Some features of viscous flow of vortices in superconducting alloys near the critical temperature

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Time equations for slow ($\omega \ll \Delta$) motion of the vortex structure in alloys with $l \ll \xi(T)$ near T_c are derived on the basis of the microscopic superconductivity theory. The major role is played by the term responsible for the slow diffusion mechanism of relaxation. The motion of vortex lines is considered for the case of a magnetic field $H \ll H_{c2}$. This relaxation mechanism leads to a large value of the viscosity coefficient.

1. INTRODUCTION

It is known that the finite resistance of a type-II superconductor in a magnetic field exceeding the lower critical field H_{c1} is due to energy dissipation that occurs when the vortex structure moves in the superconductor. This question has been under experimental study for quite some time [1], whereas the theoretical study of the motion of vortex filaments was based until recently only on phenomenological models [2]. These turned out (see [3-6]) not to cover all the significant kinetic processes that occur in superconductors. A consistent approach to this problem obviously calls for the use of microscopic time-dependent equations of the theory of superconductivity [3].

The motion of a vortex structure in strong magnetic fields (near H_{c2}), where the superconductivity is greatly suppressed, was investigated on the basis of the microscopic theory, for example, by Caroli and Maki [7], and by Baba and Maki [8]. Great interest, however, attaches to the wide range of fields $H \ll H_{c2}$. In an earlier paper [9], using alloys with paramagnetic impurities as an example, we have proposed a scheme by which to describe the vortex motion within the framework of microscopic differential time-dependent equations of the Ginzburg-Landau type.

Our result [9] is valid for the case of large paramagnetic-impurity concentrations, strictly speaking, only for an uncharged Fermi gas, since we took no account of the term due to the normal energy loss in the center of the vortex. This circumstance was pointed out by a number of authors [10,11].

It follows from [9] that at finite temperatures, when the anomalous terms in the equations for the superconductors become important, the viscosity experienced by the moving vortex increases strongly, i.e., the vortex speed is low. This is caused by a circumstance already noted by Éliashberg and one of the authors [4], that the anomalous term becomes larger in order of magnitude than the remaining terms with time derivatives when the damping of the excitations is determined by slow diffusion processes. The reason for this increase in viscosity is that to ensure stationary motion of the vortex it is essential that the perturbations produced around the moving vortex have time to attenuate within a time on the order of $\xi(T)/u$, where u is the vortex speed and $\xi(T)$ is the scale of variation of the ordering parameter. In superconductors, the times of homogeneous energy relaxation are quite long (Θ_D^2/T^3 and E_F/T^2 for electron-phonon and electron-electron interactions, respectively). The damping of the excitations is therefore determined by a diffusion mechanism which, nonetheless, is still slow enough at temperatures $T_c - T \ll T_c$.

This article is devoted to a study of these features of relaxation for vortex filaments moving at $T_c - T \ll T$ under the more favorable experimental conditions of superconducting alloys with ordinary impurities. It will be shown that the relaxation mechanism mentioned above plays the principal role in this case, so that an analytic solution of the resultant equations can be obtained.

2. DERIVATION OF EQUATIONS FOR THE ANOMALOUS GREEN'S FUNCTIONS IN ALLOYS AT A TEMPERATURE CLOSE TO T_c

The general scheme for deriving the time-dependent equations for superconductors was developed by Éliashberg and one of the authors [3]. It was shown that when the expressions for the ordering parameter Δ and for the current are analytically continued to the real frequency axis, the result consists of terms that are regular in the upper or lower half of the complex ϵ plane, as well as the so-called "anomalous terms," which do not possess this property. We write down the expressions derived in [3] and choose immediately for Δ a gauge whereby we obtain throughout only the gauge-invariant combinations

$$|\Delta|, \quad \mathbf{Q} = \mathbf{A} - \frac{c}{2e}\nabla\chi, \quad \mu = \varphi + \frac{1}{2e}\dot{\chi},$$

where χ is the phase shift of the ordering parameter (we shall henceforth omit the absolute-magnitude symbols of Δ):

$$\frac{\Delta(\mathbf{r})}{|g|} = \int \frac{d\varepsilon}{4\pi i}\left[\operatorname{th}\frac{\varepsilon-\omega}{2T}F^R_{\varepsilon,\varepsilon-\omega}(\mathbf{r},\mathbf{r}) - \operatorname{th}\frac{\varepsilon}{2T}F^A_{\varepsilon,\varepsilon-\omega}(\mathbf{r},\mathbf{r})\right] + \int \frac{d\varepsilon}{4\pi i}F^{(a)}_{\varepsilon,\varepsilon-\omega}(\mathbf{r},\mathbf{r}),$$

$$\mathbf{j} = -\frac{e}{m}\int \frac{d\varepsilon}{4\pi i}\left\{(\hat{\mathbf{p}}-\hat{\mathbf{p}}')\left[\operatorname{th}\frac{\varepsilon-\omega}{2T}G^R_{\varepsilon,\varepsilon-\omega}(\mathbf{r},\mathbf{r}') - \operatorname{th}\frac{\varepsilon}{2T}G^A_{\varepsilon,\varepsilon-\omega}(\mathbf{r},\mathbf{r}') + G^{(a)}_{\varepsilon,\varepsilon-\omega}(\mathbf{r},\mathbf{r}')\right]\right\}_{\mathbf{r}'=\mathbf{r}} - \frac{Ne^2}{mc}\mathbf{Q}_\omega(\mathbf{r}). \qquad (1)$$

Here $G^{R(A)}$ and $F^{R(A)}$ are retarded and advanced Green's functions, $\hat{\mathbf{p}} = -i\nabla$, and $\hat{\mathbf{p}}' = -i\nabla'$.

It is convenient to write down the definitions of the anomalous functions directly in matrix form. To this end, we introduce the matrices

$$\mathcal{G}^{R(A)}_{\varepsilon,\varepsilon_1}(\mathbf{r},\mathbf{r}') = \begin{pmatrix} G^{R(A)}_{\varepsilon,\varepsilon_1}(\mathbf{r},\mathbf{r}') & F^{R(A)}_{\varepsilon,\varepsilon_1}(\mathbf{r},\mathbf{r}') \\ -F^{+R(A)}_{\varepsilon,\varepsilon_1}(\mathbf{r},\mathbf{r}') & \bar{G}^{R(A)}_{\varepsilon,\varepsilon_1}(\mathbf{r},\mathbf{r}') \end{pmatrix},$$

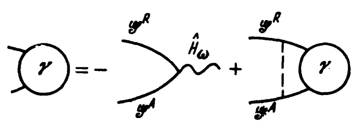
$$\mathcal{G}^{(a)}_{\varepsilon,\varepsilon-\omega}(\mathbf{r},\mathbf{r}') = \begin{pmatrix} G^{(a)}_{\varepsilon,\varepsilon-\omega}(\mathbf{r},\mathbf{r}') & F^{(a)}_{\varepsilon,\varepsilon-\omega}(\mathbf{r},\mathbf{r}') \\ -F^{+(a)}_{\varepsilon,\varepsilon-\omega}(\mathbf{r},\mathbf{r}') & \bar{G}^{(a)}_{\varepsilon,\varepsilon-\omega}(\mathbf{r},\mathbf{r}') \end{pmatrix},$$

$$\hat{H}_\omega(\mathbf{r}) = \begin{pmatrix} -e\mathbf{Q}_\omega\hat{\mathbf{p}}/mc + e\mu_\omega & -\Delta_\omega \\ \Delta_\omega & e\mathbf{Q}_\omega\hat{\mathbf{p}}/mc + e\mu_\omega \end{pmatrix}.$$

