Strong plasma turbulence at helicon frequencies

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An analogy is drawn between hydrodynamic turbulence and collisionless-plasma turbulence at frequencies between the ion and electron cyclotron frequencies, when the ions can be assumed to be stationary while the electrons drift in crossed (self-consistent) random fields. In this case the system of equations can be reduced to a single equation (for the magnetic field H) which is close in structure to the velocity-field equation. The presence of spectral transfer in the region of large wave numbers is successfully demonstrated using the closure hypothesis. The pulsation spectrum of the magnetic fields is found with the aid of the Kolmogorov hypothesis to be $\sim k^{-7/3}$. A similar situation may obtain in a collision plasma with a strong Hall current and, possibly, in metals. The microcurrent power spectrum or, what is the same thing, the pulsation spectrum of the electron-velocity field, is derived from the Maxwell equations and the spectral dependence of H.

1. INTRODUCTION

Strong turbulence develops in the presence of intense noise in a plasma, so that to separate out a δ-function dependence of the spectral function $J(k, \omega)$ on the frequency ω no longer makes sense. On the other hand, it is often assumed that the fluctuations in the electric field are largely potential fluctuations. In the present paper we shall consider the inverse situation, i.e., the other limiting case, when the electric fields are largely rotational in nature. In view of the foregoing, it is natural to expect that we can trace the analogy between such turbulence and turbulence in hydrodynamics. It is significant that both the potentiality condition and the assumption that the turbulence is weak are often violated[1]. To consider strong turbulence we could use weakly coupled integral equations in which the adiabatic interaction is separated out. However, in view of the fact that there is no rigorous method for separating out such an interaction for an arbitrary situation, of no less validity will apparently be the other method—the construction of an analogy with hydrodynamic turbulence. Notice that Kadomtsev[2,3] made an assumption about the mixing length and used it to describe the turbulent diffusion of a plasma in a trap and in a positive glow-discharge column.

In the present paper we shall use the Kolmogorov similarity hypothesis to derive the pulsation spectrum of the magnetic field. It becomes clear right from the beginning that this approach cannot at all prove to be universally applicable to strong plasma turbulence, in view of the great variety of conditions and processes that occur in this case. In this connection we can at least point out the numerous attempts that have been made to obtain by this method the magnetic-field fluctuation spectrum in magneto-hydrodynamics for large magnetic Reynolds numbers and in the presence of strong fluctuations, i.e., when it no longer makes sense to speak of waves (see the reviews[4,5]). In other cases similarity and self-similarity hypotheses are used in conjunction with the kinetic equations for the waves, and it is even possible sometimes to justify these hypotheses (e.g., in ion-sound turbulence[6]).

We consider below a collisionless low-temperature plasma of constant density (i.e., a homogeneous plasma). Let there exist at zero time an ordered electron velocity (much higher than the velocity of the ions) which produces a current and, hence, a magnetic field. We shall consider the frequencies

$$\omega_i < \omega < \omega_e, \quad \omega_i = \frac{eH}{m_i c}, \quad \omega_e = \frac{eH}{m_e c}. \qquad (1.1)$$

We should bear in mind then that ω will subsequently not be the oscillation frequency, but the reciprocal of the fluctuation-correlation time. The assumption that the ions are stationary becomes natural at these frequencies. The electrons, on the other hand, drift in crossed fields. Electric (nonpotential) fields appear because of the variation in time of the magnetic fields. A nonlinear equation for the field and the fluctuation spectrum will be derived below.

2. THE BASIC EQUATIONS

All the fields in the present problem will be self-consistent; in particular, the existence of a uniform external magnetic field is not assumed, so that the frequencies ω_i and ω_e in (1.1) should be understood as gyrofrequencies in magnetic self-fields. The situation is reminiscent of quasi-hydrodynamics[7], but in view of the fact that the ions are not magnetized (they can be considered to be stationary), the drift approximation is applicable only for the electrons.

Further, it is necessary to use Maxwell's equations and the equations of motion. The displacement current can be neglected, for we shall assume that $\omega_e \ll \omega_0$ (ω_0 is the plasma frequency), and, furthermore, the primary electron velocity is the electric-drift velocity in the crossed fields:

$$\mathbf{v}_e = c[\mathbf{EH}] / H^2 \qquad (2.1)^*$$

(all the velocity corrections, connected with the nonuniformity and nonstationarity of the fields, contain the parameter ω/ω_e, which we conventionally assume to be small; moreover, the plasma is a low-temperature plasma, and $p = 0$).

