Critical velocities at very low temperatures, and the vortices in a quantum bose fluid

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The theory of critical velocities at zero temperatures is considered. An investigation of the vortex-ring spectrum in a finite cross section channel shows that the Landau critical velocity for vortices vanishes. However, the probability for production of a large vortex is very low and hence in a ring channel and at T = 0 persistent helium currents should exist up to velocities close to the critical Landau velocity for a phonon-roton spectrum. It is shown that the region of applicability of the theory for T = 0 is restricted to comparatively low temperatures and small channel lengths. In this connection it is suggested that when the temperature and channel length are reduced to such an extent that the condition for complete disappearance of the normal component obtained in the present paper is satisfied, a considerable increase of the critical velocities may be expected such that the velocities approach the Landau critical velocity for the phonon-roton spectrum.

INTRODUCTION

Experimental measurements of the critical velocities in superfluid helium at low temperatures[1] show that they are much lower than the critical velocity determined by the Landau criterion for the known phonon-roton spectrum of elementary excitations $\epsilon(\mathbf{p})$:

$$v_L = \min \varepsilon(\mathbf{p}) / p, \tag{1}$$

where $\mathbf{p}$ is the excitation momentum.

Critical velocities close to the experimental values are obtained only if the Landau criterion is applied to the spectrum of vortex rings[2]. Such an explanation of the observed critical velocities encounters, however, a large number of difficulties. Some of them are connected with the question as to the form of the spectrum of the vortex states.

Initially, the critical velocities were determined by using energy and momentum expressions derived in classical hydrodynamics for a vortex ring with circulation h/m, situated in an unbounded liquid[3]

$$\varepsilon = \frac{\rho h^2 r}{2m^2}\left(\ln\frac{r}{\lambda} + b\right), \tag{2}$$

$$p = \pi\rho h r^2 / m, \tag{3}$$

where ρ is the density of the liquid, m is the mass of the particle, r is the radius of the vortex ring, λ is the correlation length and determines the dimension of the vortex core, and b is a constant on the order of unity and depends on the structure of the vortex core. It is seen from (2) and (3) that ϵ/p decreases monotonically with radius r. The critical velocity was assumed to be ϵ/p for r equal to the transverse dimension of the system. In this case, however, the influence of the walls must be taken into account when ϵ is determined. Calculations within the framework of classical hydrodynamics[4,5] have shown that the energy of the vortex in a cylindrical channel of radius R depends as $r \to R$ on the structure of the vortex core, and in particular can vanish[4], meaning the vanishing of the Landau critical velocity.

It seems useful in this connection to consider this question in the model of a weakly non-ideal Bose gas, a model used, starting with the papers of Pitaevskiĭ[6] and Gross[7], for quantum-mechanical calculations of the structure of the vortex core. This is done in the present paper (Sec. 1).

The question of the vortex-ring energy as $r \to R$ was considered earlier by Fetter[8], who obtained, in the weakly non-ideal Bose-gas model, a finite value for the vortex energy at $d = R - r \to 0$, which he equated to half the energy of a pair of vortices located far from the wall and separated by distance 2d. This method of estimating the vortex energy is suitable, however, only so long as $d \gg \lambda$. A variational estimate of the energy of a vortex located at a distance $d \ll \lambda$ from the wall, where the wave function vanishes, is made in Sec. 1 and shows that as $d \to 0$ the vortex energy vanishes, and consequently the Landau critical velocity also vanishes for the vortex.

Another question that called for additional analysis was that of the physical meaning of the vortex momentum defined by expression (3). The point is that when energy and momentum are defined in classical hydrodynamics, one usually considers a liquid that is at rest at infinity. With such boundary conditions it follows directly from the mass conservation law for an incompressible liquid in a channel with a finite cross section that the average velocity over any cross section of the channel vanishes both in the presence and in the absence of vortices. Therefore the total momentum of the liquid also vanishes, and expression (3) defines the Kelvin momentum, which is the time integral of the resultant system of forces that cause the liquid to go over into the corresponding vortex motion from the state of rest[3,9].

Such a definition of p in (3) is not satisfactory, since it is precisely the finite-state momentum and not the Kelvin momentum which is important in the determination of the critical velocities. Some workers[10,11] have therefore expressed doubts concerning the validity of the use of expression (3) to determine the critical Landau velocity.

The question of the vortex momentum is considered in Sec. 2 for the case of a liquid flowing in an annular channel. In quantum hydrodynamics, it is necessary in this case to satisfy a periodic boundary condition for the phase of the superfluid liquid, namely, the phase should acquire an increment equal to a multiple of 2π on going around the ring. It follows from this condition that an incompressible liquid cannot be immobile far from the vortex. The liquid undergoes translational motion, and this motion is what determines the total momentum of the vortex state of the liquid. In the general case, the latter momentum is given by expression (3) with an increment that is a multiple of the quantity $Nh/L = Sh\,\rho/m$, where N is the total number of particles and S is the channel cross section. In other words, the Kelvin momentum is the momentum of one of the possible vortical states.

