Symmetry properties of the collision integral and nonisotropic stationary solutions in weak turbulence theory

A. V. Kats and V. M. Kontorovich
Institute of Radiophysics and Electronics, Ukrainian Academy of Sciences
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Transformations of k space, consisting of rotations and extensions, are found; these transform the surface, on which the laws of conservation of energy and momentum are satisfied for colliding quasiparticles when the energy and momentum of one of the particles is fixed, into itself. These transformations are a generalization of the nonlinear Zakharov transformations in ω space and are transformations with conservation of angles, which convert geometric figures expressing conservation laws into similar figures. The symmetry properties of the collision integral for such transformations are investigated; drift solutions of the kinetic equation are found which are small nonisotropic deviations from isotropic solutions of the Kolmogorov type for weak turbulence. It is significant that these solutions cannot be obtained by inserting a Doppler-shifted frequency into the isotropic solution because of the nonequilibrium nature of the distributions. The case of a decay dispersion law is considered. Drift solutions are discussed for capillary waves and acoustic turbulence.

1. INTRODUCTION

In contrast with strong turbulence (turbulence of an incompressible liquid), the theory of weak turbulence[1,2] describes systems of waves possessing dispersion of the propagation velocity. The latter circumstance is very important and guarantees sufficiently weak interaction between the waves if their number is not too great. This also permits us to use (in the random phase approximation) the kinetic equation for the distribution function of the "quasiparticles" for a description of weak turbulence. This function, N(k), is proportional to the mean square amplitude of a wave with wave vector k:

$$\dot{N} = I_{\text{coll}}\{N\}. \qquad (1.1)$$

The collision integral $I_{\text{coll}}\{N\}$ describes the processes of interaction of the quasiparticles. On the left side of Eq. (1.1), we can introduce the component $\gamma(\mathbf{k}|N(\mathbf{k})$, which takes into account the finite lifetime of the quasiparticles, and the component $-\delta(\mathbf{k})N(\mathbf{k})$, which simulates, for example, the source associated with the instability, while the source described by δ and the sink described by γ, which are assumed to be essentially separated in k space. In what follows, we shall assume that one of them is located at zero and the other at infinity. We note that the specific form of the source and the sink is quite unimportant for what follows and the corresponding terms are included in (1.1) only for convenience in setting up the problem. Formation of the turbulence spectrum takes place as a result of the nonlinear processes of interaction of the waves in the inertial region (between the energy-containing region of the source and the dissipative region of the sink). In this region, the principal term in the kinetic equation is the collision integral. The stationary distributions of the Kolmogorov type of interest to us are solutions of the equation[3]

$$I_{\text{coll}}\{N\} = 0. \qquad (1.2)$$

As has been shown by Zakharov,[3] the collision integral I_{coll} for waves is symmetrical in frequency space in the case of isotropic distributions N_ω, which allows us to "factorize" them. This makes it possible to find stationary, nonequilibrium distributions $N_\omega = \omega^s$ as exact solutions which cause the collision integral to vanish—for surface gravitational[4] and capillary[5] waves, turbulent plasma,[6] and "acoustic" turbulence in a compressible liquid.[7] As has been noted previously,[4-7] stationary, locally isotropic, distributions can also be found for weak turbulence from dimensionality considerations and using the Kolmogorov hypothesis[8] on the constancy of the flux through the inertial region from the source region to the sink region. The solutions thus constructed are identical with the exact isotropic solutions of Eq. (1.2).

We note that, in a system of propagating waves, the spectral density of the disturbance energy $\omega(\mathbf{k})N(\mathbf{k})$ of local isotropic turbulence cannot be determined solely from considerations of dimensionality and similarity, because of the appearance of an addition local characteristic—the wave velocity. For weak turbulence however, an additional relation between the flux and the spectrum or the distribution is imposed by the kinetic equation.

To determine this relation, we write down the continuity equation for the spectral energy density $E_k = \omega_k N_k k^{d-1}$:

$$\dot{E}_k + \partial P / \partial k = 0, \qquad (1.3)$$

which we obtain by multiplying the kinetic equation (1.1) by the frequency and the density of states (d = dimensionality of k space; we omit numerical coefficients). The energy flux over the spectrum P, is determined by the equation

$$\partial P / \partial k = -\omega_k k^{d-1} I_{\text{coll}}\{N_k\}. \qquad (1.4)$$

In the present work, we limit ourselves to the case of a decay dispersion law for the quasiparticles. Here the collision integral is a quadratic functional of N. As is seen from (1.4), it then follows that

$$N \propto P^{1/2}. \qquad (1.5)$$

With the aid of dimensionality considerations, this is sufficient to find the distribution in the inertial interval:

$$N_\omega = P^{1/2} \omega^{-1/2} k^{-1/2(d+5)}(\omega) \quad (k(\omega) = \omega^{1/\alpha}). \qquad (1.6)$$

Here we have used normalization of the distribution over the energy density

$$\mathscr{E}/\rho L^d \equiv \int dk\, E_k = \int d\mathbf{k}\,\omega(\mathbf{k}) N(\mathbf{k}), \qquad (1.7)$$

where ρ is the density, L the dimension of the system, and $\mathscr{E}$ the total energy of the disturbance.

