Theory of magnetization self-oscillations on parametric excitation of spin waves

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It is shown that low-frequency magnetization self-oscillations can be explained by instability of collective degrees of freedom. The latter are space- and time-periodic variations of the amplitudes and phases of coupled pairs of parametric spin waves which in a certain sense are analogous to second-sound waves in a magnon system. The spectrum of the collective oscillations is studied and conditions are determined for appearance of the self-oscillations, particularly as applied to YIG. Nonlinear self-oscillation conditions are studied by means of an electronic computer. The theory and experiments are compared qualitatively. Some properties of the self-oscillations are explained, in particular their giant crystallographic anisotropy in almost isotropic cubic ferromagnets.

1. INTRODUCTION

It is well known that, as a rule, no stationary regime is established in ferromagnets when spin waves are parametrically generated, and that the magnetization describes complicated oscillations about a certain mean value. This phenomenon is observed in the form of amplitude and frequency modulations of the pump power passing through a resonator with a sample, or in the form of low-frequency (LF) oscillations in a coil that responds to the change of the longitudinal component of the magnetization (M_z). The self-oscillations accompanying parametric excitation of spin waves were first observed by Hartwick, Peressini, and Weiss[1]. This inexplicable phenomenon aroused lively interest and was therefore extensively investigated experimentally (see[2]). The first enthusiasm soon disappeared, however, after it turned out that the properties of the self-oscillations depend on practically all the possible parameters of the system, namely the pump power, the magnetic field intensity, the crystallographic anisotropy, the dimensions and shape of the sample, etc. Doubts were expressed concerning the possible existence of any simple regularities in this region.

The main experimental facts obtained for perfect yttrium iron garnet (YIG) crystals with parallel pumping consist in the following.

1. The self-oscillation frequencies lie in the interval from 10^4 to approximately 10^7 Hz (depending on the pump power and on the constant magnetic field). At a slight excess above threshold, the self-oscillation spectrum consists of one line, and when the pump level increases, the number of lines increases and they shift towards higher frequencies. At large excesses above threshold, the spectrum has a noiselike character.

2. The threshold of the self-oscillations is usually quite small, 0.1—1 dB relative to the threshold of the parametric excitation, with the exception of the region of small wave vectors ($H > H_c$), where the threshold increases noticeably. The threshold also increases when internal inhomogeneities are introduced into the crystal[3].

3. A giant crystallographic anisotropy of the self-oscillation properties is observed; this anisotropy greatly exceeds the anisotropy of the spin-wave spectrum. The intensity of the self-oscillations in YIG with magnetization oriented along the [111] axis exceeds the intensity of these oscillations along the [100] axis by approximately 100 times[2].

The physical nature of the self-oscillations was considered in the literature from different points of view. Many authors (see, for example,[4]), have assumed that these oscillations are due mainly to changes in the magnetization following parametric excitation of the spin waves. Owing to the decrease of the magnetization (on the value of which the natural frequency of the wave depends) the waves that build up during the course of generation become gradually detuned and attenuate. The process then repeats, etc. This model was quantitatively analyzed by Monosov[2], who demonstrated the possible existence of self-oscillations in those cases when the change of the magnetization lags the change in the spin-wave amplitude, in accordance with the Bloch-Bloembergen relaxation model. Within the framework of these representations and certain additional assumptions, Monosov and co-workers succeeded in explaining a number of properties of the self-oscillations, for example, the excitation threshold and the initial oscillation frequency. However, the inertia of the magnetization, as shown by Bar'yakhtar and Urushadze[5], may turn out to be appreciable only in that part of the spectrum where the following decay processes are allowed:

$$\omega_p/2=\omega_k=\omega_{k+k'}+\omega_{k'}.$$

For YIG ($T = 300°K$, $\omega_p = 2\pi\cdot 10^{10}\ \mathrm{sec}^{-1}$) this is a narrow region near the saturation field H_S[6], $H_S < H_0 < H_S - 100$ Oe. In addition, the appreciably arbitrary character of the assumptions made leaves open the question of the conditions of applicability of the model of "inertial" self-oscillations even in the decay part of the spectrum.

