Universality of the power spectra of relativistic electrons produced in a turbulent plasma

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It is demonstrated that the stationary energy distributions of relativistic electrons in a turbulent plasma located in a magnetic field can be described by a power law $\sim\epsilon^{-\gamma}$. An equation for γ is obtained, which takes into account the anisotropy of the electron distribution and of the electromagnetic radiation interacting with it in a uniform magnetic field. It is shown that for a distribution function not averaged over the directions of the magnetic field there is a single solution of the equation, which yields for γ, the universal value 3.

1. In[1,2] it was shown that the stationary distribution of relativistic electrons in a turbulent plasma is a power-law distribution $\sim 1/\epsilon^{\gamma}$ and that $\gamma = 3$ when there is no magnetic field. Because of a number of mathematical difficulties, we considered in[1,2] a simplified problem in which the effect of the reabsorbed radiation on the particles was averaged over the direction of the magnetic field. This way of stating the problem is of interest for astrophysical applications, since cosmic magnetic fields often are random. In the limit of very weak magnetic fields the result $\gamma = 3$ did, of course, not depend on this averaging. It is of considerable interest to have the exact solution of the problem for the case of a uniform external magnetic field. The present paper is devoted to the solution of that problem. Assuming that the magnetic field is uniform means that the direction of the magnetic field does not change along the path length of the appropriate reabsorption of the electromagnetic radiation of the relativistic electrons, which in the actual cases of a number of astrophysical objects can correspond to rather large sizes. (These sizes depend, of course, on many parameters, in particular, on the plasma density n, the density n_* of the relativistic particles, the magnetic field strength, and so on; see[3].) Moreover, the reabsorption length is the larger the higher the frequency, and hence also the energy of the particles interacting with the radiation. The requirement of the uniformity of the magnetic field may thus be a very stringent one for high-energy particles.

The present investigation shows that the distribution of relativistic electrons in a uniform magnetic field becomes anisotropic. The distribution of the reabsorbed radiation also becomes anisotropic. Therefore, if the direction of the magnetic field would change over distances less than the reabsorption length (for instance, for high-energy particles) the equilibrium form of the distribution would, even though the anisotropy may be weak, not correspond in one section with a uniform magnetic field to the equilibrium form of a neighboring section of the magnetic field. This shows that the distribution function of the relativistic electrons in the case of a random magnetic field must be anisotropic over a length characteristic for the change in direction of the magnetic field. The authors consider that taking this "local" non-uniformity into account for an isotropic distribution function of the relativistic electrons which is averaged over directions is one of the most important problems for further studies.

The present paper refers only to the rigorous solution of the problem of the form of the distribution function of the relativistic electrons in a strictly uniform magnetic field. We give here a proof that the distribution function of the relativistic electrons under those conditions is a universal function of the energy, $f_\epsilon \sim 1/\epsilon^3$, i.e., $\gamma = 3$. The value $\gamma = 3$, obtained here, lies very close to the average $\gamma = 2.7$ for the majority of cosmic radio-sources. The dependence of the electron distribution on the angle with the magnetic field remains an arbitrary one, but we assume that this distribution does not change significantly over a characteristic angle $\Delta\theta \approx mc^2/\epsilon$ and, in particular, we exclude the case when all electrons move at a small angle $\Delta\theta \ll mc^2/\epsilon$ along the direction of the magnetic field (this case was shown in[3] to lead to $\gamma = 2$).

2. The assumed limitation on the anisotropy, $\Delta\theta \gg mc^2/\epsilon$, allows us to neglect in the kinetic equation for the electrons the derivatives with respect to the angular variables. We shall write down that equation, taking these derivatives into account (here and henceforth $c = 1$):

$$\frac{\partial f_{\varepsilon,x}}{\partial t} + \mathbf{v}\frac{\partial f_{\varepsilon,x}}{\partial \mathbf{r}} = \sum_{\sigma=1,2}\int_0^\infty \omega\, d\omega \int_{-1}^{1} dy\, I^\sigma_{\omega,y} \times\left(\frac{\partial}{\partial\varepsilon} + \frac{\partial}{\partial x}\frac{y-x}{\varepsilon}\right) W^\sigma_{\varepsilon,\omega,x,y}\,\varepsilon^2\left(\frac{\partial}{\partial\varepsilon} + \frac{y-x}{\varepsilon}\frac{\partial}{\partial x}\right)\frac{f_{\varepsilon,x}}{\varepsilon^2} + \sum_{\sigma=1,2}\int_0^\infty \frac{\omega^3}{(2\pi)^2}\, d\omega \int_{-1}^{1} dy \left(\frac{\partial}{\partial\varepsilon} + \frac{\partial}{\partial x}\frac{y-x}{\varepsilon}\right) W^\sigma_{\varepsilon,\omega,x,y} f_{\varepsilon,x}. \quad (1)$$