As is seen from (1.4), the distribution with constant flux over the spectrum $P = \text{const}$ should cause the collision integral to vanish, i.e., it should be a solution of Eq. (1.2). The value of the constant is determined by the power of the source, and in this case, when the distribution $\tilde{N}$ is introduced over all the axes, $P = \int d\mathbf{k}\,\omega\,\delta\tilde{N}_k = \int d\mathbf{k}\,\omega\gamma\tilde{N}_k$. The latter equation is satisfied by virtue of the conservation of energy in the collisions ($\int d\mathbf{k}\,\omega I_{\text{coll}}\{N\} = 0$).

We now turn to the analytic solution of Eq. (1.2). This nonlinear integral equation can be solved only by use of the properties of self-similarity and symmetry. Zakharov uses in his transformation[3] essentially the symmetry in the space of the frequencies ω on the class of isotropic distributions N_ω has been used in a significant way. It turns out that a more general transformation symmetry can be demonstrated in $\mathbf{k}$ space, which allows us in particular to obtain nonisotropic turbulence spectra. This is the subject of our paper. It considers symmetry transformations, and the drift stationary distribution is obtained in an approximation that is linear in the drift parameters[9] for a decay dispersion law (Secs. 2 and 3). The local character of the resultant nonisotropic distributions is discussed. Nonequilibrium distributions are also obtained, which correspond to a small deviation from the Rayleigh-Jeans distribution (Sec. 4).

2. THE COLLISION INTEGRAL AND SYMMETRY TRANSFORMATIONS

We consider the collision integral for a decay dispersion law $\omega(\mathbf{k}) = k^\beta$, $\beta = 1/\alpha > 1$. We omit proportionality coefficients, which are easily inserted in the final result. A dispersion law of such a type exists, for example, in capillary waves, where $\omega = (\sigma k^3/\rho)^{1/2}$ (σ is the surface tension, ρ the density), and for sound waves, where $\omega = ck$, for a suitable form of small dispersion of the sound velocity. In such processes, the collision integral takes the form[5,7]

$$I = \int d\tau_k [W_{\mathbf{k}|\mathbf{k}_1\mathbf{k}_2} f(\mathbf{k}|\mathbf{k}_1\mathbf{k}_2) - W_{\mathbf{k}_1|\mathbf{k}_2\mathbf{k}} f(\mathbf{k}_1|\mathbf{k}_2\mathbf{k}) - W_{\mathbf{k}_2|\mathbf{k}\mathbf{k}_1} f(\mathbf{k}_2|\mathbf{k}\mathbf{k}_1)], \quad (2.1)$$

where $d\tau_k \equiv d\mathbf{k}_1 d\mathbf{k}_2$. W_k is the transition probability:[1)]

$$W_{\mathbf{k}|\mathbf{k}_1\mathbf{k}_2} \equiv \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\delta(\omega - \omega_1 - \omega_2) U_{\mathbf{k}|\mathbf{k}_1\mathbf{k}_2}, \quad \omega_1 \equiv \omega(\mathbf{k}_1) \text{ etc.} \quad (2.2)$$

and f_k is bilinear in the distribution function $N(\mathbf{k})$:

$$f(\mathbf{k}|\mathbf{k}_1\mathbf{k}_2) = N_1 N_2 - NN_1 - NN_2, \quad N \equiv N(\mathbf{k}), \quad N_1 \equiv N(\mathbf{k}_1) \text{ etc.} \quad (2.3)$$

The functions $U_k(W_k)$ and f_k are symmetric with respect to two successive arguments. (We note that the distribution functions are grouped together in the quantity f_k, which contains terms describing both the arrival and the departure of particles, owing to the principle of detailed balancing.) We limit ourselves in the following to the consideration of isotropic media only. Here the transition probability W_k is invariant to the simultaneous rotation g of all three vector arguments:

$$W_{\hat{g}\mathbf{k}|\hat{g}\mathbf{k}_1\hat{g}\mathbf{k}_2} = W_{\mathbf{k}|\mathbf{k}_1\mathbf{k}_2} \quad (W_{\hat{g}k} = W_k). \quad (2.4)$$

We shall be interested in systems in which there are no isolated time and length scales (self-similarity). For this reason, the collision integral has an additional symmetry connected with the homogeneity of the functions $\omega(\mathbf{k})$, U_k and W_k:

$$\omega(\lambda k) = \lambda^\beta \omega(k), \quad U_{\lambda\mathbf{k}|\lambda\mathbf{k}_1\lambda\mathbf{k}_2} = \lambda^m U_{\mathbf{k}|\mathbf{k}_1\mathbf{k}_2} \quad (U_{\lambda k} = \lambda^m U_k). \quad (2.5)$$