The equation for the anomalous functions $\mathscr{G}^{(a)}_{\epsilon,\epsilon-\omega}$ contain retarded and advanced functions $\mathscr{G}^{R(A)}_{\epsilon,\epsilon_1}$ that depend on two frequencies (see [3]). In view of the slowness of the considered processes ($\omega \ll \Delta$), the regular functions can be replaced by the equilibrium functions $\mathscr{G}^{R(A)}_{\epsilon} 2\pi\delta(\epsilon-\epsilon_1)$, and we can omit from $\hat{H}_\omega$ the μ-containing terms, which are of the order of ω. Ultimately we get

$$\mathscr{G}^{(a)}_{\varepsilon,\varepsilon-\omega}(\mathbf{r},\mathbf{r}') = -\frac{\omega}{2T}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}\int \mathscr{G}_\varepsilon^R(\mathbf{r},\mathbf{r}_1)\hat{H}_\omega(\mathbf{r}_1)\mathscr{G}^A_{\varepsilon-\omega}(\mathbf{r}_1,\mathbf{r}')\,d^3\mathbf{r}_1$$
$$+(2\pi\tau)^{-1}\int \mathscr{G}_\varepsilon^R(\mathbf{r},\mathbf{r}_1)\frac{2\pi^2}{mp_0}\mathscr{G}^{(a)}_{\varepsilon,\varepsilon-\omega}(\mathbf{r}_1,\mathbf{r}_1)\mathscr{G}^A_{\varepsilon-\omega}(\mathbf{r}_1,\mathbf{r}')\,d^3\mathbf{r}_1. \tag{2}$$

This formula corresponds to the diagram equation shown in the figure. The dashed lines denoting averaging over the impurities corresponds, as usual, to the factor $(2\pi\tau)^{-1}$, where τ is the free path time.



We introduce also the notation

$$\hat{\gamma}(\mathbf{r}) = \begin{pmatrix} \gamma_1 & \gamma_2 \\ -\gamma_2^+ & \bar{\gamma}_1 \end{pmatrix} = \frac{2\pi^2}{mp_0}\mathscr{G}^{(a)}_{\varepsilon,\varepsilon-\omega}(\mathbf{r},\mathbf{r}),$$

$$\hat{\boldsymbol{\gamma}}(\mathbf{r}) = \begin{pmatrix} \boldsymbol{\gamma}_1 & \boldsymbol{\gamma}_2 \\ -\boldsymbol{\gamma}_2^+ & \bar{\boldsymbol{\gamma}}_1 \end{pmatrix} = \frac{2\pi^2}{mp_0}\left[\frac{\hat{\mathbf{p}}-\hat{\mathbf{p}}'}{2p_0}\mathscr{G}^{(a)}_{\varepsilon,\varepsilon-\omega}(\mathbf{r},\mathbf{r}')\right]_{\mathbf{r}'=\mathbf{r}} \tag{3}$$

p_0 is the Fermi momentum.

Expansions of the regular parts of (1) in powers of Δ/T are well known [6]. We can use them to write down expressions for Δ and for the current:

$$\frac{\pi}{8T_c}\left[-\frac{\partial}{\partial t}+D_1\left(\nabla^2-\frac{4e^2}{c^2}Q^2\right)\right]\Delta+\left[\frac{T_c-T}{T_c}-\frac{7\zeta(3)}{8\pi^2T_c^2}\Delta^2\right]\Delta$$
$$+\int_{-\infty}^{\infty}\frac{d\varepsilon}{4\pi i}\frac{(\gamma_2+\gamma_2^+)}{2}=0, \tag{4a}$$

$$\frac{i\pi}{8T_c}D_1\operatorname{div}\left[\frac{2e}{c}\mathbf{Q}\Delta^2\right]=\int_{-\infty}^{\infty}\frac{d\varepsilon}{4\pi i}\frac{(\gamma_2-\gamma_2^+)\Delta}{2}, \tag{4b}$$

$$\mathbf{j}=\mathbf{j}^{(a)}-\frac{mp_0e^2}{2\pi c}\frac{\Delta^2}{T_c}D_1\mathbf{Q}, \tag{4c}$$

$$\mathbf{j}^{(a)}=-\frac{ep_0^2}{2\pi^2}\int_{-\infty}^{+\infty}\frac{d\varepsilon}{4\pi i}(\boldsymbol{\gamma}_1-\bar{\boldsymbol{\gamma}}_1). \tag{4d}$$

Here

$$D_1=\frac{v_0^2\tau}{3}y(\tau T),\qquad y(x)=\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2[(2n+1)2\pi x+1]}.$$

At $\tau T \ll 1$, we have $y(\tau T) = 1$ and at $\tau T \gg 1$ we get $y(\tau T) = 7\zeta(3)/2\pi^3\tau T$.

We mention immediately (and will demonstrate later on) that the condition that $|\Delta|$ be real (Eq. (4b)) follows from the expression for the current with allowance for the electroneutrality of the superconductor, $\operatorname{div}\mathbf{j} = 0$. The problem thus reduces to a solution of (2) for the anomalous functions.

We confine ourselves to alloys in which $l \ll \xi(T)$, where $l = v_0\tau$ is the mean free path. In this region we can draw the following conclusions concerning the spatial behavior of the functions $\mathscr{G}^{R(A)}_\epsilon(\mathbf{r},\mathbf{r}')$. We represent them in the form

$$\mathscr{G}_\varepsilon^{R(A)}(\mathbf{r},\mathbf{r}')=\frac{m\exp(-|\mathbf{r}-\mathbf{r}'|/2l)}{2\pi|\mathbf{r}-\mathbf{r}'|}\{\hat{g}_+^{R(A)}(\mathbf{r},\mathbf{r}')\exp(ip_0|\mathbf{r}-\mathbf{r}'|)$$
$$+\hat{g}_-^{R(A)}(\mathbf{r},\mathbf{r}')\exp(-ip_0|\mathbf{r}-\mathbf{r}'|)\}. \tag{5}$$

Applying to the right and to the left of (5) the operator

$$\mathscr{G}_\varepsilon^{-1R(A)}(\mathbf{r})=\begin{pmatrix} -\varepsilon+\dfrac{\hat{\mathbf{p}}^2}{2m}-\dfrac{e}{mc}\mathbf{Q}\hat{\mathbf{p}}-E_F & -\Delta \\ \Delta & \varepsilon+\dfrac{\hat{\mathbf{p}}^2}{2m}+\dfrac{e}{mc}\mathbf{Q}\hat{\mathbf{p}}-E_F \end{pmatrix}-\Sigma_\varepsilon^{R(A)}(\mathbf{r}),$$

where

$$\Sigma_\varepsilon^{R(A)}(\mathbf{r})=\frac{1}{2\pi\tau}\frac{2\pi^2}{mp_0}\mathscr{G}_\varepsilon^{R(A)}(\mathbf{r},\mathbf{r})$$

is the impurity self-energy part, we can easily obtain the conditions satisfied by the function $\hat{g}^{R(A)}_\pm(\mathbf{r},\mathbf{r}')$. We introduce the vector $\mathbf{n} = (\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|$; then

$$\int\{\hat{g}_+(\mathbf{r},\mathbf{r}')+\hat{g}_-(\mathbf{r},\mathbf{r}')\}^{R(A)}_{\mathbf{r}'=\mathbf{r}}\frac{do_\mathbf{n}}{4\pi}=1, \tag{6}$$

$$\left[\mp iv_0\mathbf{n}\nabla-\Sigma_\varepsilon^{R(A)}(\mathbf{r})\pm\frac{i}{2\tau}\right]\hat{g}_\pm^{R(A)}(\mathbf{r},\mathbf{r}')+\begin{pmatrix}-\varepsilon_\pm & -\Delta \\ \Delta & \varepsilon_\pm\end{pmatrix}\hat{g}_\pm^{R(A)}(\mathbf{r},\mathbf{r}')=0,$$

$$\hat{g}_\pm^{R(A)}(\mathbf{r},\mathbf{r}')\left[\pm iv_0\mathbf{n}\nabla'-\Sigma_\varepsilon^{R(A)}(\mathbf{r}')\pm\frac{i}{2\tau}\right]+\hat{g}_\pm^{R(A)}(\mathbf{r},\mathbf{r}')\begin{pmatrix}-\varepsilon_\pm & -\Delta \\ \Delta & \varepsilon_\pm\end{pmatrix}=0,$$

where $\epsilon_\pm = \epsilon \pm v_0 e\mathbf{n}\cdot\mathbf{Q}/c$ and the operator ∇' acts on $\mathbf{r}'$.

