frac{\chi}{\eta}\right)^2 = \frac{e^2}{4\pi}\frac{(eHT)^2}{192}. \tag{36}$$

It is evident from the above expression that the intensity of the radiation is independent of the mass and energy of the particles involved. In case B, we obtain

$$I_B^{(0)} = \frac{2e^2m^2\chi^2}{\pi^3\eta^2}\Lambda_4 = \frac{1}{4}I_A^{(0)}, \tag{37}$$

$$\Lambda_m = \sum_{k=0}^{\infty} \frac{3 - 4\cos(\frac{1}{2}k + \frac{1}{4})\pi}{(2k+1)^m}.$$

For spinor particles we obtain the following expressions

$$I_A^{(\frac{1}{2})} = I_A^{(0)}\left(\ln\xi - \frac{5}{6} + \frac{96}{\pi^4}L_A\right),$$

$$L_A = \sum_{k=0}^{\infty} \frac{\ln(2k+1)}{(2k+1)^4} = 0.017, \tag{38}$$

$$I_B^{(\frac{1}{2})} = I_B^{(0)}\left(\ln\xi - \frac{5}{6} + \frac{384}{\pi^4}L_B\right),$$

$$L_B = \mathscr{L}_4 = 0.094, \qquad \mathscr{L}_m = \sum_{k=0}^{\infty} \frac{[3 - 4\cos(\frac{1}{2}k + \frac{1}{4})\pi]\ln(2k+1)}{(2k+1)^m}.$$

In the case of scalar particles, the total intensity is dominated by the contribution of the range of angles $\vartheta \sim \sqrt{\eta/\gamma}$, which are large compared with $1/\gamma$. In the case of spinor particles, the whole integration domain contributes to the logarithmic term.

In the range of high values of η, the radiation is concerned more strongly in the first harmonics than in the classical case. It would be interesting to determine the probability of the emission of photons of energy comparable with the electron energy. The necessary expression for the intensity must next be multiplied by $1/\omega = (1+u)/u\epsilon$. Then the probability of emission of photons of frequency ω is

$$dW^{(0,\frac{1}{2})} = \sum_{k=1}^{\infty} F \frac{e^2m^2\chi^2 du}{\pi^3\epsilon\eta^2k^4(1+u)^2}\left(f^{(0,\frac{1}{2})}(u) + 2\left(\frac{u}{\eta k}\right)^2 - \frac{2u}{\eta k}\right)\theta(\eta k - u), \tag{39}$$

and the angular distribution is of the form

$$dW^{(0,\frac{1}{2})} = \sum_{k=1}^{\infty} F \frac{e^2m^2\chi^2}{\pi^3k^3\epsilon\eta} \frac{dx}{(1+\eta kx)^2}(f^{(0,\frac{1}{2})}(\eta kx) - 2x + 2x^2). \tag{40}$$

If $\eta \ll 1$, we find that the total probability of emission is

$$W_A^{(0)} = \frac{8}{3}\frac{e^2m^2\chi^2}{\pi^3\epsilon\eta} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} = \frac{7}{3}\zeta(3)\frac{e^2m^2\chi^2}{\pi^3\epsilon\eta},$$

$$W_B^{(0)} = \frac{8}{3}\frac{e^2m^2\chi^2}{\pi^3\epsilon\eta}\Lambda_3, \quad \Lambda_3 \approx 0.445, \tag{41}$$

where $\zeta(n)$ is the Rieman zeta function. If $\eta \gg 1$, we find that

$$W_A^{(0)} = \frac{4e^2m^2\chi^2}{\pi^3\epsilon\eta^3} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^5} = \frac{31}{8}\zeta(5)\frac{e^2m^2\chi^2}{\pi^3\epsilon\eta^3}, \tag{42}$$

$$W_B^{(0)} = \frac{4e^2m^2\chi^2}{\pi^3\epsilon\eta^3}\Lambda_5, \quad \Lambda_5 \approx 0.197.$$

The probability of emission of radiation from particles with half-integral spin in the $\eta \gg 1$ case is

$$W_{A,B}^{(\frac{1}{2})} = \tfrac{1}{2}W_{A,B}^{(0)}\left(\ln\eta + \tfrac{1}{2} + \delta_{A,B}\right). \tag{43}$$

Here,

$$\delta_A = \frac{32}{31\zeta(5)} \sum_{k=0}^{\infty} \frac{\ln(2k+1)}{(2k+1)^5} = 0.005,$$

$$\delta_B = \mathscr{L}_5/\Lambda_5 \approx 0.148.$$

If we substitute $\delta = 0$ in Eq. (43), we find that the expression obtained agrees, to within a numerical factor, with the probability of Compton scattering in the case when the incident photon is "soft". This is due to the fact that an undulator can be represented by a superposition of standing waves of definite frequencies and these, in turn, can be regarded as sums of plane waves "traveling" in opposite directions, the dominant contribution being made by that wave whose wave vector is antiparallel to the electron velocity. The "softness" of the quanta of this wave follows from the condition $\hbar\Omega \ll \epsilon$ (this is the criterion of the quasiclassical case).

[1] L. D. Landau and E. M. Lifshitz, Teoriya polya, Nauka, 1967 (The Classical Theory of Fields, 3rd ed., Pergamon Press, Oxford, 1971).

[2] V. N. Baĭer, V. M. Katkov, and V. S. Fadin, Izluchenie relyativistskikh élektronov (Radiation of Relativistic Electrons), Atomizdat, M., 1972.

[3] V. L. Ginzburg, Izv. Akad. Nauk SSSR, Ser. Fiz. 11, 165 (1947).

[4] G. I. Budker, Vestn. Akad. Nauk No. 1, (1972).

[5] N. A. Korkhmazyan and S. S. Élbakyan, Dokl. Akad. Nauk SSSR 203, 791 (1972) [Sov. Phys.-Dokl. 17, 345 (1972)].

[6] D. F. Alferov, Yu. A. Bashmakov, and E. G. Bessonov, Preprint No. 23, Institute of Physics, Academy of Sciences of the USSR, M., 1972.

[7] V. N. Baĭer and V. M. Katkov, Zh. Eksp. Teor. Fiz. 53, 1478 (1967) [Sov. Phys.-JETP 26, 854 (1968)].

[8] V. N. Baĭer and V. M. Katkov, Zh. Eksp. Teor. Fiz. 55, 1542 (1968) [Sov. Phys.-JETP 28, 807 (1969)].

Translated by A. Tybulewicz
230