Recoil corrections in a strong nuclear field

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Corrections to atomic levels of the order of the reciprocal of the nuclear mass are considered for high nuclear charges ($Z\alpha \sim 1$). An explicit expression is found for the additional interaction and the self-interaction of electrons due to the slow motion of the nucleus. A simple closed expression is derived for corrections of the above order to the atomic levels in the lowest order in α.

1. INTRODUCTION

It is well known that, in the limit as the mass of the charged nucleus tends to infinity, its interaction with light particles (electrons) can be accurately and completely described by introducing the appropriate external potential into the usual quantum electrodynamics. In particular, when radiation corrections for a single electron are ignored, this leads to the Dirac equation with a given external potential. When the finite mass of the nucleus is taken into account, this leads to corrections which, in the lowest approximation, are of the order of $1/M$, where M is the nuclear mass. These corrections were first calculated by Salpeter[1] for the hydrogen atom. The calculation was based on the fact that the electromagnetic field of the proton was small ($Z\alpha \ll 1$), so that the analysis could be confined to the simplest Feynman diagrams for the expansion in terms of $Z\alpha$.

It would be interesting to consider the same corrections (of order $1/M$) for a strong nuclear field $Z\alpha \sim 1$. This case involves the summation of an infinite sequence of Feynman diagrams describing the interaction of the electrons with the nucleus. The evaluation of these corrections for the ordinary heavy atoms is not at present a particularly urgent problem because the corrections are, in fact, small. However, they may turn out to be substantial for more exotic objects such as, for example, mesic atoms.

In this paper we derive a closed expression for all corrections of order $1/M$ for a system consisting of a nucleus and a number of light particles (electrons) interacting with the quantized electromagnetic field. These corrections can be associated with the Hamiltonian for a nonrelativistic nucleus interacting with the electromagnetic field

$$\mathscr{H} = \frac{(\mathbf{p} + Ze\mathbf{A})^2}{2M} + \frac{Ze}{2M}\sigma\mathbf{H} \tag{1}$$

(**A** is the vector potential and **H** the magnetic field) and, accordingly, divide into four groups. The recoil corrections correspond to the term $\mathbf{p}^2/2M$, and for a nonrelativistic system of two particles they reduce to the introduction of an effective mass. Corrections for the nuclear magnetic moment correspond to the last term in Eq. (1). The remaining corrections correspond to the interaction of a spinless charged particle with the electromagnetic field, and split into a part which is linear in the field and another part which is quadratic in the field. Explicit expressions for these corrections are given by Eqs. (18), (24), (27), and (32), respectively.

2. THE LIMIT AS $M \to \infty$

In this section, which is of an auxiliary nature, we shall discuss the method of calculation, and will illustrate it by finding the limiting expressions for Green's functions when the nuclear mass tends to infinity. As a result, we shall find that the nucleus can be described by introducing an external field. As already pointed out, this is hardly a new or physically obvious result, but since our subsequent calculations will use the same methodology, there is some point in considering a simple example first.

We shall start by splitting the heavy-particle (nuclear) propagator into principle and correction terms when $M \to \infty$. Let

$$S(x) = \frac{1}{(2\pi)^4}\int d^4p \frac{1}{M - \hat{p} - i0} e^{-ipx}. \tag{2}$$

To within terms of the order of $1/M$, we can then show that

$$S(x) = \sum_{i=0}^{5} S_i(x);$$

$$\begin{aligned} S_0 &= {}^1/_2(1+\gamma_0)\sigma^{(+)}, & S_1 &= -i\mathbf{p}^2 x_0 S_0/2M; \\ S_2 &= -\mathbf{p}\gamma\sigma^{(+)}/2M, & S_3 &= {}^1/_2(1-\gamma_0)\sigma^{(-)}, \\ S_4 &= i\mathbf{p}^2 x_0 S_3/2M, & S_5 &= -\mathbf{p}\gamma\sigma^{(-)}/2M. \end{aligned} \tag{3}$$

In this expression

$$\sigma^{(\pm)}(x) = i\theta(\pm x_0)e^{\mp iMx_0}\delta^3(x) \tag{4}$$

and $\mathbf{p} = -i\nabla$, where $\boldsymbol{\gamma}$ and γ_0 are the Dirac matrices. When S(x) is substituted into the Feynman graph, transition from particles to antiparticles, i.e., a change in sign in the argument of $\exp(-iMx_0)$, leads to an additional reduction in $1/M$ because of the integration with respect to x_0. Therefore, if we associate $\exp(-iMx_0)$ with the nucleus, then in terms containing $\exp(iMx_0)$ we can neglect terms $\sim 1/M$. Consequently, S_4 and S_5 need not be taken into account.