Thus, the investigation carried out in Secs. 1 and 2 shows that the vortex states always include some states in which the transitions from the ground state are energywise allowed for arbitrarily small velocity of the liquid, i.e., the critical Landau velocity for the vortices vanishes.

On the other hand, in order for such transitions to be able to lead to damping of superfluid flow, their probability must be quite high. However, as was first shown by Vinen[12,13], the production of a large vortex ring is an event of rather small probability, since it is connected with the change of the state of the liquid in a large volume. An estimate of the matrix element of the transition to the vortical state, which is given in the present paper (Sec. 2), confirms Vinen's argument.

It should be noted that the wave function of the vortical state is usually chosen from the condition that the energy be at minimum (see Sec. 1) at a fixed position of the vortex line (i.e., the line where the density of the superfluid component vanishes and the bypassing of which causes the phase to increase by 2π). This vortical state will be called stationary, since the time derivative of the density is equal to zero, and the spatial distribution of the phase is the same as in an incompressible quantum liquid. Since the appearance of a vortex in such a liquid, according to Sec. 2, is accompanied by the appearance of motion in the entire volume of the liquid, the probability of a transition to such a state is very low. Therefore the lifetime of the flows state at velocities lower than the Landau critical velocity is determined by those transitions to the vortical states for which the spatial distribution of the phase is determined not by the minimum of the energy, but by the maximum probability of the transition from the ground state, provided that this transition is energywise allowed at the specified velocity of the liquid. In a vortex state defined in this manner, the liquid is in motion only near the vortex line, and the time derivative of the density differs from zero, although the density does remain constant everywhere except at the core of the vortex. Such vortex states will therefore be called nonstationary or deformed. Although the transition probability is much higher in these states than in stationary vortex states, it is still much too small to explain the experimentally observed critical velocities.

A way out of the situation produced here can be sought in either of two directions. First, we can attempt to seek other types of excitations capable of yielding the experimentally observed critical velocities. At the present time, however, neither theory nor experiment is in possession of data on the presence of such excitations.

Second, one can hope to obtain sufficiently low critical velocities by forgoing the simple Landau theory proposed for absolute zero temperature and taking into consideration the existence of a certain number of excitations in the liquid at temperatures corresponding to the experimentally measured critical velocities. Such a possibility was already pointed out by Cooper[14]. This raises the question of the temperature limit up to which the simple Landau theory can be used.

This question is discussed in Sec. 4 for a liquid in an annular channel. For such a channel, the present author obtained the dependence of the free energy on the momentum at $T = 0$ and $T > 0$ in an earlier study[15]. An analysis of these relations shows that the normal component of the liquid can be disregarded only at very low temperatures, or as long as

$$\rho_n / \rho \ll \lambda / L, \tag{4}$$

where L is the length of the annular channel and ρ_n and ρ are the normal and total densities of the liquid.

If condition (4) is satisfied, it is legitimate to use the Landau theory, which attributes the destruction of the superfluidity to transitions from the ground state with transfer of the momentum to the channel walls. However, all the critical-velocity measurements known to this author were performed at higher temperatures, when condition (4) is not satisfied. In this case the lifetime of the flow state can be determined by interactions between already existing excitations. If this is indeed the case, then one should expect that lowering the temperature will lead to an increase of the critical velocities up to the values of the Landau critical velocity for the phonon-roton spectrum.

The concluding section of the article discusses conditions under which the indicated increase in the critical velocities might be observed.

1. WAVE FUNCTION OF VORTICAL STATE OF WEAKLY NON-IDEAL BOSE GAS, AND VORTEX ENERGY NEAR THE WALL

To describe the condensed state in the model of the non-ideal Bose gas, we introduce the N-particle wave function

$$\Psi^N = \prod_{i=1}^{N} \frac{\Psi(\mathbf{r}_i)}{\sqrt{N}}, \tag{5}$$

where the condensate wave function $\Psi(\mathbf{r}) = f(\mathbf{r})\exp(i\varphi(\mathbf{r}))$ is normalized to the total number of particles, and the equations for the real amplitude $f(\mathbf{r})$ and for the phase $\varphi(\mathbf{r})$ (the Gross-Pitaevskiĭ equations)

$$-\frac{\hbar^2}{2m}\nabla^2 f - \frac{\hbar^2}{2m} f(\nabla\varphi)^2 + gf^3 - \mu f = 0, \tag{6}$$

$$2f\nabla f\nabla\varphi + f^2\nabla^2\varphi = 0 \tag{7}$$

are obtained by minimizing the functional

$$\Phi = E - \mu N = \int\left\{\frac{\hbar^2}{2m}(\nabla f)^2 + f^2(\nabla\varphi)^2 + \frac{g}{2}f^4\right\}d\tau - \mu\int f^2\,d\tau, \tag{8}$$

where

$$g = \int V(\mathbf{r})\,d\tau,$$

$V(\mathbf{r})$ is the short-range two-particle interaction potential, $d\tau$ is the volume element, and μ is the chemical potential. For the vortical state, the amplitude $f(\mathbf{r})$ vanishes on the vortex line, and the phase $\varphi(\mathbf{r})$ acquires an increment 2π on going around this line.