Wang and co-workers[7] have proposed a different self-oscillation model, based on the postulated existence of a packet of parametric spin waves that are close in frequency (near $\omega_p/2$). In this case a sample of finite dimensions acts like a Fabry-Perot resonator that selects a number of discrete frequencies separated by an amount $v_g\cdot 2\pi/d$ from the nonmonochromatic packet (v_g is the group velocity of the spin waves and d is the sample dimension). In the author's opinion, it is the "beats" between these frequencies which are the cause of the oscillations.

Much experimental work was done[7,8] for the purpose of confirming the dependence of the oscillation frequency on the constant magnetic field as given by the beat model. The authors note surprisingly good agreement. The beat picture, however, completely ignores the experimentally observed dependence of the frequency on the pump power. At the same time, the theoretical arguments concerning the parametric generation of spin waves, which were developed by Zakharov and one of us[9], favor the idea that the parametric waves have a high degree of monochromaticity, and at any rate that the scatter of their frequencies is much smaller than the reciprocal spin-wave relaxation time. It is shown in[9] that the low-frequency self-oscillations of the magnetization, which are observed in the course of parametric generation of spin waves, are due to the presence of new branches of collective excitations in the energy spectrum of the ferromagnet. These excitations constitute periodic changes, in time and in space, of the amplitudes of pairwise coupled parametric and spin waves (analogous in a certain sense to second-sound waves on magnons); their characteristic frequencies are determined by the magnitude of the interaction of the pairs with one another and usually lie in an interval $0-10^7$ sec^{-1}.

The existence of collective oscillations was experimentally observed[10] in a study of the reaction of a parametric spin system to a weak signal. In these experiments, resonant absorption of a weak electromagnetic field, due to excitation of the zeroth (homogeneous) mode of the collective oscillations, was observed against a background of parallel spin-wave pumping.

In addition to the homogeneous mode, in which all pairs oscillate in unison, there are many inhomogeneous oscillation modes and waves. As a rule, some of them turn out to be unstable and are self-excited even in the absence of a driving force. This instability occurs when the frequency of the collective oscillations vanishes, and in this sense it is similar to the instability of the "soft mode" in ferroelectrics. In ferroelectrics (under thermodynamic equilibrium!) the instability of the soft mode leads to a phase transition. In our case, which is far from thermodynamic equilibrium, there is no other stable state, and an instability of the soft-mode type leads to oscillations of the system about the ground state.

In the present paper we study the spectrum of the collective oscillations and obtain the conditions under which self-oscillations occur, principally in YIG. Next, a computer is used to analyze the nonlinear regime of the self-oscillations for the simplest modes. The depth of modulation of the ground state then turns out to be far from small and can reach ~100% in certain cases, thereby characterizing the excitation of the self-oscillation as a phenomenon of strong turbulence. We investigate the properties of such turbulence, namely the period and waveform of the oscillations, the character of motion of the system in k-space, and the transition to the stochastic regime (noise).

2. COLLECTIVE OSCILLATIONS

The interaction of spin waves plays a decisive role in parametric excitation: it limits the level of excitation of the spin waves and leads, as we shall show, to the appearance of collective oscillations against its background. In S-theory (see[11,12]), this interaction is described with the aid of a self-consistent field of coupled spin waves $a_{\mathbf{k}}$, $a_{-\mathbf{k}}$. The Hamiltonian of such a system is given by

$$\mathscr{H}=\int(\omega_{\mathbf{k}}-\omega_p/2)a_{\mathbf{k}}a_{\mathbf{k}}^*d\mathbf{k}+{}^1/_2\int(hV_{\mathbf{k}}a_{\mathbf{k}}^*a_{-\mathbf{k}}^*+\text{c.c.})d\mathbf{k}$$
$$+\int T_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}}a_{\mathbf{k}}^*a_{\mathbf{k}'}a_{\mathbf{k}'}^*d\mathbf{k}\,d\mathbf{k}'+{}^1/_2\int S_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}}a_{-\mathbf{k}}a_{\mathbf{k}'}^*a_{-\mathbf{k}'}^*\,d\mathbf{k}'. \tag{1}$$