Combining the corresponding matrix elements of these equations, we easily find that the functions $\hat{g}^{R(A)}_\pm(\mathbf{r},\mathbf{r}')$ with coinciding points ($\mathbf{r}' \to \mathbf{r}$) are connected with the functions

$$\hat{g}_\varepsilon^{R(A)}(\mathbf{v}_0,\mathbf{k})=\int\mathscr{G}_\varepsilon^{R(A)}(\mathbf{p},\mathbf{p}-\mathbf{k})\frac{d\xi}{\pi i},$$

which are integrated with respect to the energy variable $\xi = v_0(p-p_0)$, by the relations

$$[\hat{g}_\pm^{R(A)}(\mathbf{r},\mathbf{r}')]_{\mathbf{r}'\to\mathbf{r}}={}^1/_2\{1\pm\hat{g}_\varepsilon^{R(A)}(\pm\mathbf{v}_0,\mathbf{r})\},$$
$$[\hat{g}_\pm^{R(A)}(\mathbf{r}',\mathbf{r})]_{\mathbf{r}'\to\mathbf{r}}={}^1/_2\{\hat{1}\pm\hat{g}_\varepsilon^{R(A)}(\mp\mathbf{v}_0,\mathbf{r})\}, \tag{7}$$

where $\mathbf{v}_0 = v_0\mathbf{n}$. Indeed, the functions

$$\hat{g}_\varepsilon^{R(A)}(\mathbf{v}_0,\mathbf{r})=\begin{pmatrix} g_\varepsilon(\mathbf{v}_0,\mathbf{r}) & f_\varepsilon(\mathbf{v}_0,\mathbf{r}) \\ -f_\varepsilon^+(\mathbf{v}_0,\mathbf{r}) & \bar{g}_\varepsilon(\mathbf{v}_0,\mathbf{r})\end{pmatrix}^{R(A)}$$

satisfy the equations derived by Eilenberger [12] and the corresponding boundary conditions at infinity:

$$-i\mathbf{v}_0\nabla g_\varepsilon+\Delta(f_\varepsilon^+-f_\varepsilon)+\frac{i}{2\tau}(f_{0\varepsilon}f_\varepsilon^+-f_\varepsilon f_{0\varepsilon}^+)=0,$$
$$-i\mathbf{v}_0\nabla\bar{g}_\varepsilon-\Delta(f_\varepsilon-f_\varepsilon^+)+\frac{i}{2\tau}(f_{0\varepsilon}^+f_\varepsilon-f_\varepsilon^+f_{0\varepsilon})=0,$$
$$\left[-2\varepsilon-i\mathbf{v}_0\left(\nabla-i\frac{2e}{c}\mathbf{Q}\right)\right]f_\varepsilon+\Delta(g_\varepsilon-\bar{g}_\varepsilon)$$
$$+\frac{i}{2\tau}(g_\varepsilon-\bar{g}_\varepsilon)f_{0\varepsilon}+\frac{i}{2\tau}(\bar{g}_{0\varepsilon}-g_{0\varepsilon})f_\varepsilon=0, \tag{8}$$
$$\left[-2\varepsilon+i\mathbf{v}_0\left(\nabla+i\frac{2e}{c}\mathbf{Q}\right)\right]f_\varepsilon^++\Delta(g_\varepsilon-\bar{g}_\varepsilon)$$
$$+\frac{i}{2\tau}(g_\varepsilon-\bar{g}_\varepsilon)f_{0\varepsilon}^++\frac{i}{2\tau}(\bar{g}_{0\varepsilon}-g_{0\varepsilon})f_\varepsilon^+=0,$$

The subscript 0 designates throughout functions averaged over the direction of $\mathbf{v}_0$.

It follows from (8) that

$$g_\varepsilon^{R(A)}(\mathbf{v}_0,\mathbf{r})+\bar{g}_\varepsilon^{R(A)}(\mathbf{v}_0,\mathbf{r})=0,$$
$$[g_\varepsilon^{R(A)}(\mathbf{v}_0,\mathbf{r})]^2-f_\varepsilon^{R(A)}(\mathbf{v}_0,\mathbf{r})f_\varepsilon^{+R(A)}(\mathbf{v}_0,\mathbf{r})=1. \tag{9}$$

We have used here the fact that as $\mathbf{r} \to \infty$ we get

$$g_\varepsilon^{R(A)}(\mathbf{v}_0,\mathbf{r})=\frac{\varepsilon}{\lambda_\infty^{R(A)}},\quad f_\varepsilon^{R(A)}(\mathbf{v}_0,\mathbf{r})=\frac{\Delta_\infty}{\lambda_\infty^{R(A)}},\quad \lambda_\infty=\sqrt{\varepsilon^2-\Delta_\infty^2},$$

where $\Delta = \Delta_\infty = \text{const}$; $\lambda^{R(A)}$ are so defined that when $\epsilon = \epsilon_0 + i\delta$, $\epsilon_0 > \Delta > 0$, and $\delta \to +0$ we have $\lambda^{R(A)} = \pm\lambda$.

In the limit when $l \ll \xi(T)$, the functions $\hat{g}^{R(A)}(\mathbf{r}, \mathbf{r}')$, which vary over distances on the order of $\xi(T)$, can be regarded as slow in comparison with $\exp[-|\mathbf{r} - \mathbf{r}'|/2l]$, while the functions $g_\epsilon^{R(A)}(\mathbf{v}_0, \mathbf{r})$ can be regarded in first approximation as isotropic with respect to the directions of the vector $\mathbf{v}_0$ (cf.[13]), i.e., they can be represented in the form

$$\hat{g}_\epsilon^{R(A)}(\mathbf{v}_0, \mathbf{r}) = \hat{g}_{0\epsilon}^{R(A)}(\mathbf{r}) + \frac{\mathbf{v}_0}{v}\hat{\mathbf{g}}_\epsilon^{R(A)}(\mathbf{r}),$$

where $|\mathbf{g}| \ll g_0$. We see from (6) and (7) that the last assumption is the consequence of the slowness of the variation of $g_\pm$. Indeed, the functions $\hat{g}_{0\epsilon}$ drop out from the large terms $\pm i(2\tau)^{-1}\hat{g}_\pm - \Sigma\hat{g}_\pm$ of (6) as $r \to r'$, and it turns out that $|\mathbf{g}| \sim (l\Delta/\xi_0 T_c)g_0$ (an important factor in this estimate is that the principal role will henceforth be assumed by frequencies $\epsilon \sim \Delta$).

Using the representation (5), we could calculate directly the kernel $\mathcal{G}_\epsilon^R(\mathbf{r}, \mathbf{r}')\mathcal{G}_{\epsilon-\omega}^A(\mathbf{r}', \mathbf{r})$ in (2) and obtain equations for the functions $\hat{\gamma}(\mathbf{r})$. The determinant of the matrix equation for $\hat{\gamma}$, taken accurate only to zeroth order terms in l, vanishes in this case. It becomes necessary to expand the kernel in (2) up to terms of second order in l, but this entails extremely cumbersome calculations. We therefore choose a somewhat different method, which is also valid so long as it can be assumed that $l \ll \xi(T)$, an assumption that can always be made in a sufficiently small vicinity of T_c. Our results in this region are valid not only for "dirty" alloys with $l \ll \xi_0$, but also for relatively pure superconductors, where

$$v_0/T_c \ll l \ll \frac{v_0}{T_c}(1 - T/T_c)^{-1/2}.$$

We write down the equations that are satisfied by the functions

$$\hat{g}^{(a)}_{\epsilon,\epsilon-\omega}(\mathbf{v}_0, \mathbf{k}) = \int \mathcal{G}^{(a)}_{\epsilon,\epsilon-\omega}(\mathbf{p}, \mathbf{p} - \mathbf{k})\, d\xi, \qquad \xi = v_0(p - p_0),$$

using the same procedure as Eilenberg[12] (see above) and Eliashberg[14]. We apply to Eq. (2) the operator $\mathcal{G}_\epsilon^{-1R}$ from the left, the operator $\mathcal{G}_\epsilon^{-1A}$ from the right, and subtract one resultant equation from the other. After integrating with respect to ξ we obtain

