Rate of energy transfer in a plasma

V. N. Alyamovskiĭ
P. N. Levedev Physics Institute
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An expression which does not contain any cut-off parameters is obtained for the rate of variation of the mean energy of nonrelativistic homogeneous plasma components due to Coulomb collisions. If the light component is the hotter one, the effective electron-ion interaction range significantly exceeds the Debye ion length due to exchange of ion-sound waves.

The rate of energy transfer from one charged plasma component to another was calculated with logarithmic accuracy by different authors[1-3]. In the present communication we calculate this quantity accurate to terms that are linear in the density, inclusive, using a previously obtained collision integral[4,5]. Like other authors[1-3], we assume a Maxwellian theoretical velocity distribution.

1. In an exact description of the short-range Coulomb interaction and with allowance for the dynamic polarization of the plasma, the collision integral for a tenuous plasma takes the form[4,5]

$$\left(\frac{\partial f_i}{\partial t}\right) = e_i^2 \frac{\partial}{\partial \mathbf{n}} \sum_j n_j e_j^2 \int d\mathbf{p}_j \, d\Omega_{\mathbf{q}} L_{ij} \mathbf{Q}_{ij} + 4e_i^2 \sum_j n_j e_j^2 R_i^{(j)}$$

$$R_i^{(j)} \equiv -\int d\mathbf{p}_j \int_0^{2\mu_{ij}w_{ij}} d\Delta \ln \frac{\Delta}{2\mu_{ij}w_{ij}} \frac{d}{d\Delta} \int \frac{d\Omega_\Delta}{\Delta} \cdot$$

$$\times \delta\left(\Delta \mathbf{w}_{ij} - \frac{\Delta^2}{2\mu_{ij}}\right) [f_i(\mathbf{p}_i - \Delta) f_j(\mathbf{p}_j + \Delta) - f_i(\mathbf{p}_i) f_j(\mathbf{p}_j)],$$

$$\mathbf{Q}_{ij} \equiv \delta(\mathbf{q}\mathbf{w}_{ij}) \left[\mathbf{q}\left(\frac{\partial}{\partial \mathbf{p}_i} - \frac{\partial}{\partial \mathbf{p}_j}\right) f_i(\mathbf{p}_i) f_j(\mathbf{p}_j)\right] \mathbf{q},$$

$$L_{ij} \equiv \ln|2\mu_{ij} w_{ij} / \zeta(|e_i e_j / \hbar w_{ij}|) e_i e_j \varkappa_{\mathbf{q}}(\mathbf{q}\mathbf{v}_i)|^2 - \frac{\operatorname{Re} \varkappa_{\mathbf{q}}^2(\mathbf{q}\mathbf{v}_i)}{|\operatorname{Im} \varkappa_{\mathbf{q}}^2(\mathbf{q}\mathbf{v}_i)|} \operatorname{arcctg} \frac{\operatorname{Re} \varkappa_{\mathbf{q}}^2(\mathbf{q}\mathbf{v}_i)}{|\operatorname{Im} \varkappa_{\mathbf{q}}^2(\mathbf{q}\mathbf{v}_i)|}$$

$$\varkappa_{\mathbf{q}}^2(u) \equiv 4\pi \sum_i n_i e_i^2 \int \frac{d\mathbf{p}_i}{u - \mathbf{q}\mathbf{v}_i + i\delta} \mathbf{q} \frac{\partial f_i}{\partial \mathbf{p}_i}, \qquad \delta \to +0;$$

$$\mu_{ij} \equiv m_i^{-1} + m_j^{-1}, \quad \mathbf{w}_{ij} \equiv \mathbf{v}_i - \mathbf{v}_j, \quad \mathbf{v}_i \equiv \mathbf{p}_i / m_i, \quad |\mathbf{q}| = 1,$$

e_i, m_i, and n_i are the charge, mass, and density of the particles of type i. The arc cotangent in the formula for L_{ij} is chosen between zero and π (the term with the exchange interaction is immaterial in this case and is not presented). The form of the function ζ was given in an earlier paper[5]; we present here only the limiting values

$$\zeta(\eta) \cong \begin{cases} \eta^{-1}, & \eta \ll 1 \\ e^C, \quad C = 0.577\ldots, & \eta \gg 1 \end{cases}.$$