The degree of homogeneity of the transition probability depends on the dimensionality d of $\mathbf{k}$ space: $W_{\lambda k} = \lambda^{m-d-\beta} W_k$. We shall not need the explicit form of the quantities U_k, and the homogeneity exponents are given in the table. We also note that $U_k \geq 0$, inasmuch as this is the square of the modulus of the matrix element of the interaction Hamiltonian.[10]

As is known, at a momentum different from zero in a system of quasiparticles, the collision integral causes the equilibrium distribution function $N^0(\mathbf{k})$ to vanish:

$$N^0(\mathbf{k}) = (\omega - \mathbf{k}\mathbf{u})^{-1}, \quad (2.6)$$

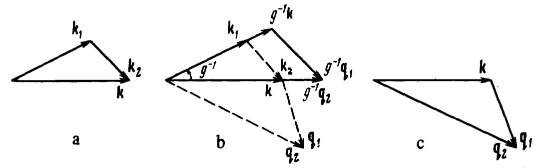
which corresponds to the Rayleigh-Jeans distribution in a drift with velocity $\mathbf{u}$. Here, as is seen from (2.3), the function $N^0(\mathbf{k})$ brings about the vanishing of each of the functions f_k by virtue of the conservation of energy and momentum (2.2):

$$f(\mathbf{k}|\mathbf{k}_1\mathbf{k}_2) = N(\mathbf{k})N(\mathbf{k}_1)N(\mathbf{k}_2)[N^{-1}(\mathbf{k}) - N^{-1}(\mathbf{k}_2) - N^{-1}(\mathbf{k}_1)]. \quad (2.7)$$

However, it is important that the equation $I_{\text{coll}}\{N\} = 0$ has also other solutions for which f_k is already different from zero.[5,7] These solutions describe stationary but nonequilibrium distributions (distributions of the Kolmogorov type with flux over the spectrum), while, in contrast with the equilibrium solutions, their properties are determined by the symmetry properties (2.4) and (2.5).

We consider separately each of the three components in the collision integral (2.1). The surfaces on which the conservation laws are satisfied are different in each of the components; however, there exist transformations that convert these surfaces into one another. These are the triangle similarity transformations, which express the law of momentum conservation (extensions and rotations). For example, let us consider the third component in (2.1) and denote provisionally the integration variables by $\mathbf{q}_1$, $\mathbf{q}_2$. The law of momentum conservation, corresponding to the third component, is expressed by

Turbulent system	Parameters and distributions							
	Dimensionality of k space	$\omega = k^\beta$	$U_{\lambda k} = \lambda^m U_k$	$U_{k\,\vert\,kk_1} \sim k_1^{m_1} k^{m_2}$, $k_1 \ll k$, $m_2 = m - m_1$	Turbulent spectrum $N = \omega^{s_1}[1 + \omega^{p_1} \varkappa \delta u]$		Deviation from equilibrium distribution $N = \omega^{-1}[1 + \omega^p \varkappa\delta u + \omega^t \delta\mu]$	
	d	$\beta \equiv 1/\alpha$	m	m_1	s_1 [5,7]	p_1	p	t
Capillary waves	2	3/2	9/2	7/2	−17/6	1/3	−10/3	−11/3
Acoustic Turbulence (decay dispersion law)	3	1	3	1	−9/2	0	−7	−7



Transformation b which converts two triangles (c and a) into one another; the triangles express the laws of conservation of energy and momentum in the decay $\mathbf{q}_2 = \mathbf{k} + \mathbf{q}_1$ (c) and $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$ (a) ($\mathbf{q}_2 = (\lambda g)^2 \mathbf{k}_1$, $\mathbf{q}_1 = \lambda g \mathbf{k}_2$, $\lambda = k/k_1$). The energy and the momentum of one of the particles, $\omega(k)$ and $\mathbf{k}$, are the same for both processes.

the triangle shown in Fig. c. Then the transformation which converts triangle c into a, which expresses the momentum conservation law in the first component (2.1), is conveniently represented as the result of two successive operations: initially, we turn the triangle $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{k}$ so that after the rotation $\hat{g}^{-1}$ the vector $g^{-1}\mathbf{q}_2$ is directed along $\mathbf{k}$ (see Fig. b), after which we perform the extension λ so that $\hat{G}^{-1}\mathbf{q}_2\,(\lambda\hat{g})^{-1}\mathbf{q}_2 = \mathbf{k}$. As a result, the triangle b formed by the vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{k}$ ($\mathbf{q}_2 = \mathbf{k} + \mathbf{q}_1$) goes over into the similar triangle $\mathbf{k}$, $\mathbf{k}_1$, $\mathbf{k}_2$ ($\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$), while the vector $\mathbf{k}$ remains invariant in the given transformation. It is evident that $\hat{g} = \hat{g}_1$ is the rotation from the vector $\mathbf{k}_1$ (see Fig. a) to the vector $\mathbf{k}$, and the extension $\lambda = \lambda_1 = k/k_1$. In the integral, this transformation corresponds to the change of variables[2)]

$$G_1:\quad \mathbf{q}_2 = (\lambda_1 g_1)^2\mathbf{k}_1,\quad \mathbf{q}_1 = \lambda_1 g_1 \mathbf{k}_2,\quad \lambda_1 = k/k_1. \tag{2.8}$$