In this equation we have taken into account terms of order m/ϵ, but neglected terms of order m^2/ϵ^2. Moreover, in (1) x is the cosine of the angle between the direction of motion of the relativistic electron and the magnetic field, y is the cosine of the angle between the electromagnetic wave vector and the magnetic field, and $W^\sigma_{\epsilon,\omega,x,y}$ the sum of the probabilities for the emission of a wave in Compton scattering by any turbulent oscillations and in the synchrotron mechanism. The relativistic electron distribution function is normalized as follows:

$$\int_0^\infty d\varepsilon \int_{-1}^{1} dx\, f_{\varepsilon,x} = n_*, \quad (2)$$

where n_* is the total relativistic electron density.

The second term on the right-hand side of (1) describes spontaneous losses and the first term, which is proportional to the radiation intensity $I^\sigma_{\omega,y}$ describes the general case are distributed anisotropically. (8) is thus a very general relation and Λ^{σ} in (8) must correspond to the sum of all mechanisms described in which we must yet include Compton scattering by low-frequency radiation.

The domain of applicability of (8) is restricted by the assumption that the ϵ-dependence of U is strictly the one indicated. This assumption loses its validity for steeply anisotropic particles when their distribution changes appreciably within a solid angle m^2/ϵ^2. In that case the maximum emitted frequencies are also not proportional to ϵ^2 but to the first power of ϵ (see[3]). However, for the case considered Eq. (8) includes practically all cases of interest. We note that the quantities $\Lambda^{\sigma}(q, x)$ can be written down for practically all turbulent waves if we use the tables in[3].

4. We now give a proof of the power-law nature of the distribution of the relativistic electrons with $\gamma = 3$, using (8). We write

$$R_{\nu}^{\sigma}(x)=\int q^{\nu/2}\Lambda^{\sigma}(q,x)\,dq \tag{15}$$

and we shall look for a solution of the equilibrium kinetic equations (4)–(7)

$$\frac{\partial}{\partial\varepsilon}\frac{f_{\varepsilon,x}}{\varepsilon^2}=-\frac{A(\varepsilon,x)}{D(\varepsilon,x)}\frac{f_{\varepsilon,x}}{\varepsilon^2},\qquad I^{\sigma}_{\omega,y}=-\frac{Q^{\sigma}_{\omega,y}}{\gamma^{\sigma}_{\omega,y}} \tag{16}$$

in the following form

$$f_{\varepsilon,x}=\frac{\text{const}}{m}\left(\frac{m}{\varepsilon}\right)^{\gamma(x)}\varphi(x)n_*, \tag{17}$$

where $\gamma(x)$ is the exponent of the power-law spectrum, which may depend on the cosine x of the angle with the direction of the magnetic field, while $\varphi(x)$ is the angular part of the distribution function of the relativistic electrons. One sees easily that $Q^{\sigma}_{\omega,y}$ and $\gamma^{\sigma}_{\omega,y}$ can then be expressed in terms of the function R^{σ}_{ν} which we just introduced, in the following way:

$$\begin{aligned}Q^{\sigma}_{\omega,y}&=\frac{\text{const}}{4\pi^2}\varphi(y)\,\omega_{pe}^{3}\left(\frac{2\omega_{pe}}{\omega}\right)^{(\gamma-1)/2}R^{\sigma}_{\gamma-1}(y)\,n_*,\\ \gamma^{\sigma}_{\omega,y}&=\frac{-\text{const}}{4m}\varphi(y)\,(\gamma+2)\left(\frac{2\omega_{pe}}{\omega}\right)^{(\gamma+4)/2}\omega_{pe}R^{\sigma}_{\gamma}(y)\,n_*.\end{aligned} \tag{18}$$

This proves the universality of the $^5/_2$ law for the intensity of the radiation under reabsorption conditions:

$$I^{\sigma}_{\omega,y}=\frac{\omega_{pe}^{2}}{\pi^2(\gamma+2)}\left(\frac{\omega}{2\omega_{pe}}\right)^{5/2}\frac{R^{\sigma}_{\gamma-1}(y)}{R^{\sigma}_{\gamma}(y)}m. \tag{19}$$