Let us write out the equations which will be needed below:

$$\operatorname{rot} \mathbf{H} = -\frac{4\pi}{c} ne\mathbf{v}_e, \qquad (2.2)$$

$$\operatorname{rot} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}. \qquad (2.3)$$

The interaction matrix V_{ijf} for (2.4) has the form

$$V_{ifm}(\mathbf{k}, \mathbf{q}, \mathbf{p}) = {}^1/_2 k_e \{\varepsilon_{ief} q_m + \varepsilon_{iem} p_j\} \beta. \tag{3.6}$$

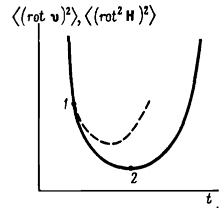
Substituting (3.6) into (3.5), we obtain (here C(k, t) is the spectral function of the **H** field)

$$\frac{\partial^2 C(k)}{\partial t^2} + \frac{4}{3} k^4 C(k) \chi_0 \beta^2 = \beta^2 \int \left(1 - \frac{(\mathbf{qp})^2}{q^2 p^2}\right) [q^4 + p^4 + (\mathbf{pq})] \times (p^2 + q^2) C(q) C(p)\, d\mathbf{q}, \quad \chi_0 = \int C(k)\, d\mathbf{k} = {}^1/_2 \langle H^2 \rangle. \tag{3.7}$$

It is not difficult to show (by integrating (3.7) over **k**) that (3.7) conserves energy. In hydrodynamics an equation of the type (3.7) is derived by equating the fourth-order semi-invariants to zero. The equation obtained also conserves energy and yields at the same time a growing vorticity:

$$\frac{d^2 \langle (\operatorname{rot} \mathbf{v})^2 \rangle}{dt^2} = \frac{1}{3} \langle (\operatorname{rot} \mathbf{v})^2 \rangle \tag{3.8}$$

(see[11] Sec. 19.3). The growth of $\langle(\text{curl } v)^2\rangle$ is not apparent from (3.8), but can be seen from the plot of the solution (see the figure). Even if the beginning (t = 0) of the process corresponds to the point 1, i.e., the derivative is negative at t = 0, the vorticity eventually increases—after the point 2. This circumstance illustrates the transfer of energy into the region of large wave numbers.



From (3.7) we obtain an equation similar to (3.8):

$$\frac{d^2 \chi_4}{dt^2} = \frac{4}{3} \beta^2 \chi_4^2 + \frac{64}{15} \beta^2 \chi_2 \chi_6;$$
$$\chi_2 = \int k^2 C(k)\, d\mathbf{k}, \quad \chi_4 = \int k^4 C(k)\, d\mathbf{k}, \tag{3.9}$$
$$\chi_6 = \int k^6 C(k)\, d\mathbf{k}.$$

It can be seen from (3.9) that $d\chi_4/dt$ increases in time (just like $d < (\text{curl } v)^2\rangle/dt$ from (3.8)), so that if at t = 0 the derivative $d\chi_4/dt \geq 0$, then the growth of χ_4 is guaranteed. If, however, at t = 0 the quantity $d\chi_4/dt < 0$, i.e., the function is at a point such as 1 (see the figure), then, since the second term on the right hand side of (3.9) is positive definite, the plot of χ_4 as a function of t will lie above the plot of $\langle(\text{curl } v)^2\rangle$ (naturally, it is assumed that $\langle(\text{curl } v)^2\rangle$ and χ_4 have been reduced to the same dimensions); see the dashed curve in the figure. Therefore $d\chi_4/dt$ will pass through zero and χ_4 will subsequently increase. Thus, we have obtained from (3.7) that at large t

$$\frac{d}{dt} \frac{\chi_4}{\chi_2} > 0,$$

i.e., we have found a transfer to the region of large wave numbers.