The law of momentum conservation in the new variables has the form $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$ and, as a consequence of the homogeneity and isotropy of the dispersion law, we obtain the following relation for the frequencies:

$$\omega(q_2) - \omega(k) - \omega(q_1) = \lambda_1^{\beta}[\omega(k) - \omega(k_1) - \omega(k_2)], \tag{2.9}$$

whence it is seen that in the new variables the energy conservation law has the same form as in the first term in (2.1). A similar transformation $\hat{G}_2$ for the second component in (2.1) is brought about by the replacement of the variables (we again denote the integration variables in the first integral by $\mathbf{q}_1'$, $\mathbf{q}_2'$ and the new variables, $\mathbf{k}_1$, $\mathbf{k}_2$):

$$G_2:\ \mathbf{q}_1' = (\lambda_2 g_2)^2\mathbf{k}_2,\quad \mathbf{q}_2' = \lambda_2 g_2 \mathbf{k}_1,\quad \lambda_2 = k/k_2 \tag{2.10}$$

where $\hat{G}_2 = \lambda_2\hat{g}_2$ transforms $\mathbf{k}_2$ into $\mathbf{k}$. Under this transformation, the conservation law $\mathbf{q}_1' = \mathbf{q}_2' - \mathbf{k} = 0$ and $\omega(\mathbf{q}_1') - \omega(\mathbf{q}_2') - \omega(\mathbf{k}) = 0$ transforms into $\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 = 0$ and $\omega(\mathbf{k}) = \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2) = 0$.

We now consider the transformation of the transition probabilities. Using (2.8) and the properties (2.4) and (2.5), we obtain

$$U_{\mathbf{q}_2|\mathbf{k}\mathbf{q}_1} = U_{(\lambda_1 g_1)^2\mathbf{k}_1|(\lambda_1 g_1)(\lambda_1 g_1)^{-1}\mathbf{k},\,\lambda_1 g_1\mathbf{k}_2} = \lambda_1^{m} U_{\lambda_1 g_1\mathbf{k}_1|(\lambda_1 g_1)^{-1}\mathbf{k},\,\mathbf{k}_2} = (k/k_1)^m U_{\mathbf{k}|\mathbf{k}_1\mathbf{k}_2}, \tag{2.11}$$

where we have taken it into account that $\lambda_1 g_1 \mathbf{k}_1 = \mathbf{k}$. By virtue of (2.8), the transformation Jacobian is equal to λ_1^{3d}. Finally, the collision integral is obtained in the following form, as a result of the transformations (2.8) and (2.10);

$$I\{N\} = \int d\mathbf{k}_1\, d\mathbf{k}_2 W_{\mathbf{k}|\mathbf{k}_1\mathbf{k}_2}\{f(\mathbf{k}|\mathbf{k}_1\mathbf{k}_2) - \lambda_1^{r} f((\lambda_1 g_1)^2\mathbf{k}_1|\mathbf{k}, \lambda_1 g_1\mathbf{k}_2) - \lambda_2^{r} f((\lambda_2 g_2)^2\mathbf{k}_2|\lambda_2 g_2\mathbf{k}_1, \mathbf{k})\}, \tag{2.12}$$

where $r = 2d + m - \beta$, $\lambda_{1,2} = k/k_{1,2}$. Or, in condensed form,

$$I\{N\} = \int d\tau_k W_k\{f_k - \lambda_1^{r} f_{G_1 k} - \lambda_2^{r} f_{G_2 k}\}. \tag{2.12'}$$

Before discussing the properties of the transformations any further, we use them to find isotropic power-law solutions of Eq. (1.2), $N(k) = N_\omega = \omega^s$. These distributions were first obtained (in another way) by Zakharov.[3]

For an isotropic distribution, the function $f_k = f_k^0$ is invariant to rotations and is homogeneous:

$$f^0_{\hat{g}k} = f_k^0,\quad f^0_{\lambda k} = \lambda^{2\beta s} f_k^0, \tag{2.13}$$

i.e., its symmetry properties are identical with the properties of W_k. The collision integral (2.12) is "factorized":

$$I\{N_\omega\} = \omega^{\nu}\int d\mathbf{k}_1\, d\mathbf{k}_2\,\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\,\delta(\omega - \omega_1 - \omega_2)\, U_{\mathbf{k}|\mathbf{k}_1\mathbf{k}_2} \times f^0(\mathbf{k}|\mathbf{k}_1\mathbf{k}_2)\{\omega^{-\nu} - \omega_1^{-\nu} - \omega_2^{-\nu}\}, \tag{2.14}$$

$$f_k^0 \equiv (\omega\,\omega_1\omega_2)^s(\omega^{-s} - \omega_1^{-s} - \omega_2^{-s}),\quad \nu \equiv \nu(s) = 2s + \alpha(2d + m) - 1. \tag{2.15}$$