We shall be interested in the contribution to an arbitrary Feynman graph for Green's function describing our system (nucleus plus electrons) which is due to the nuclear line. The initial and final states of the nucleus will be assumed to be given and will be respectively described by the bispinors $\psi_1(x)$ and $\bar{\psi}_2(x)$, which satisfy the free Dirac equation $(p - M)\psi_1 = 0$, and similarly for ψ_2. Expanding in terms of $1/M$, we obtain

$$\begin{aligned} \psi_1 &= \left(1 - i\frac{\mathbf{p}^2}{2M}x_0 - \frac{\mathbf{p}\gamma}{2M}\right)e^{-iMx_0}\begin{pmatrix}1\\0\end{pmatrix}\varphi_1, \\ \bar{\psi}_2 &= \varphi_2^*\begin{pmatrix}1\\0\end{pmatrix}e^{iMx_0}\left(1 + i\frac{\mathbf{p}^2}{2M}x_0 - \frac{\mathbf{p}\gamma}{2M}\right). \end{aligned} \tag{5}$$

In these expressions $\varphi_1(x)$ and $\varphi_2(x)$ are nonrelativistic wave functions (three-dimensional spinors). In the second line in Eq. (5) it is assumed that the operator $\mathbf{p}$ acts in the left direction.

Let us now consider the case $M \to \infty$ and include in the Feynman graphs only those terms which do not contain $1/M$ at all. We shall take an arbitrary Feynman graph for our system and consider only that part of it which is connected with the nuclear line (Fig. 1). The points at which the photon lines are attached to the nuclear line will be indicated by $x_1, \dots, x_n$ in the direction of motion of the nucleus. The other ends of the photon lines, which are indicated by $y_1, \dots, y_n$, are attached to electron lines. The number of electron lines, and the presence and mutual disposition of other lines which are not directly joined to the nuclear line, are of no significance for our purpose and can be quite arbitrary. Because of the projectors $\frac{1}{2}(1 + \gamma_0)$, and the absence of the lower components in the principal terms for the initial and final states in Eq. (5), only the scalar component of the electromagnetic field can be attached to the nuclear line.

Consider the Coulomb gauge for all the electromagnetic propagators connected with the nucleus. The expression

$$A = \int d^3x\varphi_2^*(\mathbf{x})\,V(\mathbf{x}-\mathbf{y}_n)\,V(\mathbf{x}-\mathbf{y}_{n-1})\dots V(\mathbf{x}-\mathbf{y}_1) \times\varphi_1(\mathbf{x})\,\theta(y_{n0}-y_{n-1,0})\,\theta(y_{n-1,0}-y_{n-2,0})\dots\theta(y_{20}-y_{10}). \qquad (6)$$

can then be associated with the part of the graph shown in Fig. 1, where $V(x) = Ze/4\pi|\mathbf{x}|$ is the Coulomb potential of the nucleus.

Let us now consider further graphs which differ from Eq. (1) only by the distribution of the x points along the nuclear line. The y points and the remainder of the diagram are fixed (Fig. 2). The matrix element for Fig. 2 will differ from Eq. (6) only by the θ functions and the first factor will be common. Let us take the sum of all the graphs of this type. The result is a product of the θ functions and is equal to unity. In fact, the θ functions restrict the region of the integration of the variables y_0, arranging them in a definite order corresponding to the order of the x points along the nuclear line. However, when all the possible sequences of the x points have been taken into account, it is clear that all the y_0 will run over the entire range between $-\infty$ and $+\infty$, i.e., there will be no θ functions left.

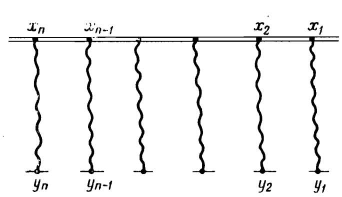

FIG. 1

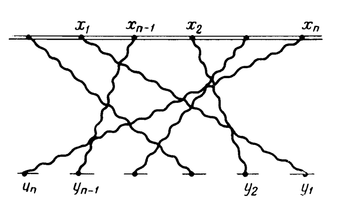

FIG. 2

Thus, after summation over all the possible sequences of the x points along the nuclear line, the part of the Green function connected with it is found to be

$$A_t = \int d^3x\varphi_2^*(\mathbf{x})\,V(\mathbf{x}-\mathbf{y}_n)\dots V(\mathbf{x}-\mathbf{y}_1)\,\varphi_1(\mathbf{x}). \qquad (7)$$

If we now take $\varphi_2(\mathbf{x}) = \varphi_1(\mathbf{x})$ and $|\varphi_1(\mathbf{x})|^2 = \delta^3(x)$, which corresponds to the nucleus at the origin, we obtain

$$A_t = V(\mathbf{y}_n)\,\dots\,V(\mathbf{y}_1). \qquad (8)$$

Hence, it is clear that, in our approximation ($M \to \infty$), the inclusion of the interaction with the nucleus is equivalent to the introduction of an external Coulomb field V into the quantum electrodynamics of the electron. This can be verified by direct comparison of the contributions due to the corresponding Feynman diagram.

