Quantum vortices and phase transitions in Bose systems

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The continual integral formalism for describing quantum vortices is illustrated by applying it to a two-dimensional Bose system. It is shown that a two-dimensional system of phonons and vortices is equivalent to relativistic electrodynamics. The phase transition from the superfluid to the normal state is connected with the dissociation of coupled pairs of vortices of opposite sign. A similar approach to three-dimensional Bose systems leads to the conclusion that here the phase transition to the normal state is accompanied by the appearance of long vortex filaments.

1. In the present paper we propose a method for describing quantum vortices in Bose systems and discuss their effect on the phase transition to the superfluid state. The method is described in Secs. 2 and 3 with the two-dimensional model for a Bose gas as an example. The possibility of superfluidity in this model has been validated by different methods in a number of papers [1-4]. Qualitative conclusions for the three-dimensional systems are drawn in Sec. 4, and general conclusions in Sec. 5.

The proposed method is based on the continual integral. It allows us to study both the static and dynamical properties of the system and is convenient, in particular, for the computation of the excitation energy spectrum. An important distinctive feature of the developed procedure is integration first over the fast variables and then over the slowly varying fields (the functions ψ and $\bar{\psi}$), using a different perturbation-theory scheme at each of the two stages. In the integration over the slowly varying fields we go over to an integral over the density $\rho = |\psi|^2$ and phase φ, which is a function of ψ, as well as over the coordinates of the zeroes of the functions ψ and $\bar{\psi}$ corresponding in the two dimensional case to the cores of the quantum vortices. The variables ρ and φ are convenient for describing the phonon excitations.

Neglecting the dispersion in the phonon spectrum and neglecting the phonon-phonon interaction, the two-dimentional phonon-vortex system turns out to be equivalent to two-dimensional relativistic electrodynamics. Phonons play the role of photons and vortices the role of charged particles.

Vortices in a two dimensional Bose system exist at low temperatures in the form of pairs of opposite sign coupled by a long-range logarithmic potential. As the temperature increases the number of pairs increases, while the mean distance between them decreases. The phase transition from the superfluid to the normal state reduces to a dissociation of the coupled pairs. Above the transition we deal with a plasma-type system. The characteristic (for a plasma) Debye screening leads to the disappearance of the long-range correlations.

The low-frequency second sound (light in the electromagnetic analog) cannot propagate in the plasma state. The corresponding branch of the energy spectrum goes over into the plasma-oscillation branch. The description of the system in terms of phonons and vortices above the transition temperature is suitable as long as the radius of the vortex core is not comparable with the mean intervortex distance.

In the three-dimensional Bose system the analog of the coupled pairs is the vortex rings and the analog of the isolated vortices are long vortex filaments that originate from and terminate on the walls of the vessel. The phase transition from the superfluid to the normal state is connected with the appearance of long vortex filaments.

2. The method expounded below for describing quantum vortices in the continual-integral formalism is in the framework of the scheme worked out previously [5,6] for the construction of the low-frequency asymptotic form of the Green function and the kinetic equations in the microscopic procedure.

Continual integrals for Bose systems are evaluated over the space of the complex functions $\psi(\mathbf{x}, \tau)$ and $\bar{\psi}(\mathbf{x}, \tau)$ defined in the volume V and periodic in the "time" τ with the period $\beta = (k_B T)^{-1}$.[1)] The Green functions are defined as the products of the fields ψ and $\bar{\psi}$ averaged over the indicated space with the weight e^S, where

$$S = \int_0^\beta d\tau \int dx\, \bar{\psi}(x, \tau)\, \partial_\tau \psi(x, \tau) - \int_0^\beta H'(\tau)\, d\tau \tag{2.1}$$

is the action and $H'(\tau)$ is the Hamiltonian

$$\begin{aligned} H'(\tau) = \int dx \left(\frac{1}{2m} \nabla \bar{\psi}(x, \tau) \nabla \psi(x, \tau) - \lambda \bar{\psi}(x, \tau) \psi(x, \tau) \right) \\ + \frac{1}{2} \int dx\, dy\, U(x-y)\, \bar{\psi}(x, \tau) \bar{\psi}(y, \tau) \psi(y, \tau) \psi(x, \tau). \end{aligned} \tag{2.2}$$

Thus, the Green functions to be determined coincide (possibly apart from the sign) with the usual temperature Green functions [7].