Let us consider a straight-line vortex parallel to a plane wall located at a distance d away. Let the wall coincide with the yz plane, let the z axis be parallel to the vortex, and let the vortex lie in the xz plane, i.e., we have a two-dimensional problem with coordinates $x = 0$ for the wall and $x = d$ and $y = 0$ for the vortex.

The boundary condition at the wall is used in the form $\Psi = 0$, i.e., $f = 0$.[1)] In the absence of a vortex, i.e., in the ground state, Eqs. (6) and (7) have an exact solution[16]

$$\varphi = \text{const}, \quad f_0 = (\rho/m)^{1/2} \operatorname{th}(x/\sqrt{2}\lambda), \tag{9}$$

where ρ is the density of the liquid far from the boundary and the correlation length λ and the chemical potential μ are determined by the expressions

$$\lambda = \hbar/\sqrt{2g\rho}, \quad \mu = g\rho/m. \tag{10}$$

If the vortex is far from the wall ($d \gg \lambda$) and the liquid at infinity is at rest, then the phase φ is the imaginary part of a function that is analytic in $w = x + iy$:

$$\varphi = \operatorname{Im}\{\ln(w-d) - \ln(w+d)\}. \tag{11}$$

The asymptotically exact solution for the amplitude f at distances much larger than λ from the wall and from the vortex line is of the form

$$f = (\rho/m)^{1/2}(1 - 2d^2\lambda^2/r_1^2r_2^2), \tag{12}$$

where $r_1 = [(x-d)^2 + y^2]^{1/2}$ and $r_2 = [(x+d)^2 + y^2]^{1/2}$ are the distances to the vortex and to its image. Matching the solution (12) to the numerical solution of Ginzburg and Pitaevskiĭ[16] for an isolated vortex, we obtain the energy of the vortex

$$\varepsilon = \frac{1}{4\pi} H \frac{\hbar^2\rho}{m^2}\left(\ln\frac{2{,}92d}{\lambda} + O\left(\frac{\lambda}{d}\right)\right), \tag{13}$$

where H is the length of the vortex and $O(\lambda/d)$ is a quantity of the order of λ/d.

In the opposite limiting case $d \ll \lambda$, it is impossible to obtain an exact expression for the vortex energy, but for our purposes it suffices to estimate the upper bound of the energy of the vortex state by a variational method, by choosing suitable trial functions for f and φ:

$$f = f_0(1 - \exp(-r_1/l)) \approx (x/\sqrt{2}\lambda)\sqrt{\rho/m}[1 - \exp(-r_1/l)], \tag{14}$$

$$\varphi(w) = \operatorname{Im}\{\ln(w-d) - \ln(w+b) + (b+d)/(w+a)\}, \tag{15}$$

where a, b, and l are constants subject to variation and f_0 is the amplitude f without the vortex as defined by (9).

Substituting (14) and (15) into (8) and subtracting the value of Φ for the ground state, we obtain an expression for the vortex energy in which the main contribution is made at $d \ll \lambda$ by the first two terms in (8), with gradients ∇f and $\nabla\varphi$:

$$\begin{aligned}\varepsilon = \frac{H\hbar^2\rho}{4m^2\lambda^2}\int_0^\infty dx \int dy \Big\{\Big[1 - \exp\left(-\frac{r_1}{l}\right)\Big]^2 - 1 \\ + \frac{2x(x-d)}{lr_1}\Big[\exp\left(-\frac{r_1}{l}\right) - \exp\left(-\frac{2r_1}{l}\right)\Big] + \frac{x^2}{l^2}\exp\left(-\frac{2r_1}{l}\right) \\ + x^2\Big[1 - \exp\left(-\frac{r_1}{l}\right)\Big]^2\left|\frac{1}{w-d} - \frac{1}{w+b} - \frac{b+d}{(w+a)^2}\right|^2\Big\} \\ = \frac{H\hbar^2\rho d^2}{2m^2\lambda^2} G\left(\frac{l}{d}, \frac{a}{d}, \frac{b}{d}\right),\end{aligned} \tag{16}$$

where $g(l/d, a/d, b/d)$ is a function that depends only on the dimensionless constants l/d, a/d, and b/d.