$$\begin{pmatrix} A_1 & A_2 \\ -A_2^+ & \bar{A}_1 \end{pmatrix} = \Sigma^R(\mathbf{r})\hat{g}^{(a)}(\mathbf{v}_0, \mathbf{r}) - \hat{g}^{(a)}(\mathbf{v}_0, \mathbf{r})\Sigma^A(\mathbf{r})$$
$$+ i(2\tau)^{-1}[\hat{\gamma}(\mathbf{r})\hat{g}^A(\mathbf{v}_0, \mathbf{r}) - \hat{g}^R(\mathbf{v}_0, \mathbf{r})\hat{\gamma}(\mathbf{r})] \tag{10}$$
$$- \pi i \frac{\omega}{2T}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}[\hat{H}_\omega \hat{g}^A(\mathbf{v}_0, \mathbf{r}) - \hat{g}^R(\mathbf{v}_0, \mathbf{r})\hat{H}_\omega],$$

where

$$\begin{Bmatrix} A_1 \\ \bar{A}_1 \end{Bmatrix} = (-i\mathbf{v}_0\nabla \mp \omega)\begin{Bmatrix} g^{(a)}(\mathbf{v}_0, \mathbf{r}) \\ \bar{g}^{(a)}(\mathbf{v}_0, \mathbf{r}) \end{Bmatrix} \pm \Delta(f^{+(a)}(\mathbf{v}_0, \mathbf{r}) - f^{(a)}(\mathbf{v}_0, \mathbf{r})),$$

$$\begin{Bmatrix} A_2 \\ A_2^+ \end{Bmatrix} = \left[\mp(2\varepsilon - \omega) - i\mathbf{v}_0\left(\nabla \mp i\frac{2e}{c}\mathbf{Q}\right)\right]\begin{Bmatrix} f^{(a)}(\mathbf{v}_0, \mathbf{r}) \\ f^{+(a)}(\mathbf{v}_0, \mathbf{r}) \end{Bmatrix}$$
$$\pm \Delta(g^{(a)}(\mathbf{v}_0, \mathbf{r}) - \bar{g}^{(a)}(\mathbf{v}_0, \mathbf{r})).$$

For the sake of brevity, we shall henceforth drop the subscripts ϵ and $\epsilon - \omega$, bearing in mind that $\hat{g}^R$ refers to the frequency ϵ, g^A to the frequency $\epsilon - \omega$, and $\hat{g}^{(a)}$ to both frequencies.

Averaging (10) over the directions of the vector $\mathbf{v}_0$, we obtain

$$-iv_0\mathbf{d}\hat{\gamma} + \begin{pmatrix} \Delta(\gamma_2^+ - \gamma_2) - \omega\gamma_1 & -2\varepsilon\gamma_2 + \Delta(\gamma_1 - \bar{\gamma}_1) \\ -2\varepsilon\gamma_2^+ + \Delta(\gamma_1 - \bar{\gamma}_1) & \Delta(\gamma_2 - \gamma_2^+) + \omega\bar{\gamma}_1 \end{pmatrix}$$
$$= \pi i\frac{\omega\Delta_\omega}{2T}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}\begin{pmatrix} f_0^R - f_0^{+A} & -g_0^R + \bar{g}_0^A \\ \bar{g}_0^R - g_0^A & f_0^{+R} - f_0^A \end{pmatrix} \tag{11}$$
$$+ \pi i\frac{\omega}{2T}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}\frac{v_0}{3}\frac{e}{c}\mathbf{Q}\begin{pmatrix} -\mathbf{g}^R + \mathbf{g}^A & \mathbf{f}^R + \mathbf{f}^A \\ \mathbf{f}^{+R} + \mathbf{f}^{+A} & \bar{\mathbf{g}}^R - \bar{\mathbf{g}}^A \end{pmatrix}.$$

We denote by $\mathbf{d}\hat{\gamma}$ the gauge-invariant differentiation

$$\begin{pmatrix} \nabla\boldsymbol{\gamma}_1 & \left(\nabla - i\frac{2e}{c}\mathbf{Q}\right)\boldsymbol{\gamma}_2 \\ -\left(\nabla + i\frac{2e}{c}\mathbf{Q}\right)\boldsymbol{\gamma}_2^+ & \nabla\bar{\boldsymbol{\gamma}}_1 \end{pmatrix}$$

The terms containing τ^{-1} drop out after averaging.

To derive the equations satisfied by the quantities $\hat{\gamma}$, we proceed as follows. With the aid of the representation (5), using the fact that the functions $\hat{g}_\pm^{R(A)}$ vary slowly over distances on the order of l, we calculate the vector $\hat{\boldsymbol{\gamma}}$ directly from the definition (3). It suffices to do so accurate to first order in l. We then obtain the sought equations by substituting the values of $\hat{\boldsymbol{\gamma}}$ in (11).

We note that the method used by Usadel[13] to obtain the equations for the functions f and g averaged over the angles v_0 are not valid here. Indeed, multiplying (10) by $\mathbf{v}_0$ and averaging over the directions of $\mathbf{v}_0$ we would obtain one more system of relations between $\hat{\gamma}$ and $\hat{\boldsymbol{\gamma}}$. Since $\Sigma^{R(A)} = i(2\tau)^{-1}\hat{g}_0^{R(A)}$, the principal terms with $\hat{\gamma}$ are contained in the combination $i(2\tau)^{-1}[\hat{g}_0^R\hat{\underline{\gamma}} - \hat{\boldsymbol{\gamma}}\hat{g}_0^A]$. Were it possible to express the quantities $\boldsymbol{\gamma}_1$, $\bar{\boldsymbol{\gamma}}_1$, $\boldsymbol{\gamma}_2$, $\boldsymbol{\gamma}_2^+$ with the aid of this system in terms of $\hat{g}_0^{R(A)}$ and $\hat{\mathbf{g}}^{R(A)}$, then, by substituting them in (11), we would obtain equations for $\hat{\gamma}$. However, the determinant of the four-row matrix for the quantities $\boldsymbol{\gamma}_1$, $\bar{\boldsymbol{\gamma}}_1$, $\boldsymbol{\gamma}_2$, $\boldsymbol{\gamma}_2^+$ is equal to zero.

Substituting (2) and (5) in (3), we get

$$\frac{mp_0}{2\pi^2}\hat{\boldsymbol{\gamma}} = -\frac{\omega}{2T}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}\int\left(\frac{m}{2\pi R}\right)^2 e^{-R/l}\mathbf{n}\{\hat{g}_+^R(\mathbf{r}, \mathbf{r}')\hat{h}_-(\mathbf{r}')\hat{g}_-^A(\mathbf{r}', \mathbf{r})$$
$$- \hat{g}_-^R(\mathbf{r}, \mathbf{r}')\hat{h}_+(\mathbf{r}')\hat{g}_+^A(\mathbf{r}', \mathbf{r})\}d^3\mathbf{r}'$$
$$+ (2\pi\tau)^{-1}\int\left(\frac{m}{2\pi R}\right)^2 e^{-R/l}\mathbf{n}\{\hat{g}_+^R(\mathbf{r}, \mathbf{r}')\hat{\gamma}(\mathbf{r}')\hat{g}_-^A(\mathbf{r}', \mathbf{r}) \tag{12}$$
$$- \hat{g}_-^R(\mathbf{r}, \mathbf{r}')\hat{\gamma}(\mathbf{r}')\hat{g}_+^A(\mathbf{r}', \mathbf{r})\}d^3\mathbf{r}',$$

$$\hat{h}_\pm(\mathbf{r}') = \begin{pmatrix} \pm v_0 e\mathbf{n}\mathbf{Q}_\omega(\mathbf{r}')/c & -\Delta_\omega(\mathbf{r}') \\ \Delta_\omega(\mathbf{r}') & \mp v_0 e\mathbf{n}\mathbf{Q}_\omega(\mathbf{r}')/c \end{pmatrix}, \qquad \mathbf{R} = \mathbf{r} - \mathbf{r}'.$$