It is easy to verify that the following equation holds

$$\int d\mathbf{p}_i \psi(\mathbf{p}_i) R_i^{(j)} = -\int d\mathbf{p}_i \, d\mathbf{p}_j f_i f_j \int_0^\infty d\Delta \ln \frac{\Delta}{2\mu_{ij}w_{ij}}$$

$$\times \frac{d}{d\Delta} \int \frac{d\Omega_\Delta}{\Delta} \delta\left(\Delta \mathbf{w}_{ij} - \frac{\Delta^2}{2\mu_{ij}}\right) [\psi(\mathbf{p}_i - \Delta) - \psi(\mathbf{p}_i)],$$

where ψ is an arbitrary function (in view of the presence of the δ-function, the integral with respect to Δ in $R_i^{(j)}$ can be extended to $\Delta = \infty$). At $\psi(\mathbf{p}_i) = a_0 + \mathbf{a}_1\mathbf{p}_i + a_2 p_i^2$, the right-hand side vanishes. Thus, we have for the rate of change for the average energy of the component i as a result of collisions

$$\Gamma_i \equiv \left(\frac{\partial}{\partial t} \int d\mathbf{p}_i \frac{p_i^2}{2m_i} f_i\right)_c = -e_i^2 \sum_{j \neq i} n_j e_j^2 \int d\mathbf{p}_i \, d\mathbf{p}_j \, d\Omega_{\mathbf{q}} L_{ij} \mathbf{v}_i \mathbf{Q}_{ij}. \qquad (1)$$

2. We consider the case of a two-component two-temperature plasma. Breaking L up into terms that depend only on w or $\mathbf{q} \cdot \mathbf{v}$, we get from (1)

$$\Gamma_1 = \frac{8\sqrt{\pi} n_2 e_1^2 e_2^2 (T_2 - T_1)}{m_1 m_2 u^3} (I + I'),$$

$$I \equiv \int_0^\infty dx \, e^{-x} \ln \left| \frac{2\mu_{12} u^2 x}{e_1 e_2 \varkappa} \zeta^{-1}\left(\frac{|e_1 e_2|}{\hbar u x^{1/2}}\right)\right|^2,$$

$$I' \equiv -\frac{4}{\sqrt{\pi}} \int_0^\infty dx \, x^2 e^{-x^2} \left(\ln|\Phi| + \frac{\operatorname{Re}\Phi}{\operatorname{Im}\Phi} \operatorname{arcctg} \frac{\operatorname{Re}\Phi}{\operatorname{Im}\Phi}\right)$$

$$\Phi(x) \equiv \frac{\varkappa_2^2}{\varkappa^2} \varphi\left(\frac{u_1}{u} x\right) + \frac{\varkappa_1^2}{\varkappa^2} \varphi\left(\frac{u_2}{u} x\right),$$

$$\varphi(y) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{dz \, z e^{-z^2}}{z - y - i\delta},$$

$$u \equiv (u_1^2 + u_2^2)^{1/2}, \quad u_i \equiv (2T_i / m_i)^{1/2}, \quad \varkappa \equiv (\varkappa_1^2 + \varkappa_2^2)^{1/2},$$
$$\varkappa_i \equiv (4\pi n_i e_i^2 T_i^{-1})^{1/2}.$$

If I in (2) is replaced by $2\ln(k_{max}/\kappa)$, where k_{max} is a parameter chosen from physical considerations, then we obtain the results of Ramazashvili et al.[3] (in a somewhat different form). The integral I′ was tabulated in[3] for a hydrogen plasma. We present below analytic estimates of the integrals I and I′. For I we obtain

$$I \cong \begin{cases} 2\ln|2\mu_{12}u^2/\varkappa e_1 e_2| - 4C, & |e_1 e_2|/\hbar u \gg 1, \\ 2\ln(2\mu_{12}u/\hbar\varkappa) - C, & |e_1 e_2|/\hbar u \ll 1. \end{cases} \qquad (3)$$

We estimate I′ for the most important case of a large difference between the average particle velocities. Let

$$\rho \equiv \frac{u_2}{u_1} \ll 1, \qquad \alpha \equiv \frac{n_1 e_1^2 m_2}{n_2 e_2^2 m_1} \gg 1.$$

We then have for Φ:

$$\Phi(x) \cong \frac{\varkappa_2^2}{\varkappa^2} [\varphi(x) + \alpha\rho^2 + i\sqrt{\pi}\alpha\rho^3 x]. \qquad (4)$$