For the purposes of the present work, there is no need to calculate the integrals that determine $G(l/d, a/d, b/d)$ (they all converge) and then to minimize with respect to a, b, and l. Expression (16) shows that ϵ tends to zero like d^2 as $d \to 0$. The same can be stated also concerning an annular vortex of radius r in a cylindrical tube of radius R as $d = R - r \to 0$, the energy of which is determined by expression (16) at $H = 2\pi r$.

Thus, the energy of the vortex can vanish at a finite value of the momentum, i.e., the critical Landau velocity is zero for vortices.

2. MOMENTUM OF VORTEX IN QUANTUM HYDRODYNAMICS

Let us determine the momentum of the vortex for a quantum liquid in a ring with average circumference L. If the thickness of the ring is much smaller than L, then we can replace the annular channel, with sufficient accuracy, by a straight channel of length L, with periodic boundary conditions imposed on its two ends. The total momentum of the liquid at a constant density of the liquid is

$$\mathbf{P} = N\hbar\int \nabla\varphi d\tau. \tag{17}$$

Let a vortex ring be present initially in the channel, and let the phase φ acquire an increment 2π on going around the vortex line. To make the phase φ unambiguous, we draw a cut in the form of a piece of the surface passing through the vortex line and lying inside it. The volume integral in (17) can then be transformed into a surface integral over the boundary of the region and over the cut, and as a result we obtain for the momentum component along the channel axis[10]

$$P = \frac{\rho}{m}\hbar[(\varphi_2 - \varphi_1)S \pm 2\pi S_v], \tag{18}$$

where the sign of S_V depends on the sign of the circulation, φ_1 and φ_2 are the values of the phase at the ends of the channel, and S_V is the area of the projection of the cut on the plane yz (the x axis coincides with the channel axis).

If the incompressible liquid is at rest at infinity, then $P = 0$ (see the Introduction), whence $\varphi_2 - \varphi_1 = \pm 2\pi S_V/S$. Thus, $\varphi_2 - \varphi_1$ is not an exact multiple of 2π. To satisfy the periodic boundary conditions, it is necessary to impart translational motion to the liquid, and it is this motion which determines the momentum of the vortical state

$$P = 2\pi(k \pm S_v/S)\frac{S\hbar\rho}{m}, \tag{19}$$

where k is an integer.

If $k = 0$, then we obtain the Kelvin momentum from (19). If $S_V \to S$, i.e., the vortex vanishes on the wall, then the only thing present in the entire volume is translational motion that differs by a single momentum quantum $2\pi S\rho\hbar/m$ from the state at $S_V = 0$.

Expression (19) shows that purely translational motion of the Bose condensate can appear only at quantized values of P that are multiples of $2\pi S\rho\hbar/m$; this is a manifestation of the quantization of the angular momentum of the liquid in the vortical channel, and is analogous to the quantization of the magnetic moment in superconductors.

The non-quantized values of the total momentum P correspond to vortical states. In order that the energy of such a state be minimal at a given P, the length of the vortex line should be minimal at a given area S_V of the projection of the cut connected with this vortex line, since the energy is proportional to the length of the vortex line. Therefore vortices that begin and end on the walls of a channel whose vortex lines are circular arcs perpendicular to the walls of the channel and lie in a section perpendicular to the channel axis have the lowest energies.

It is obvious that expression (19) determines the summary momentum at any spatial distribution of the phase φ. It can be only continuous and obtain an increment 2π on going around a vortex filament and $2\pi k$ on going around a ring. In particular, it is applicable also to a nonstationary state of a compressible liquid for those instants of time when the density of the liquid is constant in space, but the average velocity of the liquid is different in different cross sections of the channel, and therefore the time derivative of the density differs from zero. Among these states are the deformed vortical states considered in Sec. 3. For these states, the velocity of the liquid differs from zero only near the vortex line, and there is no translational motion far from this line.

3. LIFETIME OF CURRENT STATE AT T = 0 AND AT LOW VELOCITIES

We consider a liquid that moves in a closed annular channel with a certain velocity v and is in the ground state in a reference frame that moves with the same velocity. If we use the model of the non-ideal Bose gas to describe such a liquid, then stagnation can occur in such a model as a result of two different types of quantum-mechanical transitions.

First, quasiparticles (bogolons) can be produced in the gas, and then the change of the summary momentum is accompanied by a decrease in the number of particles of the condensate.

Second, the average velocity of the superfluid component can change, in which case all N particles remain in the condensate but the condensate function that determines the N-particle wave function (5) is altered.