Substituting this expression into the diffusion coefficient $D(\epsilon, x)$ we can also express the resulting integral in terms of R^{σ}_{γ}:

$$D(\varepsilon,x)=\frac{\omega_{pe}^{4}}{\pi^2(\gamma+2)}\frac{\varepsilon^3}{m^2}\mathcal{R}_1(x),\quad \mathcal{R}_1(x)=\sum_{\sigma=1,2}R_3^{\sigma}(x)\frac{R^{\sigma}_{\gamma-1}(x)}{R^{\sigma}_{\gamma}(x)}. \tag{20}$$

We similarly find the spontaneous losses:

$$A(\varepsilon,x)=\frac{\omega_{pe}^{4}}{\pi^2}\frac{\varepsilon^2}{m^2}\mathcal{R}_2(x),\quad \mathcal{R}_2(x)=\sum_{\sigma=1,2}R_2^{\sigma}(x). \tag{21}$$

Hence, Eq. (16) for the distribution function

$$\frac{\partial}{\partial\varepsilon}\frac{f_{\varepsilon}}{\varepsilon^2}=-(\gamma+2)\frac{\mathcal{R}_2(x)}{\mathcal{R}_1(x)}\frac{f_{\varepsilon}}{\varepsilon^3} \tag{22}$$

gives, indeed, the power-law solution (17) where $\gamma(x)$ must satisfy the equation.

$$\mathcal{R}_1(x)=\mathcal{R}_2(x). \tag{23}$$

The solution of this equation is a γ-value which is independent of x and equal to 3. We have thus proven our statement.

In connection with the theorem just proven the problem may arise whether this solution is the unique solution of Eq. (23). This problem can be considered by using concrete expressions for the R^{σ}_{γ}.

5. Let us illustrate this by the example of the sum of synchrotron emission and Compton scattering by plasma oscillations with high phase velocities. In that case the evaluation of R^{σ}_{ν} gives

$$R_{\nu}^{1,2}=\frac{2e^2\pi^2}{3\omega_{pe}^{2}}\left(\varkappa a_{\nu}+(2\xi)^{(\nu+2)/2}b_{\nu}^{1,2}\right), \tag{24}$$

and Eq. (23) can be written in the form

$$C_3(\gamma)\varkappa^3+C_2(\gamma,\xi)\varkappa^2+C_1(\gamma,\xi)\varkappa+C_0(\gamma)(2\xi)^{\gamma+4}=0. \tag{25}$$

The coefficients C_{ν} can be expressed in terms of the earlier introduced a_{ν}, $b_{\nu}^{1,2}$, and ξ for the case $\xi \ll 1$ as follows:

$$\begin{aligned}&C_3=2a_{\gamma}(a_3a_{\gamma-1}-a_2a_{\gamma}),\quad C_2=C_2'(2\xi)^{(\gamma+1)/2}+C_2''(2\xi)^2,\\ &C_1=C_1'(2\xi)^{3+\gamma/2}+C_1''(2\xi)^{5/2+\gamma},\\ &C_1'=a_{\gamma}[b^{1}_{\gamma-1}b_3^{1}+b^{2}_{\gamma-1}b_3^{2}-(b_{\gamma}^{1}+b_{\gamma}^{2})(b_2^{1}+b_2^{2})],\\ &C_1''=a_3(b^{1}_{\gamma-1}b_{\gamma}^{2}+b^{2}_{\gamma-1}b_{\gamma}^{1}),\\ &C_2'=a_3a_{\gamma}(b^{1}_{\gamma-1}+b^{2}_{\gamma-1}),\quad C_2''=-a_{\gamma}^{2}(b_2^{1}+b_2^{2}),\\ &C_0=b_3^{1}b^{1}_{\gamma-1}b_{\gamma}^{2}+b_3^{2}b^{2}_{\gamma-1}b_{\gamma}^{1}-b_{\gamma}^{1}b_{\gamma}^{2}(b_2^{1}+b_2^{2}).\end{aligned} \tag{26}$$

Finally, a_{ν} and $b_{\nu}^{1,2}$ have in this case the following actual form:

$$\begin{aligned}a_{\nu}&=\frac{\nu^2+6\nu+16}{(\nu+2)(\nu+4)(\nu+6)},\qquad b_{\nu}^{1,2}=\frac{\sqrt{3}}{2\pi}\Gamma\left(\frac{\nu}{4}+\frac{1}{6}\right)\cdot\\ &\cdot\Gamma\left(\frac{\nu}{4}+\frac{5}{6}\right)\left[\frac{\nu}{4}+\frac{5}{6}\pm\frac{\nu+2}{4}\right]\frac{1}{\nu+2}.\end{aligned} \tag{27}$$