3. EFFECT OF THE RECOIL

We shall now consider the corrections for the finite mass of the nucleus which are of the order of $1/M$. It is clear at the outset that we need not take into account any graphs in which the ends of the photon lines lie on the nuclear line. All these are of order not less than the order of $1/M^2$. Therefore, all the remaining graphs are topologically the same as Figs. 1 and 2, but instead of one of the nuclear lines we substitute $S_i(x)$, $i = 1, 2, 3$ [see Eq. (3)] or, instead of the outer ends, we substitute terms $\sim 1/M$ from Eq. (5).

Recoils correspond to terms containing the operator $x_0\mathbf{p}^2/2M$, which describes the kinetic energy of the slow nucleus, i.e., the propagator S_1 and the second terms $\sim 1/M$ in Eq. (5).

The presence of the propagator $x_0\mathbf{p}^2/2M$ again ensures that all the photons interacting with the nucleus are Coulomb photons. Assuming that in the graph of Fig. 1 the propagator S_1 is introduced between x_k and x_{k+1}, we obtain the following expression for the corresponding contribution:

$$B_k = \int d^3x\varphi_2^*(\mathbf{x})\,V(\mathbf{x}-\mathbf{y}_n)\dots V(\mathbf{x}-\mathbf{y}_{k+1})\left(-i\frac{\mathbf{p}^2}{2M}\right)V(\mathbf{x}-\mathbf{y}_k)\dots \dots V(\mathbf{x}-\mathbf{y}_1)\,\varphi_1(\mathbf{x})\,(y_{k+1,0}-y_{k0})\,\theta(y_{n0}-y_{n-1,0})\dots\theta(y_{20}-y_{10}). \qquad (9)$$

In compact notation

$$B_k = \langle 2|V_n\dots V_{k+1}(-i\mathbf{p}^2/2M)\,V_k\dots V_1|1\rangle \times(y_{k+1,0}-y_{k0})\,\theta(y_{n0}-y_{n-1,0})\,\dots\,\theta(y_{20}-y_{10}), \qquad (10)$$

where $\langle 2|\dots|1\rangle$ represents the matrix element between the nuclear nonrelativistic wave functions, and $V_k \equiv V(\mathbf{x}-\mathbf{y}_k)$.

We shall start by considering the sum of the contributions of graphs of the kind shown in Fig. 1 with fixed x and y, which differ only by the point at which the propagator S_1 is introduced along the nuclear line. In other words, we shall consider the sum of the B_k over k between $k = 1$ and $k = n - 1$. Collecting together terms with the same factors y_{k0} in this sum, we have

$$B' = \sum_{k=1}^{n-1} B_k = -\frac{i}{2M}\,\theta(y_{n0}-y_{n-1,0})\dots\theta(y_{20}-y_{10})\,\langle 2|V_n\mathbf{p}^2V_{n-1}\dots V_1 y_{n0} - V_n\dots V_2\mathbf{p}^2V_1 y_{10} + \sum_{k=2}^{n-1} y_{k0}V_n\dots V_{k+1}[V_k, \mathbf{p}^2]\,V_{k-1}\dots V_1|1\rangle. \qquad (11)$$

It is now readily shown that when the recoil corrections at the outer ends (i.e., terms containing $\pm ix_0\mathbf{p}^2/2M$ in Eq. (5)) are taken into account, this reduces the first two terms in the matrix element in Eq. (11) to the same structure as is present under the summation sign between k = 2 and k = n − 1. The final expression for the recoil correction for a graph of the type shown in Fig. 1 is

$$B=-\frac{i}{2M}\theta(y_{n0}-y_{n-1,0})\dots\theta(y_{20}-y_{10})$$
$$\times\sum_{k=1}^{n}y_{k0}\langle 2|V_n\dots V_{k+1}[V_k,\mathbf{p}^2]V_{k-1}\dots V_1|1\rangle. \qquad (12)$$

This expression is not yet in the most convenient form for our purposes because the operator $[V_k, \mathbf{p}^2]$ does not commute with V_l and, therefore, summation of graphs such as those shown in Figs. 1 and 2 cannot be carried out. We must therefore transform the matrix element. We shall write

$$[V_k,\mathbf{p}^2]=\mathbf{p}[V_k,\mathbf{p}]+[V_k,\mathbf{p}]\mathbf{p}$$

and transfer the operator **p** in the first term to the left, and in the second to the right. It then turns out that

$$\langle 2|V_n\dots V_{k+1}[V_k,\mathbf{p}^2]V_{k-1}\dots V_1|1\rangle$$
$$=\sum_{\substack{l=1\\ l\neq k}}^{n}\langle 2|V_n\dots[V_l,\mathbf{p}]\dots[V_k,\mathbf{p}]\dots V_1|1\rangle\operatorname{sign}(l-k)$$
$$+\langle 2|V_n\dots[V_k,\mathbf{p}]\dots V_1\mathbf{p}|1\rangle+\langle 2|\mathbf{p}V_n\dots[V_k,\mathbf{p}]\dots V_1|1\rangle. \qquad (13)$$