For the transitions of the first type, the critical Landau velocity is large enough and equal to the velocity of sound. Therefore at velocities lower than the Landau critical velocity for the quasiparticles, the time during which the stagnation of the liquid takes place is determined by transitions of the other type, which, as follows from the preceding sections, are energywise allowed for an arbitrary low liquid velocity. The estimate presented below, however, shows that the probability of such transitions decreases rapidly with decreasing velocity.

Transitions between states corresponding to different momenta can occur only via interaction with the surrounding medium; this interaction violates the translational invariance of the Hamiltonian. Such an interaction can be represented with sufficient degree of generality in the form of the sum of M single-particle potentials acting on the N particles of the liquid:

$$H_{int}=\sum_{i=1}^{N}\sum_{j=1}^{M}V_j(\mathbf{r}_i), \tag{20}$$

$V_j(\mathbf{r}_i)$ is the potential of the j-th scattering center acting on the i-th particle of the liquid. The scattering centers can be impurities or roughnesses on the walls.

The transition probability is proportional to the square of the modulus of the matrix element of the transition and can be represented in the form:

$$W=A'\,|\langle\Psi_0{}^N|H_{int}|\Psi_1{}^N\rangle|^2 =A'\left|\sum_{j=1}^{M}\langle\Psi_0\,|V_j(\mathbf{r}')|\,\Psi_1\rangle\left(\frac{\langle\Psi_0|\Psi_1\rangle}{N}\right)^{N-1}\right|^2=A\exp(-\Gamma), \tag{21}$$

where the constant A' is determined by the density of the final states, Ψ_0 and Ψ_1 are the condensate functions of the initial (ground) and final (vortical) states, and the argument Γ of the exponential is determined by the overlap integral $\langle\Psi_0|\Psi_1\rangle = \int\Psi_0^*\Psi_1 d\tau$:

$$\Gamma=-2N\,\mathrm{Re}\left(\ln\frac{\langle\Psi_0|\Psi_1\rangle}{N}\right). \tag{22}$$

On the other hand, the quantity $\exp(-\Gamma)$ is none other than the square of the modulus of the scalar product $|\langle\Psi_0^N|\Psi_1^N\rangle|^2 = |N^{-1}\langle\Psi_0|\Psi_1\rangle|^{2N}$. The reason why the wave functions Ψ_0^N and Ψ_1^N are not orthogonal is that the model is approximate. The point is that, strictly speaking, Ψ_1^N is not an exact eigenfunction of the total-momentum operator, although the quantum fluctuations of this quantity are exceedingly small.

Apparently, however, at large Γ the errors connected with the non-orthogonality affect only the pre-exponential factor in (21). This is confirmed by an estimate of the matrix element for the transition from Ψ_0^N to the wave function $\tilde{\Psi}_1^N = \Psi_1^N + \alpha\Psi_0^N$, where the constant α, whose absolute value is equal to $\exp(-\Gamma/2)$, is chosen from the condition that Ψ_0^N and Ψ_1^N be orthogonal. Such an orthogonalization does not affect the argument of the exponential. Thus, the probability is determined principally by the overlap integral.

If the condensate function Ψ_1 is chosen from the condition that the energy be a minimum and satisfies the Gross-Pitaevskiĭ equation, then we obtain exceedingly small values of W, inasmuch as in accord with Sec. 2 motion is produced in this case in the entire volume, and the argument Γ of the exponential turns out to be proportional to the length of the channel.

To find the quantum-mechanical transitions that are most effective for the destruction of superfluidity, we seek a condensate function Ψ_1 that maximizes the overlap integral $\langle\Psi_0|\Psi_1\rangle$ under the condition that the transition to the corresponding vortical state[2)] Ψ_1^N is allowed by energy at the given velocity v of the liquid, i.e., the energy and the momentum of the vortical state satisfy the condition $\epsilon/p = v$. Under this condition, the largest values of the overlap integral, like the smallest values of the energy, are obtained for vortex lines that are circular arcs perpendicular to the walls. We consider below a case when the radius r of these circles is much smaller than the radius of curvature of the wall, which can therefore be regarded as plane, and the vortex line is a semicircle. The Kelvin momentum for such a vortical state is $p = (1/2)\pi r^2\rho h/m$.

The vortical state corresponding to the maximum of $\langle\Psi_0|\Psi_1\rangle$ will be called a nonstationary[3)] or deformed vortical state, in contrast to the stationary vortical state corresponding to the minimum of the energy. The condensate wave function $\Psi_1 = f_1\exp(i\varphi_1)$ of the deformed state is determined from the condition that the functional

$$\Phi_1=\left|\sqrt{\frac{\rho}{m}}\int\exp(i\varphi_1(\mathbf{r}'))f_1(\mathbf{r}')d\tau\right|-z_0\frac{\varepsilon}{p}, \tag{23}$$

where z_0 is a Lagrange multiplier, be maximal.