We shall use the procedure developed previously in [5,6] for computing the Green functions at low energies and momenta. The main idea consists in first integrating over the fast variables and then over the slowly-varying fields, using different perturbation-theory schemes at each of the two stages. The slowly varying part $\psi_0(\mathbf{x}, \tau)$ of the field $\psi(\mathbf{x}, \tau)$ is, by our definition, the sum of the terms in the Fourier-series expansion of ψ [2)]

$$\psi(x, \tau) = (\beta V)^{-1/2} \sum_{\mathbf{k}, \omega} e^{-i(\mathbf{k}x - \omega\tau)} a(\mathbf{k}, \omega) \tag{2.3}$$

with momenta $\mathbf{k}$ smaller than some k_0; the rapidly varying part ψ_1 is defined as $\psi - \psi_0$.

The integral of e^S over the rapidly varying fields

$$\int e^S\, d\bar{\psi}_1\, d\psi_1 = \exp S_0[\bar{\psi}_0, \psi_0] \tag{2.4}$$

can be evaluated as a statistical sum for a system of "fast" particles in the slowly varying field ψ_0. The functional S_0 has the meaning of a "hydrodynamic action"[6]. Knowing it, we can easily find the system's hydrodynamic Hamiltonian which determines the low-frequency energy spectrum.

After computing the integral (2.4) we should further integrate over the fields ψ_0 and $\bar{\psi}_0$. Let us go over in the integral over ψ_0 and $\bar{\psi}_0$ to the variables ρ and φ:

$$\psi_0 = \sqrt{\rho}\, e^{i\varphi}, \quad \bar{\psi}_0 = \sqrt{\rho}\, e^{-i\varphi}. \tag{2.5}$$

These variables are convenient for describing the phonon excitations. We shall write down the hydrodynamic action S_0 precisely in terms of these variables. For the two-dimensional model of a low-density Bose gas the expression for S_0 has been computed in [4].

In the limit as $T \to 0$ it has the form

$$\int d\tau\, d^2\mathbf{x} \left(-\frac{p_\lambda}{2m}(\nabla\varphi)^2 - \frac{1}{2} p_{\lambda\lambda}(\partial_\tau\varphi)^2 + i p_{\lambda\rho_0}\pi\partial_\tau\varphi + \frac{1}{2} p_{\rho_0\rho_0}\pi^2 - \frac{(\nabla\pi)^2}{8m\rho_0} - \frac{\pi(\nabla\varphi)^2}{2m} \right). \tag{2.6}$$

Here $\pi(\mathbf{x}, \tau) = \rho(\mathbf{x}, \tau) - \rho_0$, where ρ_0 is the density of the condensate at $T = 0$ and $p_\lambda = \bar{\rho}$ is the total density at $T = 0$. The coefficients $p_{\lambda\lambda}$, $p_{\lambda\rho_0}$, and $\rho_{\rho_0\rho_0}$ are the second derivatives (at $T = 0$) of the pressure p with respect to the chemical potential λ and the condensate density ρ_0 (with $p_{\rho_0} = 0$). Notice that the coefficient of $-(\nabla\varphi)^2/2m$ in (2.6) has the meaning of a superfluid-component density ρ_s [5,6]. In the limit as $T \to 0$, ρ_s coincides with the total density $\bar{\rho} = p_\lambda$. Bearing this observation in mind, we shall henceforth replace p_λ by ρ_s—for example, in (2.8).

The hydrodynamic action (2.6) leads to a two-fluid hydrodynamics, which is characteristic of superfluid systems. No condensate exists in a two-dimensional system at any arbitrarily small, but nonvanishing temperature. This is clear from the asymptotic form of the correlation function

$$\langle \psi(\mathbf{x}, \tau)\bar{\psi}(\mathbf{y}, \tau_1) \rangle \sim r^{-\alpha}, \quad \alpha = m / 2\pi\beta\rho_s \tag{2.7}$$

for $r = |\mathbf{x} - \mathbf{y}| \to \infty$ (see [4]). We can speak of long-range correlations in a two-dimensional Bose system at $T \neq 0$ only in the sense that the correlation functions, e.g. (2.7), decrease not exponentially, but according to a power law.