Together with the equilibrium solution $s = -1$, for which $f_k^0 = 0$, it is seen from (2.14) that, by virtue of the energy conservation law, there is also a second solution $s = s_1$, which causes the expression in the curly brackets to vanish in the collision integral ($\nu(s_1) = -1$):

$$N_\omega = \omega^s,\quad s = s_1 = -\tfrac{1}{2}\alpha(2d + m) = -\tfrac{1}{2}[1 + \alpha(d + 5)]. \tag{2.16}$$

At $s = s_1$ we have $f_k^0 \neq 0$. The last equality in (2.16) is obtained by means of the expression for the degree of homogeneity m of the square of the matrix element for decay processes, in terms of the dispersion-law parameter and the dimensionality of k-space (as a consequence of the dimensionality considerations)[3)]:

$$m = \beta + 5 - d. \tag{2.17}$$

After averaging (2.14) over the angles and changing to integration over the frequency, the collision integral is reduced to the form obtained in[5,7] by means of the Zakharov transformation[3] in k-space. It is not difficult to establish the fact[5,7] that the distribution (2.16) corresponds to a constant flux of energy over the spectrum (compare (2.16) with (1.6)) and the energy $\int dk\,\omega N_\omega$ diverges for small k at $d + \beta < 5$; i.e., the distribution is not normalizable, but is nevertheless local.[5,7]

3. STATIONARY DRIFT SOLUTIONS

In the same way as the more general distribution (2.6) $N^0(\mathbf{k}) = (\omega - \mathbf{k}\cdot\mathbf{u})^{-1}$ corresponds to the isotropic Rayleigh-Jeans distribution $N^0_\omega = \omega^{-1}$ (here $\mathbf{u}$ is the "drift" velocity, which describes the motion of a gas of quasiparticles relative to the medium), it is natural to seek similar anisotropic distributions corresponding to the isotropic turbulence spectrum (2.16). However, while the drift solution (2.6) can be obtained from the Rayleigh-Jeans distribution (for the case of an equilibrium distribution) by the Galilean substitution $\omega \to \omega - \mathbf{k}\cdot\mathbf{u}$, which corresponds to equilibrium for a non-zero total momentum of the quasiparticles), no such statement can be made in general for the nonequilibrium (cf. (3.8).)[4)] Solutions of the drift type (in the linear approximation in $\mathbf{u}$), which make the collision integral (2.1) vanish, nevertheless exist even in the case of turbulence. We seek these solutions in the form

$$N(\mathbf{k}) = \omega^s(1 + \omega^p \varkappa\delta\mathbf{u}),\quad \varkappa = \mathbf{k}/k. \tag{3.1}$$

In the linear approximation in $\mathbf{u}$, the collision integral (2.1) and the function f_k (2.3) are written in the form

$$I = I_0 + \mathbf{I}\delta\mathbf{u},\quad f_k = f_k^0 + \mathbf{f}_k\delta\mathbf{u}, \tag{3.2}$$

where I_0 contains the isotropic part of the distribution (3.1) and can be reduced to the form (2.14); but $\mathbf{I}$ reduces to the form (2.12) with this difference that in place of the scalar f_k we have the vector function $\mathbf{f}_k$, which is equal to

$$\mathbf{f} = \mathbf{f}(\mathbf{k}|\mathbf{k}_1\mathbf{k}_2) = \varkappa\omega^p\xi - \varkappa_1\omega_1^p\xi_1 - \varkappa_2\omega_2^p\xi_2,$$
$$\xi = -\omega^s(\omega_1^s + \omega_2^s), \quad \xi_1 = \omega_1^s(\omega^s - \omega_2^s), \quad \xi_2 = \omega_2^s(\omega^s - \omega_1^s). \tag{3.3}$$

It is then seen that $\mathbf{f}_k$ is a homogeneous function of degree $\beta(2s + p)$; $\mathbf{f}_{\lambda k} = \lambda^{\beta(2s+p)}\mathbf{f}_k$. Since the integral $\mathbf{I}$ depends only on the vector $\mathbf{k}$, we have $\mathbf{I} = \kappa I_u$, $I_u = \mathbf{I}$, where

$$I_u = \int d\tau_k W_k\{\varkappa\mathbf{f}_k - \lambda_1{}^r\varkappa\mathbf{f}_{G_1k} - \lambda_2{}^r\varkappa\mathbf{f}_{G_2k}\}. \tag{3.4}$$

As is seen from (2.8) and (3.3), $\kappa\cdot\mathbf{f}_{G,k}$ is equal to

$$\varkappa\mathbf{f}(G_1\mathbf{k}|G_1\mathbf{k}_1G_1\mathbf{k}_2) = \lambda_1^{\beta(2s+p)}(\hat{g}_1\varkappa_1)\mathbf{f}(\hat{k g}_1\varkappa|\hat{k_1 g}_1\varkappa_1, \hat{k_2 g}_1\varkappa_2) = \lambda_1^{\beta(2s+p)}\varkappa_1\mathbf{f}(\mathbf{k}|\mathbf{k}_1\mathbf{k}_2). \tag{3.5}$$