Variation of the amplitude f_1 of the condensate function cannot lead to any noticeable increase of $\langle\Psi_0|\Psi_1\rangle$. We shall therefore vary only the phase φ_1, assuming that f_1 assumes the same values as for a stationary vortical state.

Without loss of generality, we can assume φ_1 to be an odd function of x (the x axis is directed along the axis of the vortex ring) that vanishes far from the vortex and has a cut over a flat semicircle bounded by the vortex ring. From the condition of the maximum of Φ_1, we obtain

$$\nabla^2\varphi_1 = \sin\varphi_1 / l^2, \quad l^2 = \hbar^2 z_0 / pm. \tag{24}$$

We can solve (24) and find Γ by using the condition $\lambda \ll l \ll r$, which is valid at $u \ll c$ ($c = \hbar/\sqrt{2}\lambda m$ is the speed of sound).

The constant-phase lines in a section passing through the channel axis are shown for this case in Fig. 1. The main contribution to the overlap integral $\langle\Psi_0|\Psi_1\rangle$ and to the energy are made by regions I and II:

I. Here $\varphi_1(\mathbf{r}')$ varies only along the x axis, and Eq. (24) becomes one-dimensional; when the boundary conditions at $x = 0$ and $x = \infty$ are taken into account, we have for φ_1 the following solution at $x > 0$:

$$\varphi_1 = 4\,\mathrm{arctg}\left[-\left(\mathrm{tg}\frac{\pi}{4}\right)\exp\left(-\frac{x}{l}\right)\right]. \tag{25}$$

This region makes the principal contribution to the expression for $\langle\Psi_0|\Psi_1\rangle$ and to the energy ϵ; this contribution is proportional to r^2.

II. Region near a vortex line, with a transverse dimension of the order of l. Its contribution to $\langle\Psi_0|\Psi_1\rangle$ is of the order of l/r relative to the contribution from region I, and is therefore discarded. This region, however, makes an appreciable contribution to the energy. Near the vortex line, the right-hand side of (24) can be discarded, and the spatial distribution of the phase φ_1 turns out to be the same as for a stationary vortical state. As a result, the contribution to the energy from this region II turns out to be proportional to $r\ln(l/\lambda)$.

We finally obtain the expressions

$$\langle\Psi_0|\Psi_1\rangle = N(1 - 2\pi r^2 l / LS), \tag{26}$$

$$\varepsilon = \frac{2\pi\hbar^2\rho r^2}{m^2 l} + \pi^2\rho r\frac{\hbar^2}{m^2}\ln\frac{l}{\lambda}. \tag{27}$$

The relative error of these expressions is of the order of $1/\ln(l/\lambda)$.

It is next necessary to find the extremum of the functional Φ_1 with respect to r, and then to express the Lagrange multiplier z_0 and the length l associated with it in terms of the velocity v, after which we obtain

$$l = \frac{6\hbar}{\pi m v}, \quad r = \frac{\pi l}{4}\ln\frac{l}{\lambda} = \frac{3\hbar}{2mv}\ln\frac{\hbar}{\lambda m v}, \tag{28}$$

$$\varepsilon = \frac{3\pi^2\hbar^2\rho r}{m^2}\ln\frac{r}{\lambda}, \tag{29}$$

$$\Gamma = 27\frac{\hbar^3\rho}{m^4 v^3}\left(\ln\frac{\hbar}{\lambda m v}\right)^2. \tag{30}$$

Although the formula for Γ was obtained in the model of a weakly non-ideal Bose gas, it is apparently valid also for helium[4)]. In this case the N-particle wave function is usually expressed[2,13] in the form

$$\Psi_1^N = \prod_i \exp(i\varphi_1(\mathbf{r}_i))\Psi_0^N$$

where the wave function Ψ_0^N of the ground state is no longer equal to a constant, as in the model of a weakly non-ideal Bose gas. For estimation purposes we can choose Ψ_0^N in the form a wave function that vanishes if any two particles have approached each other to a distance shorter than a, and is equal to a constant in the remaining part of configuration space. Such a wave function was used by Penrose and Onsager[20] to estimate the number of particles of the condensate in the helium, and a is of the same order as the distance between molecules.

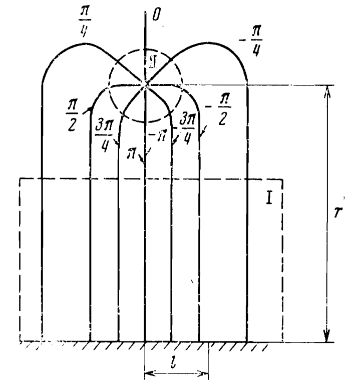

FIG. 1. Constant-phase lines in a section passing through the axis of a vortical half-ring, for a nonstationary vortical state.