We have constructed all six coefficients C_0, C_1', C_1'', C_2', C_2'', C_3 graphically as functions of γ when γ varies within the limits $1 < \gamma < 10$. The graphs of these coefficients are given in Figs. 1 and 2; they show that when the parameter γ changes within these limits these coefficients are smooth functions of γ while C_0, C_1, C_2, and C_3 vanish for $\xi \ll 1$ in the single point $\gamma = 3$ which shows the uniqueness of the solution found within these limits of γ and when $\xi \ll 1$ and for the above-mentioned sum of two emission mechanisms.

6. It is important for astrophysical applications to know which mechanisms can change this value of γ.

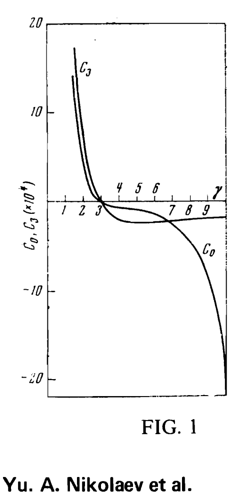

FIG. 1

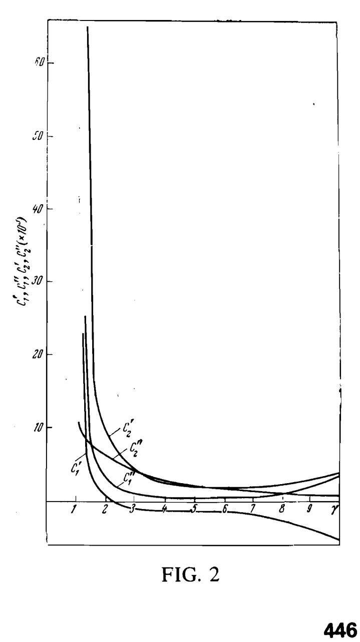

FIG. 2

the induced acceleration. The index σ describes the two possible polarizations of waves, while $I^{\sigma}_{\omega,y}$ is normalized as follows:

$$I^{\sigma}=\int_0^{\infty}d\omega\int_{-1}^{1}dy\,I^{\sigma}_{\omega,y}, \tag{3}$$

where I^{σ} is the density of the electromagnetic radiation with polarization σ ($\hbar = c = 1$, i.e., $I^{\sigma}_{\omega,y}$ differs from the radiation intensity used normally in astrophysics by a factor 2π). From the mathematical point of view it is important that the factors y-x stand next to the probability $W^{\sigma}_{\epsilon,\omega,x,y}$. The quantity y-x can in that case be estimated using the fact that the emission probability for relativistic particles is appreciable only in a narrow cone along the direction of the motion of the particles, i.e., $y-x \sim m/\epsilon$. This enables us to drop all derivatives with respect to the angular variables in (1), i.e., to write the kinetic equation in the standard form:

$$\frac{\partial f_{\varepsilon,x}}{\partial t}+\mathbf{v}\frac{\partial f_{\varepsilon,x}}{\partial \mathbf{r}}=\frac{\partial}{\partial\varepsilon}\left[\varepsilon^2 D(\varepsilon,x)\frac{\partial}{\partial\varepsilon}\frac{f_{\varepsilon,x}}{\varepsilon^2}+A(\varepsilon,x)f_{\varepsilon,x}\right], \tag{4}$$

where

$$\begin{gathered}D(\varepsilon,x)=\sum_{\sigma=1,2}\int_0^{\infty}\omega\,d\omega\int_{-1}^{1}I^{\sigma}_{\omega,y}W^{\sigma}_{\varepsilon,\omega,x,y}\,dy,\\ A(\varepsilon,x)=\sum_{\sigma=1,2}\int_0^{\infty}\frac{\omega^3}{(2\pi)^2}d\omega\int_{-1}^{1}dy\,W^{\sigma}_{\varepsilon,\omega,x,y}.\end{gathered} \tag{5}$$