We have used the weak-coupling approximation to derive the spectral transfer. It is significant, however, that in hydrodynamics the correct spectrum of the stationary fluctuations cannot be derived from an equation of the type (3.7), and, therefore, it is hardly worth while to derive C(k) from (3.7) when $\partial C/\partial t = 0$. For this reason we shall use Kolmogorov's self-similarity hypothesis, assuming that there is energy flow to the region of large k. As in hydrodynamics, let us divide the k-range into three regions: I) the energy-containing region (the source of the fluctuation magnetic field for small k, $k < k_0$); II) the inertial region, where the energy flow is realized, $k_0 < k < k_d$; III) the dissipation region, $k > k_d$. Notice, however, that we have not introduced a dissipation term into (2.4), but we have implicitly assumed the existence at large k of dissipation, whose nature is not important for the region II.

Let, following Kolmogorov, a quasi-equilibrium state be realized in the region II) (ϵ_M, the constant energy flux in the region III, has the dimension of the energy derivative of dH^2/dt^2). Since in this case the strongest interaction is that between harmonics with close k (localization), ϵ_M should be expressed in terms of the spectral density E_M and k (it should be linear in β). Hence,

$$\varepsilon_M \sim \beta H_k^3 k^2 \sim \beta E_M^{3/2} k^{7/2}. \tag{3.10}$$

The expression (3.10) should be fulfilled up to a dimensionless constant of the order of unity.

From (3.10) we can derive the following spectrum:

$$E_M = A (\varepsilon_M / \beta)^{2/3} k^{-7/3}, \tag{3.11}$$

where $A \approx 1$. We recall that the Kolmogorov spectrum has the form $E \sim k^{-5/3}$. The difference between the Kolmogorov spectrum and (3.11) is evidently due to the fact that, although Eqs. (3.2) and (3.3) are similar, they differ in their order: (3.3) contains a second derivative, whereas (3.2) contains a first derivative.

It is not difficult to show that the presence of the dimensional parameter β in Eq. (2.4) does not in itself affect the possibility of using dimensional considerations. Indeed, going over to the new system of coordinates $x = x'\beta^{1/2}$, $k = k'\beta^{1/2}$, we obtain an equation free of β:

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{rot}' [\mathbf{H} \operatorname{rot}' \mathbf{H}]$$

(rot′ represents rot in the new system). Repeating the above-presented arguments, we have

$$\varepsilon_M \sim H_M^3 k'^2 \sim E_M^{3/2}(k') k'^{7/2};$$
$$E(k') \sim \varepsilon_M^{2/3} k'^{7/3}, \quad E(k) \sim (\varepsilon_M/\beta)^{2/3} k^{-7/3},$$
$$E(k')\, dk' = E(k)\, dk,$$

which coincides with (3.11).

If now H_0 is the field intensity in the energy-containing region, then it follows from (3.10) that

$$\varepsilon_M = H_0^3 \beta k_0^2 \tag{3.12}$$

(in the Kolmogorov turbulence $\epsilon = v_0^3 l^{-1}$), so that (3.11) can be written in the form

$$E_M = H_0^2 k_0^{4/3} k^{-7/3};$$

the decay time for the field of the interaction time

$$s_M = (H_0 \beta k_0^2)^{-1} \tag{3.13}$$

(in hydrodynamics $s = v_0^{-1} k_0^{-1}$). Comparing (3.13) with (2.5), we see that S_M is the reciprocal of the frequency of the helicons with the smallest k. Finally, the characteristic frequencies for different wavelengths (or the reciprocal of the correlation time for different k)

$$\omega(k) = \varepsilon_M^{1/3} \beta^{2/3} k^{4/3} = H_0 \beta k_0^{2/3} k^{4/3}. \tag{3.14}$$

It can be seen from (3.14) that the lifetime of the pulsations falls off sharply with the scale (in hydrodynamics $\omega \sim k^{2/3}$). Using the system (2.2), we can easily

Performing the curl operation on the expression $\mathbf{E} = c^{-1}\mathbf{v}_e \times \mathbf{H}$, which is equivalent to (2.1), and using Eqs. (2.2) and (2.3), we obtain

$$\frac{\partial \mathbf{H}}{\partial t} = \frac{c}{4\pi ne}\,\mathrm{rot}[\mathbf{H}\,\mathrm{rot}\,\mathbf{H}]. \tag{2.4}$$

To concretize the physical conditions, let us write out certain characteristic field quantities (the radius ρ_e of gyration of the electrons is determined by only their ordered motion: $\rho_e = v_e/\omega_e$):

$$E \approx \omega H / ck, \quad v_e = \omega / k,$$
$$\rho_e = \omega / k\omega_e \ll 1, \quad v_i = \omega_i / k \ll v_e,$$

k is the wave vector. The condition $\rho_e \ll k^{-1}$ should naturally be fulfilled for the drift approximation, and the expression $v_i \ll v_e$ verifies the self-consistency of the problem.