Inasmuch as a noticeable change of the phase $\varphi_1(\mathbf{r}')$ occurs at distances of the order of l, then at l greatly exceeding the intermolecular distance the wave function Ψ_0^N, as a function of the coordinates of one of the particles at a fixed position of the remaining particles, goes through very many oscillations over the length l, and it can be replaced by a mean value. Therefore, performing the calculations in the limit of low velocities v, we again obtain expression (30).

Although the value of Γ for the phonon-roton spectrum in helium is not very large at velocities on the order of the Landau critical velocities, it increases quite rapidly with decreasing v. Thus, $\Gamma = 10^4$ at $v = 10$ m/sec, leading to a vanishingly small probability of such a transition for any reasonable value of the pre-exponential factor A in expression (21) for the transition probability.

4. DESTRUCTION OF SUPERFLUIDITY AT FINITE TEMPERATURES

The entire exposition in the preceding sections pertained to the case $T = 0$, when there is no normal component of the liquid. On the other hand, the presence of excitations in the liquid can greatly influence the stability of the flow states.

Let us examine the conditions necessary to realize undamped currents in a ring, both at $T = 0$ and at $T > 0$.

Undamped currents of a superfluid liquid in a ring can be connected with the presence of minima on the dependence of the free energy on the current (or the total momentum), just as in the case of superconducting rings, which were considered by Byers and Yang[21]. Such a dependence for a quantum Bose liquid was obtained in[15]. Figure 2 shows the dependences on the momentum p of the free energy F(P, v) of a subensemble of microstates with given translational superfluid velocity. Since no vortical states were considered in[15], v assumed only quantized multiple values of $\hbar/mL$. The total free energy

$$F(P) = -kT\ln\sum_v \exp(-F(P,v)/kT)$$

accurate to kT, is the envelope of the family of curves F(P, v) (the curve $m_0t_0m_1t_1m_2t_2m_3$ on Fig. 2a and the curve $m_0t_0m_1t_1m_2t_2$ on Fig. 2b). On each F(P, v) curve there is a minimum at the point m_i, so long as v does not exceed the critical Landau velocity for the quasipar-

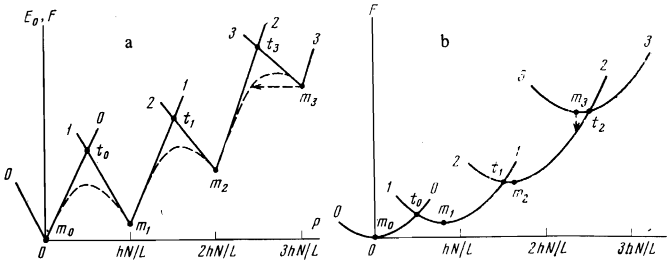

FIG. 2. Dependence of the free energy on the momentum: a) T = 0, b) T > 0. The lines 0, 1, 2, and 3 show the free energy F(P, v) respectively for the values v = 0, h/mL, 2h/mL and 3h/mL.

ticles. At T = 0, these minima are also minima on the envelope of the F(P) curve, which represents the minimum energy $E_0(P)$ at a given summary momentum P. At T > 0, however, the minima on the F(P) curve vanish at sufficiently low superfluid velocities, not exceeding the quantity $v_0 = h\rho/2mL\rho_n$.

Using the analytic expressions for F(P, v)[15], we can easily verify that the free energy F(P) has the same form as at T = 0, so long as the maximum momentum $LS\rho_n v_L$ that can be carried by the normal component in a reference frame moving with superfluid velocity stays much smaller than the distance $Nh/L = S\rho h/m$ between the minima on the $E_0(P)$ curve. If it is recognized that v_L is of the same order as $h/m\lambda$, then this condition yields the inequality (4).

Thus, when condition (4) is satisfied, the minima on the F(P) curve take place up to the critical Landau velocity v_L for the quasiparticles. These minima are also retained when the vortices are taken into account. The vortices are states with the lowest energy at a given momentum, if the vortex radius $r > \lambda$ (see the dashed curve in Fig. 2a). Near quantized values of P, however, the lowest energy $E_0(P)$ corresponds to quasiparticle excitations. Thus, allowance for the vortices is equivalent to cutting off the peak obtained on the $E_0(P)$ curve with allowance for only the quasiparticle excitations. The vortex-production process considered in Sec. 3 corresponds to a quantum-mechanical transition under the barrier on the $E_0(P)$ curve; this transition is shown by the arrow in Fig. 2a.

In the absence of minima on the F(P) curve, the metastable flow states can correspond to minima on the F(P, v) curves. The lifetime of the flow state is determined in this case by the probability of momentum transfer from the superfluid component to the normal component at a constant total momentum, i.e., by the probability of the transition into a subensemble with a lower value of v (shown by the arrow in Fig. 2b). For such transitions there is no need to introduce an interaction of the type (20), which violates the translational invariance of the system. Therefore the estimates made in Sec. 3 may turn out to be inapplicable for such a situation.