In deriving the expressions (2.6) in [4], we have essentially restricted ourselves to integration over the nonvanishing functions ψ_0 and $\bar{\psi}_0$. We now wish to take into account the contribution to the continual integral of the functions ψ_0 and $\bar{\psi}$ that vanish over some discrete set of points of the $\mathbf{x}$ plane (for each specified τ). On going around each such point the phase acquires an extra addend $2\pi n$ (n is an integer). We consider only the points with $n = \pm 1$ and speak of them as the center of quantum vortices rotating in the positive or negative direction. The points with $|n| > 1$ can be regarded as the points of confluence of $|n|$ vortices rotating in the same direction. Such formations are unstable and disintegrate into separate vortices with $|n| = 1$.

It is clear from the foregoing that allowance for the vortices leads to integration over the functions $\varphi(\mathbf{x}, \tau)$ which increase by $\pm 2\pi$ on going around the "singularities"—the zeroes of the functions ψ_0 and $\bar{\psi}_0$. We should integrate over the density ρ and the phase φ, allowing for the indicated conditions for multivaluedness, as well as over the trajectories of the vortex centers in the $(\mathbf{x}, \tau)$-space.

Let us drop from the integrand in (2.6) the last two terms $\sim(\nabla\pi)^2$ and $\pi(\nabla\varphi)^2$, which are responsible for the deviation of the phonon spectrum from the linear spectrum and for the phonon-phonon interaction. If we now evaluate the integral of e^{S_0} over π, we reduce the action S_0 to the form

$$-\int d\tau\, d^2x \frac{\rho_s}{2m}\left((\nabla\varphi)^2 + \frac{1}{c^2}(\partial_\tau\varphi)^2\right), \tag{2.8}$$

where c^2 is the square of the speed of sound. The replacement of p_λ by ρ_s has been explained above. It also ensures the validity of the formulas in the case when ρ_s becomes substantially different from p_λ. The expression (2.8) is the action of a relativistic system written in terms of Euclidean variables, where the speed of sound c plays the role of the velocity of light. Sticking to the relativistic analog, we shall show that when the vortices are taken into account the action (2.8) is essentially the action of a two-dimensional (2 + 1-dimensional) relativistic electrodynamics in which phonons play the role of photons, and quantum vortices the role of charged particles.

In terms of the variables x_1, x_2 and $x_3 = c\tau$, the expression (2.8) takes the form

$$-\frac{\rho_s}{2mc}\int (\nabla_3\varphi)^2\, d^3x, \tag{2.9}$$

where $\nabla_3\varphi$ is the three-dimensional gradient of the phase φ. The integral over φ can be evaluated with the aid of the phase shift

$$\varphi(x) \to \varphi(x) + \varphi_0(x) \tag{2.10}$$

where the shift function $\varphi_0(\mathbf{x})$, the solution of the three-dimensional Laplace equation, takes on the ambiguity in the phase. To find the function $\varphi_0(\mathbf{x})$, we note that the gradient $\nabla_3\varphi_0(\mathbf{x}) \equiv \mathbf{h}(\mathbf{x})$ is the solution of the three-dimensional magnetostatics problem defined by the equations[3)]

$$\operatorname{rot}\mathbf{h} = 2\pi\mathbf{j}, \quad \operatorname{div}\mathbf{h} = 0. \tag{2.11}$$

Here $\mathbf{j}$ is the sum of the unit linear currents flowing along the trajectories of the vortex centers. The function $\varphi_0(\mathbf{x})$ is the nonunique scalar potential of the magnetic field h produced by the system of linear currents. The square of the gradient, $(\nabla\varphi)^2$, under the integral sign in (2.9) becomes the sum $(\nabla\varphi)^2 + (\nabla\varphi_0)^2$ when the shift (2.10) is introduced. The integral of the first term describes a noninteracting field and is of no interest to us. The integral of $(\nabla\varphi_0)^2 = h^2$, on the other hand, is proportional to the energy of the magnetic field of the system of linear currents.