Here $a_{\mathbf{k}}$ and $a_{-\mathbf{k}}$ are the "slow" wave amplitudes, $V_{\mathbf{k}}$ is the coefficient of their interaction with the parallel-pumping field $h(t) = h\exp(-i\omega_p t)$, and $T_{\mathbf{k}\mathbf{k}'}$ and $S_{\mathbf{k}\mathbf{k}'}$ are nonlinear characteristics of the ferromagnet. The function $T_{\mathbf{k}\mathbf{k}'}$ describes the natural frequency of the spin waves,

$$\tilde{\omega}_{\mathbf{k}}=\omega_{\mathbf{k}}+2\int T_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}'}a_{\mathbf{k}'}^*\,d\mathbf{k}', \tag{2}$$

renormalized to account for the interaction, while the functions $S_{\mathbf{k}\mathbf{k}'}$ characterize the "collective pumping," which combines (with suitable phase shift) with the external pump $hV_{\mathbf{k}}$:

$$P_{\mathbf{k}}=hV_{\mathbf{k}}+\int S_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}'}a_{-\mathbf{k}'}d\mathbf{k}'. \tag{3}$$

In spite of the fact that the external pump $hV_{\mathbf{k}}$ exceeds the wave dissipation $\gamma_{\mathbf{k}}$ beyond the threshold, the renormalization of the pump (3) leads to establishment of a stationary state in which the energy-balance condition is satisfied for each pair of waves:

$$|P_{\mathbf{k}}|^2=\gamma_{\mathbf{k}}^2+(\tilde{\omega}_{\mathbf{k}}-\omega_p/2)^2. \tag{4}$$

From among the set of stationary states (4), the state realized is the one having external stability with respect to the production of new pairs at all points of k-space where $a_{\mathbf{k}} = 0$. We call this the ground state; for this state we have (see[11])

$$|P_{\mathbf{k}}|=\gamma_{\mathbf{k}},\quad \tilde{\omega}_{\mathbf{k}}=\omega_p/2. \tag{5}$$

The theory of the ground state is contained in the papers of Zakharov and the present authors[11,12] and reduces briefly to the following. We introduce the correlation functions

$$n_{\mathbf{k}}=\langle a_{\mathbf{k}}a_{\mathbf{k}}^*\rangle,\quad \sigma_{\mathbf{k}}=\langle a_{\mathbf{k}}a_{-\mathbf{k}}\rangle,$$

which are convenient for the description of a system of parametric waves. The angle brackets denote averaging over the time or, equivalently, over the random phases of the individual waves. The phases of the waves $a_{\mathbf{k}}$ and $a_{-\mathbf{k}}$ in the pair are rigidly correlated (within the framework of the S theory), so that

$$|\sigma_{\mathbf{k}}|=n_{\mathbf{k}}.$$

The concrete form of the functions $n_{\mathbf{k}}$ and $\sigma_{\mathbf{k}}$ depends on the structure of the coefficients $V_{\mathbf{k}}$ and $S_{\mathbf{k}\mathbf{k}'}$. In an isotropic ferromagnet, and also in YIG with orientations $\mathbf{M}_0 \parallel [111]$ and $\mathbf{M}_0 \parallel [100]$, the ground state has axial symmetry about the magnetization direction $\mathbf{M}_0$. When the threshold is not greatly exceeded (up to 6–10 dB in YIG[13]), the spin waves are located in k-space in a plane $k_z = 0$ perpendicular to the magnetization; namely,

$$n_{\mathbf{k}}^{(0)}=\frac{N_0}{2\pi k_0}\delta(k_z)\delta(k_\perp-k_0),$$
$$\sigma_{\mathbf{k}}^{(0)}=n_{\mathbf{k}}^{(0)}e^{-2i\varphi}, \tag{6}$$

where φ is the azimuthal angle of the wave vector $\mathbf{k}_0$ and N_0 is the integral amplitude of the ground state:

$$N_0=\int n_{\mathbf{k}}^{(0)}d\mathbf{k}=\frac{[(hV)^2-\gamma^2]^{1/2}}{|S_0|}, \tag{7}$$

S_0 is the value of the function $S_{\mathbf{k}\mathbf{k}'}$ averaged over the angle φ (see formula (11) below).