The latter equality follows from the linearity of $\mathbf{f}_k$ in the wave vectors (3.3) and from the definition of the rotation $\hat{g}_1$ ($\hat{g}_1\kappa_1 = \kappa$), so that $(\kappa, \hat{g}_1\kappa) = (\hat{g}_1\kappa_1, \hat{g}_1\kappa) = (\kappa_1, \kappa)$ and so on. Similarly, we get

$$\varkappa\mathbf{f}(G_2\mathbf{k}|G_2\mathbf{k}_1G_2\mathbf{k}_2) = \lambda_2^{\beta(2s+p)}\varkappa_2\mathbf{f}(\mathbf{k}|\mathbf{k}_1\mathbf{k}_2). \tag{3.6}$$

Substituting (3.5) and (3.6) in (3.4), we obtain the integral I_u in the factorized form:

$$I_u = \omega^{\nu_p}\int d\tau_k W_k\,\mathbf{f}(\mathbf{k}|\mathbf{k}_1\mathbf{k}_2)\{\varkappa\omega^{-\nu_p} - \varkappa_1\omega_1^{-\nu_p} - \varkappa_2\omega_2^{-\nu_p}\}, \tag{3.7}$$

where $\nu_p = \nu + p$, while the exponent $\nu = \nu(s)$ is given in (2.15). It is seen from this expression that, with account of the momentum conservation law $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$, which is contained in the transition probability (2.2), the integral I_u vanishes for $-\nu_p = \alpha \equiv 1/\beta$. This relation determines the exponent p, while $\nu(s)$ is determined from the condition $I_0 = 0$. We thus find the drift solution in the linear approximation.

The value $\nu = -1$ corresponds to the turbulence distribution of Zahkarov (and $s = s_1$, see (2.16)), whence we obtain the value $p_1 = 1 - \alpha$ for p. Finally, making the transformation from κ to $\mathbf{k}$. we write down the resultant distribution in the form

$$N(\mathbf{k}) = \omega^{s_1}(1 + \omega^{1-2\alpha}\mathbf{k}\delta\mathbf{u}). \tag{3.8}$$

It is seen from (3.8) that it is not possible to obtain the drift component for capillary waves ($\alpha = 2/3$) by expansion of the function $N_{\omega - \mathbf{k}\cdot\delta\mathbf{u}}$, which is thus not a solution of Eq. (1.2) (see also footnote 3).

The distribution (3.8) describes local turbulence with small anisotropy. The locality property corresponds to the fact that the region of small and large wave vectors makes a negligibly small contribution to the collision integral. The fundamental contribution to the integral is made by the inertial region, where $k_1, k_2 \sim k$. The locality of the isotropic distributions has been discussed previously.[5,7] Here we are interested in the local character of the nonisotropic distributions (3.8). We consider first the convergence of the integral (2.1) for small $k_1 \ll k$ ($k_2 \sim k$). Bearing in mind that the distribution function $N(k_1) \to \infty$, when $k_1 \to 0$, we establish the fact that the most dangerous terms are the terms $N_1N_2 - NN_1$ in the first component of (2.1) and $NN_1 - N_2N_1$ in the third. Integrating with respect to k_2 by using the momentum conservation law, we obtain for the dangerous terms

$$\int d\mathbf{k}_1\{\delta(k^\beta - k_1^\beta - |\mathbf{k}-\mathbf{k}_1|^\beta)U_{\mathbf{k}|\mathbf{k}_1\mathbf{k}-\mathbf{k}_1}[N(\mathbf{k}-\mathbf{k}_1) - N(\mathbf{k})]$$
$$- \delta(|\mathbf{k}+\mathbf{k}_1|^\beta - k^\beta - k_1^\beta)U_{\mathbf{k}+\mathbf{k}_1|\mathbf{k}\mathbf{k}_1}[N(\mathbf{k}) - N(\mathbf{k}+\mathbf{k}_1)]\}N(\mathbf{k}_1). \tag{3.9}$$

For small $k_1 \ll k$, we expand the argument of the δ function and, recognizing that $U_{\mathbf{k}+\mathbf{k}_1|\mathbf{k}\mathbf{k}_1} \approx U_{\mathbf{k}|\mathbf{k}_1\mathbf{k}-\mathbf{k}_1}$, we get for $\beta < 2$

$$\int d\mathbf{k}_1\delta(\beta\mathbf{k}\mathbf{k}_1k^{\beta-2} - k_1^\beta)U_{\mathbf{k}|\mathbf{k}\mathbf{k}_1}[N(\mathbf{k}+\mathbf{k}_1) + N(\mathbf{k}-\mathbf{k}_1) - 2N(\mathbf{k})]N(\mathbf{k}_1). \tag{3.10}$$

For $\beta > 2$, the principal terms in the δ functions will be $\mathbf{k}\cdot\mathbf{k}_1$ and k_1^2. Since their arguments are different (they are obtained by the substitution $\mathbf{k}_1 \rightleftarrows -\mathbf{k}_1$), we would have to include in the estimate of the integral the following terms of the expansion of the arguments of the δ functions and also of the matrix element U_k. Both for capillary waves and for acoustic turbulence, $\beta < 2$, and we limit ourselves only to this case.