We now expand the quantities $\hat{g}_\pm^R(\mathbf{r}, \mathbf{r}')$, $\hat{\mathbf{g}}^A(\mathbf{r}', \mathbf{r})$, $\hat{\gamma}(\mathbf{r}')$ and $\hat{h}_\pm(\mathbf{r}')$ in powers of $\mathbf{R}$ about the point $\mathbf{r}$:

$$\hat{g}_\pm^R(\mathbf{r}, \mathbf{r}') = [\hat{g}_\pm^R(\mathbf{r}, \mathbf{r}')]_{\mathbf{r}'=\mathbf{r}} - R[\mathbf{n}\nabla'\hat{g}_\pm^R(\mathbf{r}, \mathbf{r}')]_{\mathbf{r}'=\mathbf{r}}$$

etc. The derivatives $\mathbf{n}\nabla'\hat{g}_\pm(\mathbf{r}, \mathbf{r}')$, $\hat{\mathbf{r}}' \to \mathbf{r}$, can be expressed with the aid of (6) in terms of $\hat{\mathbf{g}}_\pm(\mathbf{r}, \mathbf{r})$. Integrating in (12) with respect to R and averaging over the directions of $\mathbf{n}$, we obtain with the aid of (7) after rather laborious calculations

$$\hat{\boldsymbol{\gamma}} = -\tfrac{1}{3}\{l[\hat{g}_0^R\hat{\gamma}\hat{\mathbf{g}}^A + \hat{\mathbf{g}}^R\hat{\gamma}\hat{g}_0^A] + \tfrac{1}{4}[\hat{\mathbf{g}}^R(\hat{g}_0^R\hat{\gamma} - \hat{\gamma}\hat{g}_0^A)$$
$$- (\hat{g}_0^R\hat{\gamma} - \hat{\gamma}\hat{g}_0^A)\hat{\mathbf{g}}^A] + \tfrac{1}{2}l[\hat{g}_0^R\mathbf{d}\underline{\hat{\gamma}} - (\mathbf{d}\hat{\gamma})\hat{g}_0^A]\} \tag{13}$$
$$+ \frac{\pi l}{3}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}\frac{\omega}{2T}\frac{e}{c}\mathbf{Q}_\omega\{\sigma_z - \hat{g}_0^R\sigma_z\hat{g}_0^A\},$$

where σ_z is a Pauli matrix.

Before we continue our conclusion, we turn to the determination of the regular functions $\hat{g}^{R(A)}(\mathbf{v}_0, \mathbf{r})$. We can eliminate from (8) the vectors $\mathbf{g}^{R(A)}$, $\mathbf{f}^{R(A)}$, and $\mathbf{f}^{+R(A)}$

with the aid of the conditions (8) (see [13]):

$$\mathbf{f}=l\left[\frac{f_0}{2g_0}\nabla(f_0f_0^+)-g_0\left(\nabla-i\frac{2e}{c}\mathbf{Q}\right)f_0\right],$$
$$\mathbf{f}^+=l\left[-\frac{f_0^+}{2g_0}\nabla(f_0f_0^+)+g_0\left(\nabla+i\frac{2e}{c}\mathbf{Q}\right)f_0^+\right], \qquad (14)$$
$$\mathbf{g}={}^1/_2 l\left[f_0\left(\nabla+i\frac{2e}{c}\mathbf{Q}\right)f_0^+-f_0^+\left(\nabla-i\frac{2e}{c}\mathbf{Q}\right)f_0\right],$$

and obtain equations for $f_0^{R(A)}$ and $f_0^{+R(A)}$:

$$-2\varepsilon f_0+2\Delta g_0+iD\left(\nabla-i\frac{2e}{c}\mathbf{Q}\right)\left[g_0\left(\nabla-i\frac{2e}{c}\mathbf{Q}\right)f_0-\frac{f_0}{2g_0}\nabla(f_0f_0^+)\right]=0,$$
$$-2\varepsilon f_0^++2\Delta g_0+iD\left(\nabla+i\frac{2e}{c}\mathbf{Q}\right)\left[g_0\left(\nabla+i\frac{2e}{c}\mathbf{Q}\right)f_0^+-\frac{f_0^+}{2g_0}\nabla(f_0f_0^+)\right]=0, \qquad (15)$$

where $D = v_0l/3$ is the diffusion coefficients. For the sake of brevity, the superscripts R and A to the functions f_0 and g_0 are omitted. The function g_0 is so defined that

$$g_0^2-f_0f_0^+=1.$$

The rather complicated equations in (15) become simpler in the considered temperature interval $T_c - T \ll T_c$ because the terms with the spatial derivatives and with the field are small. Indeed, as is seen from the static equations (4) ($\partial\Delta/\partial t$ and $\gamma_2+\gamma_2^+$ are equal to zero), $\nabla^2\Delta$ and $(2e\mathbf{Q}/c)^2\Delta$ are of the order of $\Delta\xi^{-2}(T)\sim\Delta^3/D_1T_c$. The combinations $D\nabla^2\Delta$ and $D(2e\mathbf{Q}/c)^2\Delta$ in (15) are therefore of order $D\Delta^2/D_1T_c$ relative to Δ. At $\tau T_c \ll 1$ (i.e., $l\ll\xi_0$) this reduces to Δ^2/T_c, which is much less than Δ at $3.3(1-T/T_c)^{1/2}\ll 1$. On the other hand, if $\tau T_c\gg 1$, then we have

$$\frac{D}{D_1}\frac{\Delta^2}{T_c}\sim\frac{l}{\xi(T)}\Delta.$$

The ratio

$$\frac{l}{\xi(T)}=1{,}3\frac{l}{\xi_0}\left(1-\frac{T}{T_c}\right)^{1/2}$$

becomes small in the vicinity of $(1-T/T_c)^{1/2}\ll\xi_0/l$.

Thus, at $l\ll\xi(T)$ we can neglect in (15) the terms with the derivative and field in comparison with Δ and ϵ. We see that at $l\ll\xi(T)$ the functions f_0 and g_0 depend adiabatically on the distance $\mathbf{r}$:

$$f_0^{R(A)}=f_0^{+R(A)}=\frac{\Delta}{\lambda^{R(A)}},\qquad g_0^{R(A)}=\frac{\varepsilon}{\lambda^{R(A)}},\qquad \lambda=\sqrt{\varepsilon^2-\Delta^2}, \qquad (16)$$

the $\lambda^{R(A)}$ are so defined that we have $\lambda^{R(A)}=\pm\lambda$ at $\epsilon=\epsilon_0+i\delta$, $\epsilon_0>\Delta>0$, and $\delta\to+0$. The matching of the expressions at $\epsilon<\Delta(\mathbf{r})$ and $\epsilon>\Delta(\mathbf{r})$ should occur, as follows from (15), near the point $\mathbf{r}_c$, where $\Delta(\mathbf{r}_c)=\epsilon$, at distances on the order of $\xi(D/\xi^2\Delta)^{2/3}$, which is much less than $\xi(T)$. The retarded and advanced functions determined in this manner are equal on the real axis in the interval $-\Delta<\epsilon<\Delta$, apart from small terms containing D.