If $\varphi_2^*=\varphi_1$, which we shall assume to be valid, the last two terms in Eq. (13) add up to zero. In fact, substituting $\mathbf{X}=V_n\dots[V_k,\mathbf{p}]\dots V_1$, we have

$$\int d^3x\varphi_2^*(x)\mathbf{p}\mathbf{X}\varphi_1(x)=-i\int d^3x\varphi_1(x)\nabla(\mathbf{X}\varphi_1(x))$$
$$=i\int d^3x\mathbf{X}\varphi_1(x)\nabla\varphi_1(x)=-\int d^3x\varphi_2^*(x)\mathbf{X}\mathbf{p}\varphi_1(x). \qquad (14)$$

Using Eq. (13), we can write the sum over k in Eq. (12) in the form of the following double sum:

$$\sum_{k=2}^{n}\sum_{l=1}^{k-1}(y_{l0}-y_{k0})\langle 2|V_n\dots[V_l,\mathbf{p}]\dots[V_k,\mathbf{p}]\dots V_1|1\rangle. \qquad (15)$$

Since $y_{l_0}-y_{k_0}$ for $l<k$ in Eq. (12), we can rewrite Eq. (15) in the form

$$-\frac{1}{2}\sum_{k\neq l}|y_{l0}-y_{k0}|\langle 2|V_n\dots[V_l,\mathbf{p}]\dots[V_k,\mathbf{p}]\dots V_1|1\rangle. \qquad (16)$$

This expression does not now depend on the sign of the difference between the time components of the variables y (or the equal time components of x) and, therefore, this expression is general for all diagrams of the type shown in Figs. 1 and 2. We can therefore consider the sum of all such graphs. As in the case of the zero-order in 1/M, the sum of all the products of the θ functions in the end turns out to be equal to unity. If we fix the nucleus at the origin, we obtain the following final expression for the contribution of the nuclear line in Figs. 1 and 2, which take into account the recoil:

$$\frac{i}{4M}\sum_{k\neq l}|y_{k0}-y_{l0}|V(\mathbf{y}_n)\dots[V(\mathbf{y}_k),\mathbf{p}]\dots[V(\mathbf{y}_l),\mathbf{p}]\dots V(\mathbf{y}_1). \qquad (17)$$

This corresponds to the appearance in the diagrams of a new vector boson line connecting the electron lines. If the corresponding propagator is denoted by $B_{\alpha\beta}(y_1, y_0)$, then we find from Eq. (17) that the only nonzero component is B_{00}, where

$$B_{00}(y_1,y_2)=\frac{i}{2M}|y_{10}-y_{20}|[V(\mathbf{y}_1),\mathbf{p}][V(\mathbf{y}_2),\mathbf{p}]. \qquad (18)$$

The formula given by Eq. (18) is the final expression for calculating the recoil corrections. The recoil corrections to the Green function are obtained by taking into account all the possible Feynman diagrams containing

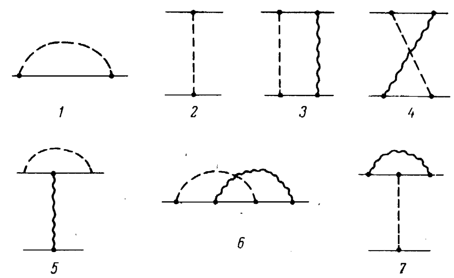

FIG. 3

one line with the propagator B, which can be attached to the same or different electron lines, and any number of other lines. Of course, the contribution due to the interaction with the nucleus in the zero order in 1/M can then be taken into account by introducing the external Coulomb potential, so that the electron propagators must be taken in the external Coulomb field (Furry representation). A series of simplest diagrams representing recoil is shown in Fig. 3. The new line with the propagator (18) is shown by the dashed lines. Diagrams 1 and 2 in Fig. 3 allow for recoil in the lowest order in α but arbitrary order in Zα. It will be shown below that the contribution of these diagrams can be represented in another way, and it will then be immediately clear that in the nonrelativistic limit for a single particle these corrections reduce to the introduction of an effective mass. Diagrams 3, 4, and 5 represent the superposition of corrections for recoil and for electron interaction one on another. Diagrams 6 and 7 show the superposition of radiative corrections onto the recoil corrections.

4. CORRECTIONS DUE TO TERMS IN THE INTERACTION BETWEEN THE NUCLEUS AND THE ELECTROMAGNETIC FIELD WHICH ARE LINEAR IN THE FIELD