Inasmuch as the matrix element U_k is a homogeneous function, its asymptotic form is

$$U_{\mathbf{k}|\mathbf{k}\mathbf{k}_1} = k_1^{m_1}k^{m_2}, \quad k_1 \ll k, \quad m_1 + m_2 = m \tag{3.11}$$

(for the exponents m and m_1, see the table). With account of (3.11), the integral (3.10) reduces to the form

$$\int d\mathbf{k}_1k_1^{m_1-1}N(\mathbf{k}_1)\frac{\partial^2N(\mathbf{k})}{\partial k_i\,\partial k_j}k_{1i}k_{1j}\delta\left(\varkappa\varkappa_1 - \alpha\left(\frac{k_1}{k}\right)^{\beta-1}\right). \tag{3.12}$$

For the drift solutions (3.1), we consider the linear term in $\delta\mathbf{u}$ in (3.12). After averaging over the angles, wherein $\kappa\cdot\kappa_1$ is replaced by $\alpha(k_1/k)^{\beta-1}$ by virtue of the δ function, we arrive at the expression

$$\int dk_1k_1^{m_1+d}\omega_1^s[1 + \alpha\omega_1^p(k_1/k)^{\beta-1}\varkappa\delta\mathbf{u}]. \tag{3.13}$$

Here we have omitted factors that do not affect the convergence. By evaluating the powers, we establish the fact that the integral converges at zero if

$$m_1 + d + 1 + \beta s > 0 \text{ и } m_1 + d + \beta(s + p + 1) > 0, \tag{3.14}$$

and the first inequality in (3.14) corresponds to convergence of the isotropic distribution. The resultant anisotropic distribution (3.8), both for capillary waves and in the case of acoustic turbulence, satisfies (3.14), which guarantees convergence of the collision integral at low wave numbers (see the table).

By considering the convergence of (2.1) at large k_2, $k_1 \gg k$ in similar fashion (the most dangerous terms as $k_{1,2} \to \infty$ are those linear in N_1 and N_2 in the second and third components), we obtain an integral of the form

$$\int dk_1k_1^{m_2+1-\beta}\varkappa\frac{\partial N(\mathbf{k}_1)}{\partial\mathbf{k}_1}\delta\left(\varkappa\varkappa_1 - \alpha\left(\frac{k}{k_1}\right)^{\beta-1}\right). \tag{3.15}$$

Substituting (3.8) in (3.15) and limiting ourselves to terms that are linear in $\delta\mathbf{u}$, we obtain the convergence condition

$$m_2 + d + 1 + \beta(s - 2) < 0 \text{ and } m_2 + d + \beta(s + p - 1) < 0, \tag{3.16}$$

where again the first condition guarantees the convergence of the integral from the isotropic distribution.

It is not difficult to show that the distribution (3.8) guarantees convergence of the collision integral both in the case of acoustic turbulence and for capillary waves. Thus, the resultant anisotropic distributions are local in character.

4. NONEQUILIBRIUM STATIONARY DEVIATIONS FROM THE ISOTROPIC RAYLEIGH-JEANS DISTRIBUTION

We consider small deviations from the equilibrium Rayleigh-Jeans solution. As before, we begin with the linearized form of the collision integral (3.2) and (3.7). We consider first equilibrium drift deviations, which can be found from the condition of the vanishing of the function f_k in (3.7). We note that for the Rayleigh-Jeans distribution ($s = -1$) the values of ξ (see (3.3)) are equal to

$$\xi = -(\omega\omega_1\omega_2)^{-1}\omega, \quad \xi_1 = -(\omega\omega_1\omega_2)^{-1}\omega_1$$

and so on, thanks to which

$$f_k = -(\omega\omega_1\omega_2)^{-1}(\varkappa\omega^{p+1} - \varkappa_1\omega_1^{p+1} - \varkappa_2\omega_2^{p+1}) \qquad (s = -1).$$

By virtue of momentum conservation, f_k vanishes if $p + 1 = \alpha$, i.e.,

$$N(\mathbf{k}) = \omega^{-1}[1 + \mathbf{k}\delta\mathbf{u}/\omega],$$

which is identical with the expansion of the equilibrium drift distribution $N^0(\mathbf{k}) = (\omega - \mathbf{k}\cdot\delta\mathbf{u})^{-1}$. It turns out that the collision integral vanishes in the presence of nonequilibrium drift deviations from the distribution $N^0_\omega = \omega^{-1}$. This results from the possibility of vanishing of the curly brackets in (3.17) at $f_k \neq 0$, which gives for the exponent ν_p the value $-\alpha$, and for the exponent p (with account of the fact that $s = -1$, see (2.15) and (2.17))

$$p = 2 - \alpha(d + 6). \qquad (4.1)$$

As is seen from (4.1), when $\alpha \lesssim 1$ the contribution falls off at large k, but can be important in the region of small k.