We now return to Eq. (11). Since, as seen from (13), $\hat{\boldsymbol{\gamma}}\propto l$, we have with the required accuracy

$$-\varepsilon(\gamma_2+\gamma_2^+)+\Delta(\gamma_1-\bar{\gamma}_1)=-\pi i\frac{\omega\Delta_\omega}{2T}(g_0^R+g_0^A)\,\mathrm{ch}^{-2}\frac{\varepsilon}{2T}. \qquad (17)$$

In addition

$$-iv_0\nabla(\boldsymbol{\gamma}_1+\bar{\boldsymbol{\gamma}}_1)-\omega(\gamma-\bar{\gamma}_1)=\pi i\frac{\omega\Delta_\omega}{T}\mathrm{ch}^{-2}\frac{\varepsilon}{2T}(f_0^R-f_0^A). \qquad (18)$$

On the other hand, $\gamma_2-\gamma_2^+$ is proportional to l, and therefore, substituting in (18) $\boldsymbol{\gamma}_1+\bar{\boldsymbol{\gamma}}_1$ from (13), we obtain with the aid of (14) and (16), retaining the principal terms in the right-hand side

$$D\nabla\left\{\frac{1}{g_0^R}\left[(f_0^R-f_0^A)(\nabla f_0^R)(\gamma_1-\bar{\gamma}_1)-\nabla(f_0^R-f_0^A)g_0^R(\gamma_2+\gamma_2^+)\right]+\frac{g_0^R-g_0^A}{2}\nabla(\gamma-\bar{\gamma}_1)-\frac{f_0^R-f_0^A}{2}\nabla(\gamma_2+\gamma_2^+)\right\}+i\omega(\gamma_1-\bar{\gamma}_1) \qquad (19)$$
$$=\pi\frac{\omega\Delta_\omega}{T}\mathrm{ch}^{-2}\frac{\varepsilon}{2T}(f_0^R-f_0^A).$$

In the derivation of (19) we have left out of the left-hand sides of the advanced functions the frequency ω, which is small in comparison with ϵ, since terms containing ω can be neglected in comparison with $i\omega(\gamma_1-\bar{\gamma}_1)$ in view of the smallness of $D\nabla^2$.

As already mentioned, there are two different regions, $|\epsilon|>\Delta$ and $|\epsilon|<\Delta$. At $|\epsilon|>\Delta$ Eq. (17) yields

$$\varepsilon(\gamma_2+\gamma_2^+)=\Delta(\gamma_1-\bar{\gamma}_1)$$

and we readily obtain from (19)

$$-\frac{\partial}{\partial t}\left[\frac{\varepsilon(\gamma_2+\gamma_2^+)}{\Delta}\right]+D\nabla^2\left[\frac{\gamma_2+\gamma_2^+}{f}\right]=\frac{2\pi i}{T}\mathrm{ch}^{-2}\frac{\varepsilon}{2T}f\frac{\partial\Delta}{\partial t}. \qquad (20)$$

We have put

$$f=\begin{cases}\Delta/(\varepsilon^2-\Delta^2)^{1/2}, & \varepsilon>\Delta\\ -\Delta/(\varepsilon^2-\Delta^2)^{1/2}, & \varepsilon<-\Delta\end{cases};\qquad g=\begin{cases}\varepsilon/(\varepsilon^2-\Delta^2)^{1/2}, & \varepsilon>\Delta,\\ -\varepsilon/(\varepsilon^2-\Delta^2)^{1/2}, & \varepsilon<-\Delta.\end{cases}$$

We see that owing to the smallness of $D\nabla^2$ (see above) the quantity $(\gamma_2+\gamma_2^+)$ is anomalously large in the region $|\epsilon|>\Delta$.

We note here the fact that at $|\epsilon|>\Delta$ the integral

$$\int(\gamma_2+\gamma_2^+)\,d\varepsilon/4\pi i$$

is determined by the frequencies $\epsilon\sim\Delta\ll T$.

At $|\epsilon|<\Delta$, the terms of first order in ω in the right and left sides of (19) vanish. In this case it is necessary to take into account in the right-hand side the difference between $f_{0\epsilon}^R$ and $f_{0\epsilon-\omega}^A$, which is due to ω. As a result we get

$$\gamma_2+\gamma_2^+=-\frac{\pi i}{T}\mathrm{ch}^{-2}\frac{\varepsilon}{2T}\frac{\varepsilon^2}{(\Delta^2-\varepsilon^2)^{3/2}}\frac{\partial\Delta}{\partial t}.$$

This expression coincides with Eq. (8) of [4] at $|\epsilon|<\Delta$. Unlike (20), it does not contain the small factor $D\nabla^2$ in the left-hand side, and therefore in the region $|\epsilon|<\Delta$ the quantity $(\gamma_2+\gamma_2^+)$ turns out to be small in comparison with its value at $|\epsilon|>\Delta$.

As to Eq. (4b), we shall now show that it is not independent, but follows from the expressions (4c, d) for the current and the continuity equation

$$\operatorname{div}\mathbf{j}+\partial\rho/\partial t=0,$$

where ρ is the charge density. The continuity equation in a superconductor, in turn, just as in any metal, means $\operatorname{div}\mathbf{j}=0$, since no charges accumulate, owing to the strong Coulomb interaction, and $\partial\rho/\partial t=0$ [3] (the quasi-neutrality condition).

We note that the last equation in our model is satisfied identically; namely, in slow nonstationary processes the deviation of the charge density from the equilibrium value is proportional to the frequency, so that $\partial\rho/\partial t$ is of second order of smallness and should be taken equal to zero. In other words, Eq. (4b) is the consequence of Maxwell's equation

$$\operatorname{rot}\mathbf{H}=\frac{4\pi}{c}\mathbf{j}$$

and of the equation for the current. Indeed, it is seen from (11) that, accurate to terms of first order in the frequency

$$v_0\nabla(\gamma_1-\bar{\gamma}_1)=2i\Delta(\gamma_2-\gamma_2^+).$$

Multiplying this equation by $(emp_0/2\pi^2)(d\epsilon/4\pi i)$ and integrating with respect to ϵ, we obtain with the aid of (4d)

$$\operatorname{div}\mathbf{j}^{(a)}-\frac{mp_0e^2}{2\pi c}D_1\operatorname{div}\left[\frac{\Delta^2}{T_c}\mathbf{Q}\right]=\frac{2iemp_0}{\pi^2}\left\{\frac{i\pi}{8T_c}D_1\operatorname{div}\left[\frac{2e}{c}\mathbf{Q}\Delta^2\right]-\int\frac{d\varepsilon}{4\pi i}\frac{\Delta(\gamma_2-\gamma_2^+)}{2}\right\}.$$

Comparing this with (4c) we obtain immediately (4b).

Thus, it suffices to write only the equations that determine $\mathbf{j}^{(a)}$. To find $\gamma_1-\bar{\gamma}_1$, it is necessary in general to solve the entire system (11). We can verify, however, that the contribution made to the energy dissipation in the vortex by the anomalous part of the current $\mathbf{j}^{(a)}$ is small in comparison with the contribution of $\gamma_2+\gamma_2^+$. Indeed, the expression for $\gamma_1-\bar{\gamma}_1$ at $|\epsilon|>\Delta$ is

$$(\gamma_1-\bar{\gamma}_1)=\frac{l}{3}\left\{-g^2\nabla\left(\frac{\gamma_1+\bar{\gamma}_1}{g}\right)+\frac{2\pi i}{T}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}g^2\frac{e}{c}\frac{\partial\mathbf{Q}}{\partial t}\right\}.$$

For $\gamma_1+\bar{\gamma}_1$ we have in turn from (11) and (13)

$$\nabla^2(\gamma_1+\bar{\gamma}_1)-\left[\frac{(\nabla f)^2}{g^2}+\left(\frac{2e}{c}\mathbf{Q}f\right)^2\right](\gamma_1+\bar{\gamma}_1)=\frac{2\pi i}{T}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}\nabla\left(g^2\frac{e}{c}\frac{\partial\mathbf{Q}}{\partial t}\right)g^{-1}.$$

Retaining in the integral

$$\int(\gamma_1-\bar{\gamma}_1)\,d\varepsilon/4\pi i$$

only the principal terms in Δ/T, we obtain

$$\mathbf{j}^{(a)}=\sigma\left\{-\frac{1}{c}\frac{\partial\mathbf{Q}}{\partial t}+\frac{1}{2e}\nabla\int(\gamma_1+\bar{\gamma}_1)\frac{d\varepsilon}{4\pi i}\right\},$$

$$\nabla^2(\gamma_1+\bar{\gamma}_1)=\frac{2\pi i}{T}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}\frac{e}{c}\operatorname{div}\left(\frac{\partial\mathbf{Q}}{\partial t}\right),$$

where $\sigma=p_0^3e^3\tau/3\pi^2m$ is the conductivity of the normal metal. Thus, the divergence of the normal current is equal to zero. As applied to the motion of vortex filaments, the last equation gives a stronger result. Namely, the vector potential Q at the center of the filament varies like ρ^{-1}, where ρ is the distance to the filament axis[15]. Stipulating that the normal current be finite in the center of the vortex $\rho\ll\xi$, we obtain for the principal terms in Δ/T

$$\nabla(\gamma_1+\bar{\gamma}_1)=\frac{2\pi i}{T}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}\frac{e}{c}\frac{\partial\mathbf{Q}}{\partial t}.$$

That is to say, the normal current is equal to zero in this case. The corrections to $\mathbf{j}^{(a)}$ in the next higher approximation in Δ/T, as seen from the equation for $\gamma_1+\gamma_1$, are of the order of Δ/T relative to $c^{-1}\sigma\partial\mathbf{Q}/\partial t$ and, consequently, make a small contribution to the energy dissipated by the moving vortex.