We shall now use the same method as in the last section to consider, in general, the corrections which are due to the introduction of a single propagator S_2 into the nuclear line, and those due to terms with $\mathbf{p}\cdot\boldsymbol{\gamma}$ at the outer ends. Let us return to Fig. 1 and suppose now that the propagator S_2 is introduced between the points x_k and x_{k+1}, while the remaining nuclear propagators are taken in the form of S_0. A transverse photon should then be emitted in one of the vertices, x_k or x_{k+1}. The combined contribution of two such graphs is

$$C_k=-\frac{Ze}{2M}\int d^4x\varphi_2^*(\mathbf{x})V(\mathbf{x}-\mathbf{y}_n)\dots V(\mathbf{x}-\mathbf{y}_{k+2})$$
$$\times\{D(x-z)\gamma^s\mathbf{p}\boldsymbol{\gamma}V(\mathbf{x}-\mathbf{y}_k)\theta(y_{k+2,0}-x_0)\theta(x_0-y_{k0})\theta(y_{k0}-y_{k-1,0})$$
$$+V(\mathbf{x}-\mathbf{y}_{k+1})\mathbf{p}\boldsymbol{\gamma}\gamma^s D(x-z)\theta(y_{k+2,0}-y_{k+1,0})\theta(y_{k+1,0}-x_0)\theta(x_0-y_{k-1,0})\}$$
$$\times V(\mathbf{x}-\mathbf{y}_{k-1})\dots V(\mathbf{x}-\mathbf{y}_1)\varphi_1(\mathbf{x})\prod_{i\neq k-1,\,k,\,k+1}\theta(y_{i+1,0}-y_{i0}). \qquad (19)$$

where D(x − z) is the electromagnetic field propagator corresponding to the transverse photon, and s = 1, 2, 3 is the vector index of the photon. Since the vertex on the electron line, which is connected with the transverse photon, is of a special character, we have denoted its coordinate by z.

Let us first sum the graphs in Fig. 1 with different positions of S_2 along the nuclear line, i.e., let us sum Eq. (19) between k = 1 and n − 1. In the resulting expression we shall combine in pairs terms in which the transverse photons are emitted from the same vertex. This will isolate two terms corresponding to emission from the extreme vertices. These are then combined with contributions due to the correction terms originating from the initial and final states and containing $\mathbf{p}\cdot\gamma$ [see Eq. (5)]. In the final expression for C all the vertices have equal weights:

$$C=-\frac{Ze}{2M}\sum_{k=1}^{n}\int dx_0\langle 2|V_n\ldots V_{k+1}(\mathbf{p}\gamma\gamma^s D(x_0)+D(x_0)\gamma^s\mathbf{p}\gamma)V_{k-1}\ldots V_1|1\rangle$$
$$\times\theta(y_{n0}-y_{n-1,0})\ldots\theta(y_{k+1,0}-x_0)\theta(x_0-y_{k-1,0})\ldots\theta(y_{20}-y_{10}). \quad (20)$$

where $D(x_0)\equiv D(x-z)$. Let us now substitute

$$\gamma\gamma^s={}^1/_2\{\gamma,\gamma^s\}+{}^1/_2[\gamma,\gamma^s]$$

and accordingly split C into two terms, C_1 and C_2. We shall begin with C_1 which will include the matrix element between the nuclear wave functions

$$\langle 2|V_n\ldots V_{k+1}\{p^s,D(x_0)\}V_{k-1}\ldots V_1|1\rangle. \quad (21)$$

We shall transform it by analogy with the transformation of the matrix element containing $(V,\mathbf{p}^2)$ in the last section [see Eq. (13)], in which case

$$C_1=-\frac{Ze}{2M}\sum_{l\neq k}\int dx_0\,\mathrm{sign}(y_{l0}-x_0)\langle 2|V_n\ldots D(x_0)\ldots[V_l,p^s]\ldots V_1|1\rangle$$
$$\times\theta(y_{n0}-y_{n-1,0})\ldots\theta(y_{k+1,0}-x_0)\theta(x_0-y_{k-1,0})\ldots\theta(y_{20}-y_{10}). \quad (22)$$

We shall now select one of the terms in the double sum over k and l, and fix the corresponding vertices x_k and x_l in the graph of Fig. 1. We next consider all other graphs with transposed vertices of the type shown in Fig. 2, but subject to the condition that x_k and x_l are fixed. Then after summing over all such graphs, the sum of the products of the θ functions will be a θ function which restricts the region of integration for the variables x_0 and y_{l0} corresponding to fixed vertices. If $l>k$, we have $\theta(y_{l0}-x_0)$, and if $l<k$ we have $\theta(x_0-y_{l0})$. Combining these two groups of terms and placing the nucleus at the origin, we obtain the following final expression for the contribution C_{1t} due to the graphs in Figs. 1 and 2:

$$C_{1t}=-\frac{Ze}{2M}\sum_{l\neq k}\int dx_0\,\mathrm{sign}(y_{l0}-x_0)D(\mathbf{z},z_0-x_0)[V(\mathbf{y}_l),p^s]\prod_{i\neq k,l}V(\mathbf{y}_i). \quad (23)$$

This corresponds to the appearance on the graphs of a line joining the electron lines. Its propagator $C_{\alpha\beta}(y_1,y_2)$ has nonzero components C_{0S} and C_{S0} (s = 1, 2, 3), where

$$C_{s0}(y_1,y_2)=-\frac{Ze}{2M}\int dx_0\,\mathrm{sign}(y_{20}-x_0)D(\mathbf{y}_1,y_{10}-x_0)[V(\mathbf{y}_2),p^s], \quad (24)$$
$$C_{0s}(y_1,y_2)=C_{s0}(y_2,y_1).$$