Similar arguments enable us to write the equation for the transfer of radiation which acts on the particles in the form

$$\begin{gathered}\frac{\partial I^{\sigma}_{\omega,y}}{\partial t}+\mathbf{v}_{\rm gr}\frac{\partial I^{\sigma}_{\omega,y}}{\partial \mathbf{r}}=I^{\sigma}_{\omega,y}\int_0^{\infty}d\varepsilon\int_{-1}^{1}dx\,W^{\sigma}_{\varepsilon,\omega,x,y}\cdot\\ \times\omega\varepsilon^2\frac{\partial}{\partial\varepsilon}\frac{f_{\varepsilon,x}}{\varepsilon^2}+\frac{\omega^3}{(2\pi)^2}\int_0^{\infty}d\varepsilon\int_{-1}^{1}W^{\sigma}_{\varepsilon,\omega,x,y}f_{\varepsilon,x}\,dx=\gamma^{\sigma}_{\omega,y}I^{\sigma}_{\omega,y}+Q^{\sigma}_{\omega,y}\end{gathered} \tag{6}$$

($\mathbf{v}_{gr}$ is the group velocity vector). It is convenient for further calculations to use the fact that the emission probabilities of relativistic particles have a steep maximum at x = y so that to a good approximation we can take out from under the integral signs the functions which stand in front of the probabilities, putting x = y in them, i.e.,

$$\begin{gathered}D(\varepsilon,x)=\sum_{\sigma=1,2}\int_0^{\infty}\omega\,d\omega\,I^{\sigma}_{\omega,x}U^{\sigma}_{\varepsilon,\omega,x},\\ A(\varepsilon,x)=\sum_{\sigma=1,2}\int_0^{\infty}\frac{\omega^3\,d\omega}{(2\pi)^2}U^{\sigma}_{\varepsilon,\omega,x},\\ \gamma^{\sigma}_{\omega,y}=\omega\int_0^{\infty}d\varepsilon\,U^{\sigma}_{\varepsilon,\omega,y}\varepsilon^2\frac{\partial}{\partial\varepsilon}\frac{f_{\varepsilon,y}}{\varepsilon^2},\\ Q^{\sigma}_{\omega,y}=\frac{\omega^3}{(2\pi)^2}\int_0^{\infty}d\varepsilon\,U^{\sigma}_{\varepsilon,\omega,y}f_{\varepsilon,y},\end{gathered} \tag{7}$$

where $U^{\sigma}_{\epsilon,\omega,x}$ is the integral of the appropriate probability over the angle y. The results of a calculation of the probabilities in the indicated approximation are the same as the well-known expressions obtained by averaging over all angles y.[3]

3. Using these formulae we can give a general proof for the existence of power-law solutions $f_\epsilon \sim 1/\epsilon^{\gamma}$ with $\gamma = 3$ using only the general expression for the probability in the form

$$U^{\sigma}_{\varepsilon,\omega,x}=\frac{m^2\Lambda^{\sigma}(q,x)}{\omega^2\varepsilon^2}\omega_{pe}^2, \tag{8}$$

where Λ^{σ} is a function of H, n, m_e, the intensity of the turbulent oscillations, and of the other plasma parameters. Of importance is here the way the probability depends on the frequency (which for the sake of convenience is written in the dimensionless form ω/ω_{pe}) and on the energy (in the dimensionless form ϵ/m). One could have included the parameters m^2 and ω_{pe}^2 in Λ^{σ}. The parameter q determines the emission condition and is proportional to ω/ϵ^2. Moreover, we shall put, introducing dimensionless quantities

$$q=\frac{\omega}{2\omega_{pe}}\frac{m^2}{\varepsilon^2}. \tag{9}$$

Written in this form, the parameter q is appropriate for the Compton emission mechanism in scattering by Langmuir oscillations with large phase velocities.[3] The emission condition in this case has the form $q < 1$. However, for all other emission mechanisms the appropriate parameters q are proportional to (9). This means that other emission mechanisms can also be described by the chosen q, but its magnitude will accordingly not vary between the limits 0 to 1 but within other finite limits, depending on the plasma parameters. We shall give two examples which illustrate this fact. For synchrotron emission

$$\Lambda^{1,2}(q,x)=\frac{\pi}{2\sqrt{3}}\frac{e^2}{\omega_{pe}^2}\left\{\int_{q/\xi}^{\infty}K_{5/3}(\lambda)\,d\lambda\mp K_{2/3}(q/\xi)\right\} \tag{10}$$

where $\xi = 3eH\sqrt{(1-x^2)}/4m\omega_{pe}$. In this case q practically changes from zero to ξ, i.e., to frequencies of the order of $\omega_{max} \sim eH\epsilon^2/m^3$. For Compton scattering by plasma oscillations with large phase velocities we have in the case of isotropic turbulence