It would have been more customary to solve the magnetostatics problem (2.11) with the aid of the vector potential $\mathbf{a}(\mathbf{x})$ ($\mathbf{h} = \operatorname{curl}\mathbf{a}$, $\operatorname{div}\mathbf{a} = 0$). For a system of linear currents the vector potential $\mathbf{a}$ is the sum of the contributions from the individual currents

$$\mathbf{a}(\mathbf{x}) = \frac{1}{2}\sum_i \int \frac{d\mathbf{l}_i(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}. \tag{2.12}$$

The action obtained from (2.9) by replacing φ by φ_0 can be represented in the form of a double sum of the contributions from the various currents:

$$-\frac{\pi\rho_s}{2mc}\sum_{i,k}\iint \frac{d\mathbf{l}_i(\mathbf{x})\, d\mathbf{l}_k(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}. \tag{2.13}$$

The terms with $i = k$ in (2.13) diverge logarithmically for $\mathbf{x}$ close to $\mathbf{y}$. This divergence is the result of the approximation in which the vortices are considered as point vortices, and the corresponding currents as linear currents. To eliminate the divergence, we must take the finite dimensions of the vortices into account. For this purpose we distinguish the vortex centers by circles of radius r_0 greater than the radius of the vortex core, but less than the mean intervortex distance. Allowance for the finite dimensions of the vortices amounts to the replacement

$$\frac{\pi\rho_s}{2mc}\sum_i \iint_{|\mathbf{x}-\mathbf{y}|<r_0} \frac{d\mathbf{l}_i(\mathbf{x})\,d\mathbf{l}_i(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \to \frac{E_v(r_0)}{c}\sum_i \int ds_i. \qquad (2.14)$$

Here $ds = |d\mathbf{l}| = \sqrt{ds^2}$ and $E_v(r_0)$ is the part of the vortex energy inside the circle of radius r_0. The expression $E_v(r_0)$ depends logarithmically on r_0:

$$E_v(r_0) = \frac{\pi\rho_s}{m}\ln\frac{r_0}{a}. \qquad (2.15)$$

It is natural to call the quantity a in (2.15) the vortex-core radius. In order of magnitude $a \sim (\lambda m)^{-1/2}$, where λ is the chemical potential and m is the mass of the Bose particle. To determine a with great accuracy, we can use, for example, the Pitaevskiĭ solution[8] to the Ginzburg-Landau equations describing the vortex structure.

Let us now transform the integral over $|\mathbf{x}-\mathbf{y}| > r_0$ in (2.13). Let us introduce a new vector potential $\mathbf{A}(\mathbf{x})$ whose expansion

$$\mathbf{A}(\mathbf{x}) = \int_{k<\tilde{k}_0} e^{i\mathbf{k}\mathbf{x}}\mathbf{a}(\mathbf{k})\,d^3k \qquad (2.16)$$

is restricted to momenta smaller than $\tilde{k}_0 \sim r_0^{-1}$. The action (2.13) can be reduced to the form

$$S_0 = -m_v(r_0)c\sum_i \int ds_i - iq\int \mathbf{A}\mathbf{j}\,d^3x - \frac{1}{2c}\int (\operatorname{rot}\mathbf{A})^2 d^3x, \qquad (2.17)$$

where $m_v(r_0) \equiv E_v(r_0)/c^2$ is the vortex "mass" and the coefficient

$$q = 2\pi\sqrt{\rho_s/mc^2} \qquad (2.18)$$

plays the role of a coupling constant. The action (2.17), which has been written in terms of Euclidean variables, describes a system of charged particles interacting with the electromagnetic field $\mathbf{A}(\mathbf{x})$ (whose momenta are cut off at the upper limit by $\tilde{k}_0 \sim r_0^{-1}$). We integrate the functional e^{S_0} over the field $\mathbf{A}(\mathbf{x})$ and along the charged-particle trajectories. We must proceed precisely along these lines in quantizing the system with the action (2.17). The integral of e^{S_0} over the field $\mathbf{A}(\mathbf{x})$ can be evaluated exactly with the aid of the shift $\mathbf{A} \to \mathbf{A} + \mathbf{A}_0$, which annihilates the linear form with respect to $\mathbf{A}$ in (2.17). We then get back to the action (2.13), which proves the correctness of the expression (2.17).