Notice that an equation of the type (2.4) can also be derived in the case of a collision plasma (it was, in particular, used by Gordeev and Rudakov in[8] (where, in addition, $p_e \neq 0$ and $n \neq$ const), and that the right hand side corresponds to the Hall current. One would think that the same equation will describe magnetic-field fluctuations in a metal, fluctuations whose frequencies will be lower than ω_i, as the ions there are stationary. It is remarkable that instead of the system (2.1)–(2.3) we have to deal with the single equation (2.4) containing a single unknown field. Knowing the spectrum of H (Eq. (2.4) alone is adequate in this case), we can easily derive from the system (2.1)–(2.3) the spectrum of the electron-velocity field (i.e., the microcurrent correlation), as well as the electric field.

In conclusion, we note that linear fluctuations for the region (1.1), namely, helicons, which are well-known from the theory of electromagnetic waves in a plasma, can also be derived from (2.4). Indeed, let $\mathbf{H} = \mathbf{H}_0 + \mathbf{h}$, where $\mathbf{H}_0$ is a uniform external field; then we obtain for the Fourier transform of h

$$-i\omega\mathbf{h} = \frac{c}{4\pi ne}[\mathbf{kh}](\mathbf{kH}_0).$$

Representing h in the form $\mathbf{h} = \mathbf{h}_1 + \mathbf{h}_2$, we easily derive the dispersion equation (the equation being the same for $\mathbf{h}_1$ and $\mathbf{h}_2$):

$$\omega = \pm\frac{\omega_e c^2 k^2}{\omega_0^2}|\cos\theta|, \quad \cos\theta = \frac{\mathbf{kH}_0}{kH_0}, \tag{2.5}$$

which coincides with the dispersion equation for helicons[9]. The spectrum for a weak helicon turbulence was obtained by M. Lifshitz and Tsytovich[10].

3. FLUCTUATION SPECTRUM IN k-SPACE

In view of the fact that no rigorous theory of strong turbulence exists, we are obliged to enlist the help of diverse sorts of hypotheses (isotropy, the similarity hypothesis, the spectral-transfer hypothesis). Let us outline the plan for the subsequent considerations. First of all, let us construct an analogy between the turbulence being investigated and hydrodynamic turbulence, which has been well investigated. Let us illustrate the existence of spectral transfer by the same methods used in hydrodynamics—no proof exists. For this purpose we use the closure hypothesis, or, equivalently, the weak-coupling approximation. In doing this we do not go over to the weak helicon turbulence approximation, for we assume that the turbulence is isotropic (whereas there exists in a weak turbulence a preferred direction—the direction of the uniform magnetic field). In essence, we shall use the hypothesis whereby we can equate the fourth-order semi-invariants to zero; the derivation of the equation for the spectral function is then mathematically analogous to the weak-coupling approximation.

Further, we shall use the Kolmogorov hypothesis to derive the fluctuation spectrum. The construction of the analogy with hydrodynamics is necessary for the justification of the introduction of the hypothesis. We shall henceforth analyze Eq. (2.4). It is important to note that (2.4) is similar in structure to the equation of motion for an incompressible fluid:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p, \quad \mathrm{div}\,\mathbf{v} = 0 \tag{3.1}$$

(we have neglected viscosity). This can be seen especially clearly if we express $\rho^{-1}\nabla p$ in terms of v using the incompressibility condition and the fact that ρ = const. It is convenient to go over into Fourier space:

$$\partial u_i/\partial t + i\sigma_{ij}(\mathbf{k})k_\alpha \int u_\alpha(\mathbf{k}^{(1)})u_j(\mathbf{k}^{(2)})\,d\lambda = 0;$$
$$\sigma_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2, \quad d\lambda = d\mathbf{k}^{(1)}d\mathbf{k}^{(2)}\delta(\mathbf{k} - \mathbf{k}^{(1)} - \mathbf{k}^{(2)}), \tag{3.2}$$

while in Fourier space (2.4) has the form

$$\partial H_i/\partial t = 2\beta\varepsilon_{[iab}k_n \int k_a^{(1)} H_b(\mathbf{k}^{(1)})H_{n]}(\mathbf{k}^{(2)})\,d\lambda, \tag{3.3}$$

$\beta = c/4\pi ne$. Here antisymmetrization with respect to the indices $[ij] = \frac{1}{2}(ij - ji)$ has been used.