$$\Lambda^{1,2}(q,x)=\frac{e^2\pi^2\varkappa}{3\omega_{pe}^2}(1-2q+2q^2), \tag{11}$$

where $\kappa = W/nm$ is the ratio of the total turbulent energy density to the electron rest energy density. In this case the probabilities are the same for both polarizations. However, the use of (8) is not restricted to the case of isotropic turbulence. In[4] it was shown that even in a weak magnetic field the plasma turbulence may be anisotropic, and, in particular, plasma oscillations are oriented parallel and antiparallel to the magnetic field direction. In this case, also for large phase velocities, the probabilities for different polarizations turn out to be different:[3]

$$\begin{gathered}\Lambda^1(q,x)=\frac{\pi^2e^2\varkappa}{2\omega_{pe}^2}(1-2q+3q^2)(1-x^2),\\ \Lambda^2(q,x)=\frac{\pi^2e^2\varkappa}{2\omega_{pe}^2}(1-2q+q^2)(1-x^2).\end{gathered} \tag{12}$$

Equation (8) covers also the case of plasma oscillations with small phase velocities when for isotropic turbulence[5]

$$\begin{gathered}\Lambda^{1,2}(q,x)=\int\frac{W_{k_1}dk_1e^2\pi^2}{6qk_1\omega_{pe}}\left[\left(1-\frac{\omega_{pe}}{k_1}q\right)^3\right.\\ \left.\pm 3\frac{\omega_{pe}^2}{k_1^2}q^2\left(1-\frac{\omega_{pe}}{k_1}q+\ln\frac{\omega_{pe}}{k_1}q\right)\right].\end{gathered} \tag{13}$$

Here $0 < q < k_1/\omega_{pe}$. It is clear that one can similarly also take the anisotropy of the turbulence into account. Then

$$\Lambda^{1,2}(q,x)=\int W_{k_1\xi}\,dk_1\,d\xi\,\Lambda^{1,2}(q,k_1,\xi,x), \tag{14}$$

where ξ is the cosine of the angle between the wave vector of the turbulent oscillations and the direction of the magnetic field or any other direction determined by the conditions for the excitation of the oscillations. It is also clear from this that Eq. (8) is retained also for all other kinds of turbulent oscillations (in particular, for Alfvén waves, whistlers, or ion-sound waves) which in

Available observations on the spectra of many cosmic sources show that the γ-value found here lies very close to the average observed value $\gamma = 2.7$ while the spread around this average value is not very large: $1.7 < 3.2$. In this connection one must emphasize that there are at least three possible causes for the spread in the observed values of γ: 1) a change in γ within the limits of the source of the relativistic electrons itself; 2) a change of γ when the particles leave the source; 3) a change of the effective value of γ due to the loss of energy of relativistic electrons beyond the limits of the source.

We understand here by the term "source" the region which is optically thick for the radiation by relativistic electrons and which produces the power-law distribution function. The first possibility is connected with the possibility that the magnetic field in the source is inhomogeneous, mentioned already in the Introduction to the present paper. A particular limiting case of this inhomogeneity is a random magnetic field, i.e., a magnetic field the magnitude and direction of which change rather often along a reabsorption length. This must also lead to an inhomogeneity of the electron distribution along a length of the order of the characteristic size of the magnetic field inhomogeneity. Formally it is already clear from Eqs. (4) and (6) that in that case one can not eliminate the gradient of the distribution function $f_{\epsilon,x}$ or the gradient in the equation for the radiation intensity. If we average, however, the distribution over distances appreciably longer than the characteristic size of the magnetic field inhomogeneity, the averaged relativistic particle distribution can be close to isotropic for a random magnetic field. Of course, this changes the value of γ. The second possibility is connected with the fact that when the relativistic particles leave the source they must diffuse strongly through the turbulence and the radiation and this may change their distribution. The problem thus consists in finding the connection between the value of γ within the source and beyond its boundaries.

We note in conclusion that the power-law solutions we obtained of the equations which we wrote down are not the only ones as there are also equilibrium solutions corresponding to the Maxwell distribution. The power-law distributions are stable only for a well-defined level of turbulence. Finally, in the general form we need a detailed analysis of the problem of the uniqueness of the solution $\gamma = 3$ taking into account all forms of turbulent oscillations and their spectra, in particular, in a strong field $\xi \gg 1$ and also taking into account different kinds of low-frequency oscillations.

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