3. MOTION OF VORTEX FILAMENTS IN AN ALLOY AT $T_c-T\ll T_c$

Just as in the earlier work[9], we assume a magnetic field $H_0\ll H_{c2}$. This means that the vortex-filament density B_0/Φ_0 is small ($\Phi_0=\pi c/e$ is the flux quantum and B_0 is the induction), i.e., the distance d between filaments is much larger than the dimension $\xi(T)$ of the core of the filament. This condition enables us to reduce the problem of the motion of a lattice of vortices to the problem of the single filament. Namely, assume that a transport current $\mathbf{j}_{tr}$ perpendicular to the magnetic field H_0 flows through the sample.

As shown by us in[9], the current produced by an individual vortex at distances $\rho,\xi\ll\rho\ll d$ from the center is equal to the transport current. It is thus necessary to establish the connection between the velocity of a single filament and the current at large distances from the filament center. Knowing the vortex velocity, we can determine from the formula $\operatorname{curl}\bar{\mathbf{E}}=-c^{-1}\partial B/\partial t$ the intensity of the average electric field

$$\bar{\mathbf{E}}=\frac{B_0}{c}[\mathbf{n}_H\times\mathbf{u}] \tag{21}$$

($\mathbf{n}_H$ is a unit vector in the magnetic-field direction and $\mathbf{u}$ is the velocity of the vortex filament), and consequently determine the resistance of the sample. We shall henceforth assume that the Ginzburg-Landau parameter is $\kappa=\delta/\xi\gg1$.[1)]

The function $\Delta(\rho)$ for an immobile vortex was obtained by Abrikosov[15]. We shall label the corresponding functions Δ and $\mathbf{Q}$ by the index 0. At low velocities it can be assumed that in first approximation the vortex moves as a unit, i.e., Δ and $\mathbf{Q}$ are the functions Δ_0 and $\mathbf{Q}_0$ of $\mathbf{r}-\mathbf{u}t$, and the deviations due to the "deceleration" are small corrections, Δ_1 and $\mathbf{Q}_1$. We choose also a cylindrical coordinate system (ρ,φ,z) in which the z axis coincides at the initial instant of time with the filament axis. We have

$$\Delta=\Delta_0(\mathbf{r}-\mathbf{u}t)+\Delta_1,\quad \mathbf{Q}=\mathbf{Q}_0(\mathbf{r}-\mathbf{u}t)+\mathbf{Q}_1. \tag{22}$$

We recall that Eqs. (4) without the anomalous term and the time derivative are static equations that are invariant relative to any displacement of the origin. The functions $\Delta_0(\mathbf{r}+\mathbf{a})$ and $\mathbf{Q}_0(\mathbf{r}+\mathbf{a})$, where $\mathbf{a}$ is an arbitrary constant vector, are therefore also solutions of (4). We put $\Delta_1^0=(\mathbf{a}\cdot\nabla)\Delta_0$ and $\mathbf{Q}_1^0=(\mathbf{a}\cdot\nabla)\mathbf{Q}_0$. Then Δ_1^0 and $\mathbf{Q}_1^0$ also satisfy the static equations linearized relative to the small deviations Δ_1 and $\mathbf{Q}_1$.

We substitute the expressions of (22) in the equations of (4). Linearizing them and recalling that $\Delta_0=t(\mathbf{u}\cdot\nabla)\Delta_0$ and $\mathbf{Q}_0-t(\mathbf{u}\cdot\nabla)\mathbf{Q}_0$ are static solutions, we obtain

$$\frac{\pi}{8T_c}D_1\left[\nabla^2\Delta_1-\frac{4e^2}{c^2}Q_0^2\Delta_1-\frac{8e^2}{c^2}\mathbf{Q}_0\mathbf{Q}_1\Delta_0\right]+\left[\frac{T_c-T}{T_c}-3\Delta^2\frac{7\zeta(3)}{8\pi^2T_c^2}\right]\Delta_1=-\Delta^{(a)}, \tag{23'}$$

$$\mathbf{j}_1+\frac{mp_0e^2D_1}{2\pi cT}[\Delta_0^2\mathbf{Q}_1+2\Delta_0\mathbf{Q}_0\Delta_1]=0;$$

$$\Delta^{(a)}=\int_0^\infty\frac{d\varepsilon}{4\pi i}(\gamma_2+\gamma_2^+). \tag{23''}$$

The quantity $\gamma_2+\gamma_2^+$ is determined from (20) where $\partial\Delta/\partial t$ is replaced by $-(\mathbf{u}\cdot\nabla)\Delta_0$ and $\Delta=\Delta_0$.

As already noted, $\Delta^{(a)}$ is large, and we can therefore omit from (23') the term $(\pi/8T_c\partial\Delta/\partial t$ in the right hand-side. As to $\mathbf{j}^{(a)}$, this quantity, as already mentioned, is of the order of $\sigma c^{-1}\partial\mathbf{Q}/\partial t(\Delta/T)$.

For a moving vortex we have therefore

$$j^{(a)}\sim\frac{\sigma\Delta}{Te\xi^2}u\sim\frac{\Delta}{T}\frac{\sigma H_{c2}u}{c}.$$

We shall show below (see (28)) that this makes a smaller contribution (by a factor Δ^2/T^2) to the conductivity than the anomalous term $\Delta^{(a)}$, so that $\mathbf{j}^{(a)}$ can be left out from the right-hand side of (23'')[2)].

To determine the current $\mathbf{j}_1$ at large distances $\rho\gg\xi(T)$ from the center of the filament, we use the procedure employed by us earlier[9] to find the integral of the system of equations for Δ_1 and $\mathbf{Q}_1$. We need also the equation stemming from the superconductor electroneutrality condition:

$$\mathrm{div}[\Delta_0^2\mathbf{Q}_1 + 2\Delta_0\mathbf{Q}_0\Delta_1] = 0. \tag{24}$$

We are not interested here in the bending of the vortex filaments, and therefore leave out the corresponding terms in (23) and (24). The results can be easily generalized to this case.

Let the velocity of the vortex $\mathbf{u}$ be parallel to the x axis. Then

$$\frac{\partial\Delta}{\partial t} = -(\mathbf{u}\nabla)\Delta_0 = -u\frac{\partial\Delta}{\partial\rho}\cos\varphi.$$

It is seen from (20) that $\Delta^{(a)}$ is also proportional to $\cos\varphi$. If we express (23) and (24) in cylindrical coordinates and separate the dependence on the angle, then we can obtain, using the same procedure as in [9],

$$[\mathbf{n}_H\mathbf{a}]\mathbf{j}_{tr} = [\mathbf{n}_H\times\mathbf{a}]\mathbf{j}_1 = \frac{mp_0e}{\pi^3}\int\Delta^{(a)}\Delta_1^0\rho\, d\rho\, a\varphi. \tag{25}$$

We now calculate $\Delta^{(a)}$. We can neglect the term with $\partial/\partial t$ in the left-hand side of (20), since the vortex velocity is low. We express (20) in cylindrical coordinates, putting

$$\gamma_2 + \gamma_2^+ = uf(\rho)w(\rho)\cos\varphi.$$

Since $\Delta_1^0 = (\mathbf{a}\cdot\nabla)\Delta_0$, the integral in (25) is proportional to the scalar product $\mathbf{u}\cdot\mathbf{a}$:

$$[\mathbf{n}_H\times\mathbf{a}]\mathbf{j}_{tr} = \frac{emp_0}{\pi^2}(\mathbf{u}\mathbf{a})\int_0^\infty\rho\, d\rho\frac{d\Delta}{d\rho}\int_{\varepsilon>0}\frac{d\varepsilon}{4\pi i}w\frac{\Delta}{(\varepsilon^2-\Delta^2)^{1/2}}, \tag{26}$$

and w is obtained from the equation

$$D\frac{a}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}(\rho w)\right] = \frac{2\pi i}{T}\frac{d}{d\rho}(\varepsilon^2-\Delta^2)^{1/2}, \quad |\varepsilon| > \Delta(\rho), \tag{27}$$

with $w = 0$ at $|\epsilon| < \Delta$.