It is easy to see that the interval $1 \gg \rho \gg \rho_0$, where ρ_0 is a solution of the equation

$$(2\alpha\rho_0^2)^{-1} = -\ln(\alpha\rho_0^3), \quad \rho_0^2 < \alpha^{-1},$$

we can neglect the third term in (4), and at $\rho \ll \rho_0$ we can neglect the second term. For ρ_0 we have

$$\rho_0^2 \cong 1/\alpha \ln \alpha, \quad \alpha \gg 1.$$

We write I′ in the form

$$I' = -\frac{2}{\sqrt{\pi}} \int_{-\infty}^\infty dx \, x^2 e^{-x^2} \frac{\operatorname{Im}(\Phi \ln \Phi)}{\operatorname{Im}\Phi}, \qquad -\pi < \arg\Phi < \pi. \qquad (5)$$

Recognizing that $\operatorname{Im} \varphi(x) = \sqrt{\pi} x \exp(-x^2)$, we represent I′ in the region $I \gg \rho \gg \rho_0$ in the form

$$I' \cong -2\ln\frac{\varkappa_1}{\varkappa} - \frac{2}{\pi} \alpha\rho^2 \lim_{\delta \to 0} \operatorname{Im} \int_{-\infty}^\infty \frac{x \, dx}{1 + \delta^2 x^2} \left(1 + \frac{\varphi(x)}{\alpha\rho^2}\right) \ln\left(1 + \frac{\varphi(x)}{\alpha\rho^2}\right).$$

Since $\varphi(z)$ is analytic in the upper half of the complex z plane, assumes no real negative in that plane, and decreases like z^{-2} at infinity, the integral is equal to the residue at $x = i\delta^{-1}$. As a result we get

$$I' \cong \ln(1 + 1/\alpha\rho^2) - 1, \quad 1 \gg \rho \gg \rho_0. \tag{6}$$

To estimate I′ at $\rho \ll \rho_0$ we write down the fraction in (5) in the form

$$-\frac{\pi\theta_+(|x|-1)}{2x^2|\operatorname{Im}\Phi|} + \left[\frac{\operatorname{Im}(\Phi\ln\Phi)}{\operatorname{Im}\Phi} + \frac{\pi\theta_+(|x|-1)}{2x^2|\operatorname{Im}\Phi|}\right],$$

where θ_+ is the step function. The contribution of the first term, accurate to quantities that vanish at $\rho = 0$, is equal to $\ln\ln(1/\alpha\rho^3)$. Putting in the second term $\rho = 0$ and using the same procedure as above, we obtain

$$-\lim_{\delta\to 0}\left\{\frac{2}{\pi}\operatorname{Im}\int_{-\infty}^{\infty}\frac{dx\,x\varphi\ln\varphi}{1+\delta^2x^2} + 2\int_{1}^{\infty}\frac{dx}{x(1+\delta^2x^2)}\right\} = \ln 2.$$

Thus

$$I' \cong \ln\ln\frac{1}{\alpha^2\rho^6}, \quad \rho \ll \rho_0. \tag{7}$$

Substituting (3), (6), and (7) in (2), we obtain ultimately for $T_1/T_2 \gg m_1/m_2$:

$$\Gamma_1 = 4\sqrt{2\pi}\frac{n_2e_1^2e_2^2m_1^{1/2}}{m_2T_1^{3/2}}(T_2 - T_1)\ln\frac{\Lambda}{\lambda};$$

$$\Lambda = \begin{cases} 1/e^{1/2}\varkappa_1, & 1 \ll u_1/u_2 \ll (\alpha\ln\alpha)^{1/2}, \\ \varkappa_2^{-1}\ln^{1/2}(u_1^6/\alpha^2u_2^6), & u_1/u_2 \gg (\alpha\ln\alpha)^{1/2}. \end{cases}$$

$$\lambda = \begin{cases} e^{2C}|e_1e_2|/2m_1u_1^2 & |e_1e_2|/\hbar u_1 \gg 1, \\ e^{C/2}\hbar/2m_1u_1, & |e_1e_2|/\hbar u_1 \ll 1. \end{cases} \tag{8}$$

Thus, at $\kappa_2/\kappa_1 \gg 1$ the effective screening radius is increased by exchange of ion-sound waves. The slowing down of the rate of this increase at $\kappa_2/\kappa_1 > \ln^{1/2}\alpha$ is due to the fact that at large velocities of these waves the electron absorption begins to predominate.

In the case of several types of heavy particles with equal mass (and temperature), it is necessary to replace κ_2^2 by

$$4\pi\sum_i n_ie_i^2/T_2$$

and sum (8) over the index 2.

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16