It is curious to note that, together with the anisotropic deviation, which is proportional to δu, a nonequilibrium isotropic deviation can also appear, which does not reduce to a temperature change:

$$N = \omega^{-1}(1 + \omega^t\delta\mu). \qquad (4.2)$$

For a non-decay dispersion law, the existence of such a deviation would have been a consequence of the conservation of the number of particles in the collisions; in our case, on the other hand, the existence of such a solution is in no way trivial. For example, deviations of this sort from turbulence distributions do not exist, nor do equilibrium distributions of the form (4.2) (at $t = -1$). Linearizing the collision integral over the distribution (4.2), we obtain

$$I = I_0 + I_\mu\delta\mu, \qquad (4.3)$$

where

$$I_\mu = \omega^{\nu_t}\int d\tau_k W_k f^\mu(\mathbf{k}\,|\,\mathbf{k}_1\mathbf{k}_2)\{\omega^{-\nu_t} - \omega_1^{-\nu_t} - \omega_2^{-\nu_t}\},$$
$$f^\mu = \omega^t\xi - \omega_1^t\xi_1 - \omega_2^t\xi_2, \quad \nu_t = \nu + t, \qquad (4.4)$$

for the values of I_0 and ξ see (2.14) and (3.3). This formula is obtained for a distribution of a more general type than (4.2): $N = \omega^s(1 + \omega^t\delta\mu)$. We first consider the turbulence distribution with $s \neq -1$. Then $f^\mu \neq 0$, and the quantity in the parentheses vanishes for $\nu_t = -1$, but since s is determined from the condition (2.16) $\nu = -1$, we then obtain the value zero for t. This means that only the constant factor in the turbulence distribution is changed. For the equilibrium solution ($f^\mu = 0$ for $s = -1$) we also obtain $t = 0$, which corresponds to a change of temperature in the Rayleigh-Jeans distribution.

We now consider the nonequilibrium deviation from the distribution $N = \omega^{-1}$, due to the vanishing of the quantity in the curly brackets in (4.4). This gives $\nu_t = -1$, whence we find, by substituting $\nu(-1)$ from (2.15) and using (2.17):

$$t = 1 - \alpha(d + 5). \qquad (4.5)$$

The obtained deviations from the equilibrium distribution fall off rapidly at large k. Therefore, at large k, the linearized collision integral converges if ($\beta < 2$)

$$m_2 + d + \beta(t - 2) = m_2 - \beta - 5 < 0,$$
$$m_2 + d - 1 + \beta(p - 1) = m_2 + \beta - 7 < 0.$$

These conditions are easily satisfied. The convergence of the collision integral at small k, on the other hand, requires satisfaction of the stringent conditions ($\beta < 2$):

$$m_1 + d + 1 + \beta(s + t) = m_1 - 4 > 0,$$
$$m_1 + d + \beta(p + s + 1) = m_1 + 2\beta - 6 > 0.$$

For capillary waves, the convergence is assured for the anisotropic contribution. In the other cases, there is no convergence at zero.

We note here that, while the drift equilibrium distributions correspond to the establishment of a local equilibrium described by the drift parameters, we are dealing here not with a partial equilibrium, but with a nonequilibrium distribution which nevertheless is also determined by a small number of drift macroscopic parameters that are formulated in correspondence with the conservation law, in the presence of flux over the spectrum.

In the present research, the anisotropic nonequilibrium distributions are found for isotropic media[5)] for a decay dispersion law. Here the properties of symmetry of the transition probability are used. The resultant symmetry transformations of the collision integral are generalized to the case of non-decay processes, which allows us to find the analogous nonequilibrium distributions in this case, too.[9]

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1)In the following, we shall also make use of the abbreviated notation $W_k \equiv W_{k|k_1k_2}$ and so on.

2)The numbering of the new variables is such that the cyclic character is preserved when the substitution $\mathbf{q}_2, \mathbf{k}, \mathbf{q}_1 \to \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}$ is made in the matrix elements of the transition probability.

3)The conditions (2.5) can be satisfied even without self-similarity, when (2.17) does not hold. Then m is an independent parameter.

4)Of course, if we consider the isotropic nonequilibrium solution N_ω (2.16) from the point of view of a system of coördinates moving with velocity v relative to the medium, then $N_\omega \to N_{\omega-\mathbf{kv}}$; however, this distribution does not cause the collision integral to vanish in the fixed system of coordinates and is not a solution that describes the drift of the quasiparticles relative to the medium.

5)We note that in the case of limiting anisotropy for ion acoustic turbulence, Kuznetsov [11] indicated transformations which are Zakharov transformations separately for the frequencies and the moduli of the transverse components of the wave vectors; such an approach corresponds to isotropy in the plane perpendicular to the magnetic field.

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