The experimental critical velocities greatly exceed v_0 in magnitude, i.e., the long-lived flow states exist where there are no minima on the F(P) curves. To estimate the critical velocities observed in the experiment, it is therefore necessary to have a theory that considers the process of momentum transfer from the superfluid to the normal component.

Such a theory, in which it is assumed that the activation energy barrier is overcome by thermal fluctuations with gradual growth of the vortex ring, was constructed in[22-26]. Near the λ point, where this barrier becomes small, the theory agrees with experiment. At lower temperatures, however, the times of production of large vortices as a result of thermal fluctuations become quite long and cannot explain the experimentally observed low critical velocities. It is possible that the discrepancy between theory and experiment at finite temperatures is caused by failure to take into account the interaction between the vortices, which is long-range and can become appreciable even at sufficiently low vortex concentrations. If this is so, then to understand the critical velocities it is necessary to construct a theory of superfluid turbulence, a phenomenological variant of which was proposed by Vinen[12]. At very low temperatures, however, when condition (4) is satisfied, neglect of the interactions between the vortices is not questionable, for in this case the momentum that can be carried by the excitations (including also the vortices) is insufficient to produce even one large vortex.

5. CONCLUSION

The estimates presented in the present paper show that at T = 0 the rate of vortex formation in a superfluid liquid remains exceedingly low all the way up to the critical Landau velocities v for the quasiparticle spectrum. This does not contradict the experimentally observed rather low critical velocities, since the measurements were performed at temperatures when the aforementioned estimates do not hold and the critical velocities are determined by the interaction between the normal and superfluid components. For experimental verification of the theory of critical velocities at absolute zero, it is necessary to perform the measurements in a region where inequality (4) is satisfied. This inequality is the condition for total vanishing of the normal component.

Let us see under what conditions the inequality (4) can be satisfied and sufficiently high critical velocities, close to v_L, can be obtained. When undamped currents are investigated in an annular channel of length L $L \sim 1$ cm, the inequality (4) is satisfied at $\rho_n/\rho < 10^{-7}$. The temperature required for this is $T < 0.2°$K.

Although we have considered mainly a liquid in a closed channel with periodic boundary conditions, all the results remain in force in the more general case when a definite phase difference is maintained at the ends of the channel, for example, if the channel joins two large vessels whose phases are specified. This takes place, for example, when liquid flows through a small hole in a thin partition. Investigations of such flows were initiated in connection with the possibility of observing the Josephson effect for helium. The effective length of the channel over which the principal change of phase takes place should be a length on the order of the hole diameter in this case[27]. It can be chosen low enough to be able to satisfy inequality (4) at higher temperatures. Thus, for the holes used in the experiments of Richards and Anderson[28] (diameter 15–20 μ), and also of Hulin et al.[29], the equality $\rho/\rho_n = L/\lambda$ can be attained at $T \sim 0.9°$K, whereas the measurements were performed at $T = 1.15°$K. The rather large critical velocities observed in such experiments may possibly be due to the start of a transition into the region of higher critical velocities. The theoretical interpretation of such experiments is, however, much more complicated than that of experiments in closed annular channels with constant cross section. In particular, the velocity of the liquid is not constant over the volume and can reach critical Landau velocities near the sharp edges and corners on the walls of the channel even at quite low average velocities.

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[1)] The condensate wave function Ψ has the same physical meaning as the complex order parameter Ψ in the phenomenological Ginzburg-Pitaevskii theory. In this story, the boundary condition $\Psi = 0$ can be used not only at a solid wall [16], but also on a free surface [17].

[2)] Since the purpose of the calculation is to determine the time of transfer of the momentum to the walls, the subsequent rate of evolution of Ψ_1^N to the state with lowest energy is immaterial.

[3)] In spite of the nonstationarity, the energy of such a state is determined sufficiently accurately, as can be verified easily by calculating the quantum fluctuations of the energy with the aid of (5). The transition to such a state can therefore be regarded as if this state were stationary.

[4)] Apart from a numerical factor, formula (30) gives lower values in $\ln(h/\lambda mv)$ than the formula obtained for the argument of the exponential by Volovik [18], who considered the production of the vortex as a result of inhomogeneities of macroscopic scale, i.e., with dimensions exceeding the length λ. Using the ideas developed in [19], Volovik solved in a quasiclassical approximation the one-dimensional quantum problem, the Hamiltonian of which constitutes the energy of the stationary vortex for a pair of canonically conjugate variables (radius of the vortex ring and its coordinate along the channel axis).

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