Let us now consider the second part of the contribution C_2, which is due to the commutator of the matrices γ. It includes the matrix element

$$-i\langle 2|V_n\ldots V_{k+1}[[\sigma\mathbf{p}],D(x_0)]V_{k-1}\ldots V_1|1\rangle. \quad (25)$$

The commutator is an ordinary function in configuration space, and commutes with all the potentials V_k. If we now sum C_2 over all the graphs in Figs. 1 and 2, we find that the θ functions in Eq. (20) will disappear. The remaining expression for the nucleus at the origin is

$$\frac{iZe}{2M}\sum_{k=1}^{n}\int dx_0[[\sigma\mathbf{p}],D(\mathbf{z},x_0)]\prod_{i\neq k}V(\mathbf{y}_i). \quad (26)$$

This contribution corresponds to an additional external vector field generated by the nucleus independently of time, which is equal to

$$\frac{Ze}{2M}\int dx_0[\sigma\nabla D(\mathbf{z},x_0)].$$

This obviously describes the interaction with the magnetic moment of the nucleus.

5. CORRECTIONS DUE TO TERMS IN THE INTERACTION BETWEEN THE NUCLEUS AND THE ELECTROMAGNETIC FIELD WHICH ARE QUADRATIC IN THE FIELD

In this section we consider terms due to the propagator S_3. Suppose that in Fig. 1 the operator is introduced between x_k and x_{k+1}. Transverse photons should therefore be emitted by these points. The corresponding vertices on the electron lines will be denoted by z_1 and z_2. The contribution of such a graph is

$$F_k=-(Ze)^2\int dx_{k+1,0}dx_{k0}\langle 2|(\delta_{sl}+ie_{slr}\sigma_r)V_n\ldots V_{k+2}D_2(x_{k+1,0})D_1(x_{k0})\cdot$$
$$\cdot V_{k-1}\ldots V_1|1\rangle\exp\{2iM(x_{k+1,0}-x_{k0})\}\theta(y_{n0}-y_{n-1,0})\ldots\theta(y_{k+2,0}-x_{k+1,0})\cdot$$
$$\cdot\theta(x_{k0}-x_{k+1,0})\theta(x_{k0}-y_{k-1,0})\ldots\theta(y_{20}-y_{10}). \quad (27)$$

where, for example, $D_2(x_{k+1,0})\equiv D(\mathbf{x}_{k+1}-\mathbf{z}_2)$. Let us consider separately the integral with respect to the time variables $x_{k+1,0}$ and x_{k0}. Substituting $x_0={}^1/_2(x_{k+1,0}+x_{k0})$ and $\xi_0=x_{k+1,0}-x_{k0}$, we find that the integral with respect to ξ_0 is

$$I=\int_{-\infty}^{+\infty}d\xi_0\theta(-\xi_0)e^{2iM\xi_0}f(\xi_0), \quad (28)$$

where $f(\xi_0)$ is a function which decreases as $\xi_0\to\infty$. When $M\to\infty$ we have, to within terms of the order of $1/M$,

$$I=f(0)/2iM. \quad (29)$$

Therefore, to the same accuracy we can rewrite Eq. (27) in the form

$$F_k=-\frac{(Ze)^2}{2iM}\int dx_0\langle 2|(\delta_{sl}+ie_{slr}\sigma_r)V_n\ldots V_{k+2}D_2(x_0)D_1(x_0)V_{k-1}\ldots V_1|1\rangle$$
$$\times\theta(y_{n0}-y_{n-1,0})\ldots\theta(y_{k+2,0}-x_0)\theta(x_0-y_{k-1,0})\ldots\theta(y_{20}-y_{10}). \quad (30)$$

Let us now again sum over all graphs of the type given in Figs. 1 and 2 with different disposition of the x points on the nuclear line, having fixed x_k and x_{k+1}. As in all the preceding cases, the sum of the θ functions will be equal to unity. If we further add graphs with interchanged x_{k+1} and x_k, this will reduce to the interchange of vector indices $s\rightleftarrows i$. Finally, we have the contribution

$$F_{kt}=i\frac{(Ze)^2}{M}\delta_{sl}\int dx_0D(\mathbf{z}_1,x_0-z_{10})D(\mathbf{z}_2,x_0-z_{20})\prod_{i\neq k,\,k+1}V(\mathbf{y}_i). \quad (31)$$

This is associated with the additional line joined to the electron lines which correspond to the vector particle with propagator $F_{\alpha\beta}$ having nonzero components F_{sl}, s, l = 1, 2, 3. We then have

$$F_{sl}(z_1,z_2)=i\frac{(Ze)^2}{M}\delta_{sl}\int dx_0D(\mathbf{z}_1,x_0-z_{10})D(\mathbf{z}_2,x_0-z_{20}). \quad (32)$$

6. RECOIL CORRECTION IN THE LOWEST ORDER ORDER IN α

In this section we shall consider corrections in the lowest order in α, i.e., without taking into account the interaction between electrons or the radiation corrections. This corresponds to graphs 1 and 2 in Fig. 3. We shall see that these corrections can be written in the form of an expression which corresponds to the Hamiltonian (1) for the nucleus interacting with the electromagnetic field. In particular, the recoil correction in the nonrelativistic limit for a single particle reduces to the well-known effective-mass correction. To be specific, we shall consider corrections to the energy levels of bound electrons. In our approximation we shall neglect the interaction between the electrons and, therefore, the level shift will be obtained directly from the graphs in Fig. 3 by replacing the outer electron propagators by the corresponding electron wave functions (Dirac functions).