To describe the motion of vortices with velocities much less than c, it is convenient to go over to the nonrelativistic approximation in the action (2.17). In this approximation we have

$$\frac{E_v(r_0)}{c}\int ds \approx E_v(r_0)\left(\int_0^\beta d\tau + \frac{1}{2c^2}\int_0^\beta \left(\frac{d\mathbf{x}}{d\tau}\right)^2 d\tau\right) \qquad (2.19)$$

$$= \beta E_v(r_0) + \int_0^\beta \frac{m_v(r_0)}{2}\mathbf{v}^2(\tau)\,d\tau.$$

The contribution of the scalar potential A_0 in (2.17) can be transformed into a term of direct interaction between the charged particles via a logarithmic potential. The action (2.17) in the nonrelativistic approximation is of the form

$$-\sum_i \left(\beta E_v(r_0) + \int_0^\beta d\tau\left(\frac{m_v(r_0)}{2}\mathbf{v}_i^2 + i\frac{g_i}{c}\mathbf{v}_i\mathbf{A}_i\right)\right)$$

$$-\frac{1}{2}\int d\tau\,d^2x\left((\partial_1 A_2 - \partial_2 A_1)^2 + \frac{1}{c^2}(\partial_\tau \mathbf{A})^2\right)$$

$$+\frac{1}{4\pi}\int d\tau\,d^2x\,d^2y\,j_0(\mathbf{x},\tau)\,j_0(\mathbf{y},\tau)\ln|\mathbf{x}-\mathbf{y}|. \qquad (2.20)$$

Here A_i is the vector potential at the point of location of the i-th vortex which moves with velocity $\mathbf{v}_i$ and has charge $g_i = \pm g$, where

$$g^2/4\pi = \pi\rho_s/m. \qquad (2.21)$$

The function $j_0(\mathbf{x}, \tau) = \sum_i g_i\delta(\mathbf{x} - \mathbf{x}_i(\tau))$ is the charge density.

3. Let us consider certain consequences of the equivalence of the "phonon-vortex" system to two-dimensional electrodynamics.

At low temperatures the vortices in the system can exist only in the form of pairs of opposite sign coupled by a long-range logarithmic potential. As the temperature is increased, the number of pairs increases and the mean distance between them decreases. Finally, at some temperature T_c dissociation of the coupled pairs occurs. Besides the coupled pairs, isolated vortices also appear in the system above the dissociation temperature, and we have a plasma-type state to deal with. It is natural to suppose that the phase transition from the superfluid to the normal state amounts precisely to the dissociation of the coupled pairs.

The quantity ρ_s, defined as the coefficient of $-(\nabla\varphi)^2/2m$ in the hydrodynamic action, does not vanish at $T > T_c$. This coefficient is analogous to the quantity ρ_s^0 introduced by Berezinskiĭ for the two-dimensional rotator system model.[3] For $T < T_c$ this coefficient practically coincides with the macroscopically determined superfluid density everywhere except in the narrow phase transition region, where an intense formation of quantum vortices begins.

Notice also that in a plasma-type state, long-range correlations vanish at $T > T_c$ as a result of the characteristic Debye screening. In particular, the correlation function $\langle\psi(\mathbf{x}, \tau)\bar{\psi}(\mathbf{y}, \tau_1)\rangle$ decreases exponentially for $T > T_c$.

The phase transition connected with the dissociation of the pairs at $T = T_c$ is accompanied by the conversion of the second-sound branch in the energy spectrum into the plasma-oscillation branch. Let us explain this in greater detail. The velocity c in the action (2.8) is close to the velocity of first sound at low temperatures and coincides with it in the limit $T \to 0$. At temperatures of the order of the phase-transition temperature T_c (but not too close to it), however, the velocity c is close to the velocity of second sound. This fact, which was previously noted in [5,9] for the three-dimensional Bose gas, is also valid for the two-dimensional system. In the case when the velocity c is close to the velocity u of second sound, the relation between c and u is the same as between the velocities of light in vacuo and in a medium:

$$u = c/\sqrt{\varepsilon}. \qquad (3.1)$$

The coefficient ϵ has the meaning of the dielectric constant of a medium. It can easily be computed in the case when the mean distance between the vortex pairs is large and the pairs can be considered as noninteracting. We use the relation

$$(\varepsilon-1)\mathbf{E}=\mathbf{D}-\mathbf{E}=\mathbf{P}=N_{\rm P}\langle\mathbf{d}\rangle_{\mathbf{E}}, \tag{3.2}$$

where $\mathbf{E}$ and $\mathbf{D}$ are the intensity and induction of the electric field, $\mathbf{P}$ is the mean dipole moment per unit volume N_p is the number of pairs in a unit volume, and $\langle\mathbf{d}\rangle_E$ is the mean dipole moment of a pair in the electric field $\mathbf{E}$. The energy of a pair in the field $\mathbf{E}$ is equal to $-\mathbf{E}\cdot\mathbf{d}$. In the limit $\mathbf{E}\to 0$ we obtain

$$\langle\mathbf{d}\rangle_{\mathbf{E}}=\langle\mathbf{d}\exp\beta(\mathbf{E}\mathbf{d})\rangle=\beta\langle\mathbf{d}(\mathbf{E}\mathbf{d})\rangle={}^{1}/{}_{2}\beta\langle d^{2}\rangle\mathbf{E}={}^{1}/{}_{2}\beta g^{2}\langle r^{2}\rangle\mathbf{E}. \tag{3.3}$$