Equation (27) has a solution that ensures that the functions $\gamma_1 - \bar{\gamma}_1$ and $\gamma_2 + \gamma_2^+$ are finite at $\rho = 0$. This solution is

$$w = \frac{2\pi i}{DT}\frac{1}{\rho}\int_0^\rho[(\varepsilon^2-\Delta^2)^{1/2} - C]\rho'\, d\rho'.$$

To determine the constant C, we recall that in our approximation $\gamma_2 + \gamma_2^+$ should be regarded as different from zero only if $|\epsilon| > \Delta(\rho)$. The condition that $\gamma_2 + \gamma_2^+$ decrease at large distances therefore determines C only when $\epsilon > \Delta_\infty$, then there exists a certain value $\rho = \rho_\epsilon$ at which $\epsilon = \Delta(\rho)$. Thus, $\gamma_2 + \gamma_2^+$ differs from zero when $\rho < \rho_\epsilon$ and equals zero when $\rho > \rho_\epsilon$.

In the approximation in which the retarded and advanced functions $f_0^{R(A)}$ and $g_0^{R(A)}$ (16) vary slowly, the rigorous boundary condition for the values of γ at $\rho = \rho_\epsilon$ necessitates that the solutions be matched together in this region. We consider it natural, however, to impose on the real quantities $(\gamma_2 + \gamma_2)/4\pi i$ etc. the requirement that they be continuous at the point ρ_ϵ. Putting $w(\rho_\epsilon) = 0$, we get

$$C = C_\varepsilon = \frac{2}{\rho_\varepsilon^2}\int_0^{\rho_\varepsilon}(\varepsilon^2-\Delta^2)^{1/2}\rho\, d\rho, \quad \varepsilon < \Delta_\infty.$$

Thus

$$C = \begin{cases}(\varepsilon^2-\Delta_\infty^2)^{1/2}, & \varepsilon > \Delta_\infty\\ C_\varepsilon, & \varepsilon < \Delta_\infty\end{cases}$$

To calculate the integral in (26), it is convenient to integrate first with respect to ρ by parts, and then again interchange the order of integration. We introduce the dimensionless quantities

$$\rho = \xi\tilde{\rho}, \quad \Delta = \Delta_\infty\tilde{\Delta}, \quad \varepsilon = \Delta_\infty\tilde{\varepsilon},$$

where, as seen from (4)

$$\xi = (\pi^3TD_1/7\zeta(3)\Delta_\infty^2)^{1/2}, \quad \Delta_\infty = (8\pi^2T_c(T_c-T)/7\zeta(3))^{1/2},$$

and $\tilde{\Delta}$ satisfies the equation (we recall that $\rho \ll \delta$, where δ is the field penetration depth):

$$\left(\frac{d^2}{d\tilde{\rho}^2} + \frac{1}{\tilde{\rho}}\frac{d}{d\tilde{\rho}} - \frac{1}{\tilde{\rho}^2}\right)\tilde{\Delta} + (1-\tilde{\Delta}^2)\tilde{\Delta} = 0$$

with boundary conditions $\tilde{\Delta} \to 1$, $\tilde{\rho} \to \infty$ and $\tilde{\Delta}(0) = 0$). From (26) we have

$$\mathbf{j}_{tr} = \alpha\left(\frac{\pi^3}{7\zeta(3)}\right)^2\frac{\sigma H_{c2}}{c}\frac{T}{\Delta_\infty}y^2(\tau T)[\mathbf{n}_H\times\mathbf{u}], \tag{28}$$

$$\alpha = \int_0^\infty\tilde{\rho}\, d\tilde{\rho}\left[\int_{\tilde{\Delta}}^1(\tilde{\varepsilon}^2-\tilde{\Delta}^2)\, d\tilde{\varepsilon} + \int_1^\infty[(\tilde{\varepsilon}^2-\tilde{\Delta}^2)^{1/2} - (\tilde{\varepsilon}^2-1)^{1/2}]^2\, d\tilde{\varepsilon}\right.$$
$$\left. - \frac{1}{2}\int_0^1\tilde{C}_\varepsilon^2\tilde{\rho}_\varepsilon^2\, d\varepsilon,\right.$$
$$\tilde{C}_\varepsilon = \frac{2}{\tilde{\rho}_\varepsilon^2}\int_0^{\tilde{\rho}_\varepsilon}(\tilde{\varepsilon}^2-\tilde{\Delta}^2)^{1/2}\tilde{\rho}\, d\tilde{\rho}.$$

To calculate the constant α we have approximated $\tilde{\Delta}(\tilde{\rho})$ by the function $\tilde{\rho}(1+\tilde{\rho}^2)^{-1/2}$, which has the correct asymptotic form as $\tilde{\rho} \to \infty$ and duplicates $\tilde{\Delta}(\tilde{\rho})$ quite well in the remaining range of $\tilde{\rho}$. All the integrals can then be easily calculated and yield $\alpha = 0.26$.

We see from (28) that in our case there is no velocity perpendicular to the current, i.e., there is no Hall effect in our approximation. Comparing (28) with expression (21) for the electric field, we obtain the effective conductivity of the superconductor with vortex filaments:

$$\sigma_{\text{eff}} = \alpha\left(\frac{\pi^3}{7\zeta(3)}\right)^2\frac{T}{\Delta_\infty}\sigma\frac{H_{c2}}{B_0}y^2(\tau T)$$
$$= 1{,}1\left(1-\frac{T}{T_c}\right)^{-1/2}\sigma\frac{H_{c2}}{B_0}y^2(\tau T).$$

We can also write down an expression for the viscosity coefficient η. The energy dissipation per unit volume and per vortex of unit length is $\sigma_{\text{eff}}E^2\Phi_0/B_0$. Equating it to ηu^2 we obtain

$$\eta = \alpha\left(\frac{\pi^3}{7\zeta(3)}\right)^2\frac{\sigma\Phi_0H_{c2}}{c^2}\frac{T}{\Delta_\infty}y^2(\tau T).$$

For alloys with $l \ll \xi_0$ we have, in particular $(y(\tau T) = 1)$:

$$\sigma_{\text{eff}} = \alpha\left(\frac{\pi^3}{7\zeta(3)}\right)^2\frac{T}{\Delta_\infty}\frac{\sigma H_{c2}}{B_0} = 1{,}1\left(1-\frac{T}{T_c}\right)^{-1/2}\frac{\sigma H_{c2}}{B_0}.$$

We call attention to the fact that the temperature dependence of the conductivity in ordinary alloys differs from the dependence of σ_{eff} in alloys with paramagnetic impurities, obtained earlier in [9], owing to the presence of the additional factor T/Δ_∞.[3)] The last circumstance reflects the slowness of the relaxation processes near T_c. The existence of such a temperature dependence is indicated by recent measurements of σ_{eff} near T_c [16].

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1)Since $\kappa \sim 1$ in real pure superconductors ($\tau T_c \gg 1$), the condition $\kappa \gg 1$ is actually satisfied in experiments only in the limit of "dirty" alloys, where $l \ll \xi_0$.

2)The vanishing of the normal current evidently allows us to disregard the normal energy loss at the center of the vortex [10,11].

3)A temperature dependence of the same kind can be obtained also from our results [9], by cutting off, in order of magnitude, the large parameter $\tau_s T_c$ as $\tau_s \to \infty$ at the value T/Δ (see [10]).

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