For graph 1 in Fig. 3, the level shift in the n-th electron state is

$$\Delta E_n = -\frac{e^2}{2\pi}\int d^3x_1\, d^3x_2\, d\omega \bar{\psi}_n(\mathbf{x}_1)\gamma_\alpha S(\omega, \mathbf{x}_1, \mathbf{x}_2)\gamma_\beta\psi_n(\mathbf{x}_2)P^{\alpha\beta}(\varepsilon_n - \omega, \mathbf{x}_1, \mathbf{x}_2). \quad (33)$$

where ψ_n and $\bar{\psi}_n$ are the electron wave functions satisfying the Dirac equation in the external Coulomb field of the nucleus for energy ϵ_n, $S(\omega, \mathbf{x}_1, \mathbf{x}_2)$ is the electron Green function in this external field for energy ω, and $P^{\alpha\beta}$ is the propagator which describes the recoil correction which in configuration space is given by Eqs. (18), (24), and (32).

For graph 2 in Fig. 3, the contribution to the level shift is

$$\Delta E_{n_1n_2} = -ie^2\int d^3x_1\, d^3x_2\, \{(\bar{\psi}_{n_1}(\mathbf{x}_1)\gamma_\alpha\psi_{n_1}(\mathbf{x}_1))(\bar{\psi}_{n_2}(\mathbf{x}_2)\gamma_\beta\psi_{n_2}(\mathbf{x}_2))P^{\alpha\beta}(0, \mathbf{x}_1, \mathbf{x}_2) - (\bar{\psi}_{n_1}(\mathbf{x}_1)\gamma_\alpha\psi_{n_2}(\mathbf{x}_1))(\bar{\psi}_{n_2}(\mathbf{x}_2)\gamma_\beta\psi_{n_1}(\mathbf{x}_2))P^{\alpha\beta}(\varepsilon_{n_1} - \varepsilon_{n_2}, \mathbf{x}_1, \mathbf{x}_2)\}. \quad (34)$$

We begin with the recoil corrections. For given energy, the propagator B is given by

$$B_{00}(\omega, \mathbf{x}_1, \mathbf{x}_2) = -\frac{i}{2M}[V(\mathbf{x}_1), \mathbf{p}][V(\mathbf{x}_2), \mathbf{p}]\left(\frac{1}{(\omega + i0)^2} + \frac{1}{(\omega - i0)^2}\right). \quad (35)$$

Substituting this in Eq. (33), we find the corresponding energy shift after integrating with respect to ω:

$$\Delta E_n^{(B)} = -\frac{e^2}{2M}\langle n|[V, \mathbf{p}](\mathscr{H} - \varepsilon_n)^{-1}\Lambda(\mathscr{H} - \varepsilon_n)^{-1}[V, \mathbf{p}]|n\rangle. \quad (36)$$

where $\mathscr{H}$ is the Hamiltonian for the Dirac equation in the Coulomb field of the nucleus, and the averaging is carried out over the state of the electron with wave function ψ_n. The operator $\Lambda = \text{sign}\,\mathscr{H}$ is equal to $+1$ for states with positive energy and -1 for states with negative energy.

Since

$$-e[V, \mathbf{p}] = [\mathscr{H} - \varepsilon_n, \mathbf{p}], \quad (37)$$

the recoil correction for graph 1 in Fig. 3 has the following final form:

$$\Delta E_n^{(B)} = {}^1\!/_2 M^{-1}\langle n|\mathbf{p}\Lambda\mathbf{p}|n\rangle.$$

For one electron in the field of the nucleus this expression exhausts all the corrections for the recoil. As can be seen, it is not simply reduced to a term of the form $\mathbf{p}^2/2M$ describing the kinetic energy of a nonrelativistic nucleus in the center of mass system and involving the effective mass correction. In fact, we have the sign operator Λ, which is due to the transition from electrons with negative energy to positrons with positive energy in the Feynman theory. The difference between Λ and unity can be readily taken into account in the case of a weak field ($Z\alpha \ll 1$), and this then leads to the correction calculated by Salpeter.[1]

The recoil corrections due to graph 2 in Fig. 3 can be found from Eq. (34). The first term in this is equal to zero because the matrix element $\langle n|[V, \mathbf{p}]|n\rangle = 0$. There remains the second (volume) term which, when Eq. (37) is taken into account, turns out to be

$$\Delta E_{n_1n_2}^{(B)} = M^{-1}\langle n_1n_2|\mathbf{p}^{(1)}\mathbf{p}^{(2)}|n_2n_1\rangle. \quad (38)$$