Let us now turn to the turbulence problem, i.e., we shall assume that both the v and the H fields are random fields. As is well known, primary turbulent processes are connected with the nonlinear interaction of fields. In hydrodynamics, if $\langle v\rangle = 0$ and the turbulence is homogeneous and isotropic, then the nonlinear interaction conserves total energy (this is easily verified: if we take the scalar product of (3.1) with v(x) at the point x and average, using the homogeneity assumption, then $\partial\langle v^2\rangle/\partial t = 0$), energy being in this case transferred to the region of large wave numbers. It is easy to show that the nonlinear interaction in (2.4) also conserves energy. Indeed, $\langle \mathbf{H}\,\mathrm{curl}[\mathbf{H} \times \mathrm{curl}\,\mathbf{H}]\rangle$ can be expressed in terms of the divergence operator and, hence, in terms of a surface integral which vanishes in virtue of the homogeneity assumption (we recall that the Hall current conserves the energy of the magnetic field). Thus, $\partial\langle H^2\rangle/\partial t = 0$.

The conclusion concerning the spectral transfer of energy in hydrodynamics cannot be arrived at via the normal methods, and we are obliged to use the closure hypotheses, or, which is often an equivalent procedure, perturbation theory. Let us illustrate this transfer, using the same method for the v and H fields. For this purpose let us construct the analog of the kinetic equation by the method used in the weak-coupling approximation. It is useful to use the model equation

$$\partial C_i(\mathbf{k})/\partial t = \int V_{ifm}(\mathbf{k},\mathbf{q},\mathbf{p})C_f(\mathbf{q})C_m(\mathbf{p})\,d\mathbf{q};$$
$$V_{ifm}(\mathbf{k},\mathbf{q},\mathbf{p}) = V_{imf}(\mathbf{k},\mathbf{p},\mathbf{q}), \quad \mathbf{p} = \mathbf{k} - \mathbf{q}, \tag{3.4}$$

which differs from Kadomtsev's model equation[1] in that it is vectorial in nature and the time dependence is retained (nonstationarity is important in this case and the transition to the frequencies ω is not made). Repeating Kadomtsev's[1] calculation, we obtain

$$\partial^2 C(k,t)/\partial t^2 = 4\int V_{ifm}(\mathbf{k},\mathbf{q},\mathbf{p})V_{fab}(\mathbf{q},\mathbf{p},\mathbf{k})\sigma_{am}(\mathbf{p})\sigma_{ib}(\mathbf{k})C(p)C(k)\,d\mathbf{q}$$
$$+ 2\int V_{ifm}(\mathbf{k},\mathbf{q},\mathbf{p})V^{*}_{iab}(\mathbf{k},\mathbf{q},\mathbf{p})\sigma_{fa}(\mathbf{q})\sigma_{mb}(\mathbf{p})C(q)C(p)\,d\mathbf{q}, \tag{3.5}$$
$$\langle C_i(\mathbf{k},t)C_j(\mathbf{k}',t)\rangle = C(k,t)\delta(\mathbf{k}+\mathbf{k}')\sigma_{ij}(\mathbf{k}).$$

derive the fluctuation spectrum E_V of the electron-velocity-pulsation field v_e:

$$E_v = H_0^2 \beta^2 k_0^{4/3} k^{-1/3}; \qquad (3.15)$$

the frequency dependence here evidently remains the same as for the magnetic fields (3.14). Naturally, the microcurrent power spectrum is similar to (3.15). It is worth noting that the energy of the electron pulsations is concentrated in the region of small scales (the integral $\int E_V dk$ diverges as $k \to \infty$); at what scales we should cut off the spectrum (3.15) is not known, since we do not consider the dissipation mechanism here.

*[EH] ≡ E × H.

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