Here $\langle r^2\rangle$ is the mean-square distance between the vortices of a pair and $g^2=4\pi^2\rho_S/m$ is the square of the charge. From (3.2) and (3.3) follows the equality

$$\varepsilon=1+\frac{g^{2}\beta}{4\pi}(N_{\rm P}\langle 2\pi r^{2}\rangle). \tag{3.4}$$

The quantity $N_p\langle 2\pi r^2\rangle$, which has the meaning of the average relative area occupied by the vortex pairs, is equal to the correction $\Delta\alpha$ to the exponent $m/2\pi\beta\rho_S$ in the asymptotic form of the correlation function (2.7) which arises when the vortices are taken into account (see [4]). As a result we obtain the formula

$$\varepsilon=1+\frac{1}{2}\frac{\Delta\alpha}{\alpha_{0}}\approx\sqrt{\frac{\alpha}{\alpha_{0}}}, \tag{3.5}$$

which gives ϵ as the square root of the ratio α/α_0 of the exponents computed with and without allowance for the vortex pairs.

As we approach the phase transition, $\epsilon\to\infty$ and $u\to 0$. This implies that second sound (light in the electromagnetic analogy) ceases to propagate. The corresponding branch of the spectrum is converted into the plasma-oscillation branch. The plasma frequency is determined from the equation

$$1-\frac{g^{2}}{k^{2}}\Pi(\mathbf{k},\omega)=0, \tag{3.6}$$

where $\Pi(\mathbf{k},\omega)$ is the "Coulombically irreducible" part of the charged-particle density correlation function. If in computing $\Pi(\mathbf{k},\omega)$ we restrict ourselves to the contribution of the simplest single-loop diagrams, we obtain in the limit as $\mathbf{k}\to 0$

$$\omega_{0}=\lim_{\mathbf{k}\to 0}\omega(\mathbf{k})=\sqrt{g^{2}\langle n_{\rm v}\rangle/m_{\rm v}}, \tag{3.7}$$

where $\langle n_V\rangle$ is the mean density of the unpaired vortices. Substituting in (3.7)

$$m_{\rm v}=\frac{E_{\rm v}}{c^{2}}=\frac{\pi\rho_{s}}{mc^{2}}\ln\frac{\bar{r}}{a}, \tag{3.8}$$

where $\bar{r}$ is the mean intervortex distance, we obtain

$$\omega=\left(\frac{4\pi c^{2}\langle n_{\rm v}\rangle}{\ln(\bar{r}/a)}\right)^{1/2} \tag{3.9}$$

In the phase-transition region the system under consideration is a system with a "strong coupling." The point is that the characteristic (for plasma theory) smallness of the charged-particle potential energy as compared to the kinetic energy does not exist here. Therefore the formulas (3.7) and (3.9) for the plasma frequency can at best be correct only with respect to the order of magnitude.

4. The above-developed method of describing quantum vortices can be extended to three-dimensional Bose systems as well. As in the two-dimensional case, to quantum vortices correspond the zeros of the functions $\psi(\mathbf{x},\tau)$ and $\bar{\psi}(\mathbf{x},\tau)$ over which we integrate. The complex functions ψ and $\bar{\psi}$ vanish on lines (on sets of dimension 1) in three-dimensional x-space for fixed τ and on plane surfaces in four-dimensional $(\mathbf{x},\tau)$-space. We should integrate over the functions ψ and $\bar{\psi}$ which vanish on plane surfaces and then over the surface configurations.

We shall not dwell at length on the distinctive features of the formalism as applied to three-dimensional systems, but limit ourselves to a qualitative consideration of the role of quantum vortices in phase transitions. The conclusions arrived at here are analogous to those drawn in Byckling's paper[10].