The remaining corrections can be found in a similar way. The propagator C_{s0} for given energy is given by

$$C_{s0}(\omega, \mathbf{x}_1, \mathbf{x}_2) = -\frac{iZe}{2M}\frac{\cos\omega|\mathbf{x}_1|}{4\pi|\mathbf{x}_1|}\frac{2P}{\omega}[V(\mathbf{x}_2), p^s], \quad (39)$$

where P represents the principal value. If we substitute this expression in Eqs. (33) and (34), we obtain, respectively, the level shifts for graphs 1 and 2 in Fig. 3:

$$\Delta E_n^{(C)} = \frac{Ze^2}{2M}\langle n|\mathbf{a}\Lambda\mathbf{p} + \mathbf{p}\Lambda\mathbf{a} + [\mathbf{b}, \mathbf{p}]|n\rangle, \quad (40)$$

$$\Delta E_{n_1n_2}^{(C)} = \frac{Ze^2}{2M}\langle n_1n_2|\mathbf{a}^{(1)}\mathbf{p}^{(2)} + \mathbf{p}^{(1)}\mathbf{a}^{(2)}|n_2n_1\rangle. \quad (41)$$

We have introduced the Hermitian operators **a** and **b** for each electron with matrix elements

$$\langle n|\mathbf{a}|m\rangle = \left\langle n\left|\boldsymbol{\alpha}\frac{\cos(\varepsilon_n - \varepsilon_m)|\mathbf{x}|}{4\pi|\mathbf{x}|}\right|m\right\rangle, \quad (42)$$

$$\langle n|\mathbf{b}|m\rangle = i\left\langle n\left|\boldsymbol{\alpha}\frac{\sin(\varepsilon_n - \varepsilon_m)|\mathbf{x}|}{4\pi|\mathbf{x}|}\right|m\right\rangle, \quad (43)$$

where $\boldsymbol{\alpha}$ represents the Dirac matrices. The operator **a** obviously has the significance of the operator representing the electromagnetic field due to an electron at the origin. In Eq. (40) the last term describes the imaginary correction to the mass, which corresponds to the instability of the excited levels in the field of the nucleus when retarded effects are taken into account.

Finally, the propagator F_{sl} for given energy is

$$F_{sl}(\omega, \mathbf{x}_1, \mathbf{x}_2) = i\frac{(Ze)^2}{M}\delta_{sl}\frac{\cos\omega|\mathbf{x}_1|\cos\omega|\mathbf{x}_2|}{16\pi^2|\mathbf{x}_1||\mathbf{x}_2|}. \quad (44)$$

Substituting this in Eqs. (33) and (34), we obtain the expressions for the corresponding energy shifts. These are given below without including the imaginary part which we have not succeeded in writing in a compact form:

$$\text{Re}\,\Delta E_n^{(F)} = \frac{Z^2e^4}{2M}\langle n|\mathbf{a}\Lambda\mathbf{a}|n\rangle, \quad (45)$$

$$\Delta E_{n_1n_2}^{(F)} = \frac{Z^2e^4}{M}(\langle n_1n_2|\mathbf{a}^{(1)}\mathbf{a}^{(2)}|n_1n_2\rangle - \langle n_1n_2|\mathbf{a}^{(1)}\mathbf{a}^{(2)}|n_2n_1\rangle. \quad (46)$$

If we compare all these expressions for the corrections, we find that in a many-electron system the total level shift in the first order in $1/M$ and the lowest order in α, due to graphs 1 and 2 in Fig. 3, can be written in the compact form

$$\text{Re}\,\Delta E = {}^1\!/_2 M^{-1}\langle(\mathbf{P} + Ze^2\mathbf{A})\Lambda(\mathbf{P} + Ze^2\mathbf{A})\rangle. \quad (47)$$

where $\mathbf{P} = \Sigma\cdot\mathbf{p}^{(i)}$ is the total momentum of all the electrons, $\mathbf{A} = \Sigma\cdot\mathbf{a}^{(i)}$ is the resultant operator corresponding to Eq. (42) for all the electrons, and $\Lambda = \prod_i \Lambda^{(i)}$. The angle brackets represent averaging over the given state of the many-electron system in the approximation in which the interaction between the electrons is not taken into account. Partial allowance for this interaction, i.e., allowance for graphs 3, 4, and 5 in Fig. 3, can be achieved by averaging over the state in Eq. (47), using the Hartree-Fock approximation. There is a clear complete correspondence between Eq. (47) and the first terms in the Hamiltonian given by Eq. (1). The operator $e\mathbf{A}$ is the resultant vector potential produced by the electrons at the point at which the nucleus is located. The appearance of the sign operator Λ is not at all trivial. As already noted, this operator represents the specific features of the description of negative levels of the Dirac equation in the Feynman picture.

[1] E. E. Salpeter, Phys. Rev. **87**, 328 (1952).

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