At low temperatures in a nonrotating Bose system excitations in the form of vortex rings can exist to which correspond closed lines where the functions ψ and $\bar{\psi}$ vanish. As the temperature increases the number of vortex rings per unit volume increases and the mean distance between them decreases. When the mean distance between the rings becomes of the order of the mean ring length, there appears a tendency to form long vortex rings. It is natural to suppose that the phase transition from the superfluid to the normal state is connected with the appearance in the system of vortex filaments of infinite length (in a real system, of filaments starting from and terminating on the walls of the vessel). A long vortex filament does not form at once but through successive union and elongation of vortex rings of finite length. Therefore, in the situation when the number of vortex rings per unit volume is sufficiently large the probability of the existence of infinitely long vortex filaments can by no means become infinitely small.

Notice that the quantity ρ_S, defined as the coefficient of $-(\nabla\varphi)^2/2m$ in the hydrodynamic action, continues to remain different from zero above the transition temperature. In other words, it is possible to describe the system above the transition in terms of normal and superfluid components perforated by quantum vortices.

Thus, the qualitative analysis shows that in a three-dimensional system, as in the two-dimensional, the phase transition is connected with quantum vortices. The closed vortex rings in the three-dimensional system can be regarded as the analog of the coupled pairs in the two-dimensional system, and the long vortex filaments as the analog of the single vortices. Indeed, a vortex ring in a plane cross section of the three-dimensional system gives two vortices of opposite sign.

Quantum vortices above the λ-transition in a rotating liquid helium were experimentally observed by Andronikashvili and co-workers[11] within 18–20 min after heating the system to 0.1–0.2 K above the λ-point. In the light of the foregoing, the fact that quantum vortices exist above the λ-point is not surprising. The collapse of the ordered vortex structure 18–20 min after heating above the λ-point is however effected by the randomly arranged long vortex filaments which form above the λ-point. A complete analysis of the problem of the lifetime of the ordered vortex structure above the λ-point is of course possible only in the framework of the kinetic approach.

5. The foregoing leads to the following conclusions.

a) The method of continual integration provides a natural description for quantum vortices. To them correspond the zeros of the functions ψ and $\bar{\psi}$ with respect to which the integration is carried out.

In the normal state the principal contribution to the continual integral is made by functions of the type of a superposition of plane waves. In the superfluid state the principal contribution is made by nonvanishing functions which describe the quantum-vortex-free states. In the vicinity of the phase transition, however, the principal contribution to the continual integral is made by function which are everywhere, except at the vortex cores, almost constant in magnitude.

b) If we neglect the phonon-phonon interaction and the dispersion of the phonon spectrum, then the two-dimensional phonon-vortex system is equivalent to relativistic two-dimensional electrodynamics.

c) The phase transition of a two-dimensional Bose system from the superfluid to the normal state amounts to the dissociation of coupled vortex pairs. The description of a two-dimensional Bose gas in terms of normal and superfluid components perforated by quantum vortices is valid both above and below the phase transition and loses meaning if, and only if, the vortex-core radius becomes of the order of the mean intervortex distance.

d) In a three-dimensional Bose system the phase transition from the superfluid to the normal state is connected with the formation of long vortex filaments. Thus, quantum vortices exist in the normal phase as well. There arises in this connection the alluring idea that the difference between the normal and the superfluid liquids is determined by the nature of the quantum vortices existing in the liquid—in the normal liquid there should be long vortex filaments besides the vortex rings.

e) The theory considered in Sec. 2 and obtained under the assumption that the phonon-phonon interaction and the dispersion of the phonon spectrum can be neglected is the relativistic theory of the complex scalar field ψ with the interaction $g|\psi|^4$ and a negative squared bare mass: we would have been dealing with tachyons if we had switched off the interaction. The interaction leads to a Bose condensation and the formation of massless particles, as well as of charged particles—quantum vortices. Since we propose to discuss this question separately, we shall not dwell on it at length here.

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[1)]We shall henceforth use a system of units with $\hbar = k_B = 1$, where $\hbar$ and k_B are the Planck and Boltzmann constants.

[2)]In (2.3), $\omega \equiv \omega_n = 2\pi n/\beta$, $k_i = 2\pi n_i/L$; n and n_i are integers.

[3)]Here we denote both two-dimensional and three-dimensional vectors by boldface type.

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74