We shall show that in a system of ferromagnetic spin waves described by the Hamiltonian (1) there exist collective oscillations against the background of the ground state, which we take for simplicity to be the state (6). This is easiest to show by separating in the Hamiltonian the part $\mathcal{H}^{(0)}$ corresponding to the ground state, and the parts $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$, which contain terms that are linear and quadratic in small deviations from the ground state. It is convenient to express the perturbations in the form

$$\sigma_{\mathbf{k}} = \sigma_{\mathbf{k}}^{(0)} + \frac{N_0^{1/2}}{2\pi k_0} \alpha_\varphi e^{-2i\varphi} \delta(k_z)\delta(\mathbf{k}_\perp - \mathbf{k}_0), \quad |\alpha_\varphi| \ll N_0. \tag{8}$$

In the study of the spectrum of the oscillations it is natural to neglect at first the dissipation (to put $\gamma = 0$) and to assume $\mathcal{H}$ to be an integral of the motion. Taking (5) into account, we verify that the energy of the ground state is extremal: $\mathcal{H}^{(1)} = 0$. The part of the Hamiltonian which is quadratic in the small perturbations (8) is given by

$$\mathcal{H}^{(2)} = \frac{N_0}{(2\pi)^2} \int_0^{2\pi} \{[2T_{\mathbf{k}\mathbf{k}'} + S_{\mathbf{k}\mathbf{k}'} e^{-2i(\varphi-\varphi')}] \alpha_\varphi \alpha_{\varphi'}^* + (T_{\mathbf{k}\mathbf{k}'} \alpha_\varphi \alpha_{\varphi'} + \text{c.c.})\} d\varphi \, d\varphi'. \tag{9}$$

Changing over to Fourier components

$$\alpha_m = \frac{1}{2\pi} \int_0 \alpha_\varphi e^{im\varphi} d\varphi$$

and using the axial symmetry

$$T_{\mathbf{k}\mathbf{k}'} = T(\varphi - \varphi'), \quad S_{\mathbf{k}\mathbf{k}'} = S(\varphi - \varphi'),$$

we obtain

$$\mathcal{H}^{(2)} = N_0 \sum_{m=-\infty}^{\infty} [(2T_m + S_m)\alpha_m \alpha_m^* + (T_m \alpha_m \alpha_{-m} + \text{c.c.})]; \tag{10}$$

$$T_m = \frac{1}{2\pi} \int_0^{2\pi} T(\varphi) e^{im\varphi} d\varphi,$$

$$S_m = \frac{1}{2\pi} \int_0^{2\pi} S(\varphi) e^{i(m-2)\varphi} d\varphi. \tag{11}$$

The quadratic Hamiltonian (10) reduces in standard fashion to the diagonal type

$$\mathcal{H}^{(2)} = 2N_0 \sum_m [S_m(2T_m + S_m)]^{1/2} \beta_m \beta_m^*, \tag{12}$$

which is the sum of the energies of the oscillators with frequencies

$$\Omega_m = 2[S_m(2T_m + S_m)]^{1/2} N_0. \tag{13}$$

These oscillators (normal modes) describe the collective excitations of the system of parametric spin waves. The coefficients S_m and T_m, which determine the spectrum of the collective excitations, depend on the pump frequency, magnetization, shape of the sample, and other experimental conditions (see formula (18) below). Under certain conditions there can occur an unstable situation in which $\Omega_m^2 < 0$ for any m. This instability, as will be shown in Sec. 4, leads to oscillations of the distribution functions $n_{\mathbf{k}}$ and $\sigma_{\mathbf{k}}$ about the ground state (6). It should be borne in mind that during the nonlinear stage of the development of the instability, the concept of individual normal modes becomes meaningless to a considerable degree. We are dealing with a strong nonlinearity of the collective oscillations, when the energy of their interaction with one another ($\mathcal{H}^{(3)}$ and $\mathcal{H}^{(4)}$) is of the same order as the self-energy $\mathcal{H}^{(2)}$. As a rule, therefore, the oscillation waveform deviates noticeably from harmonic.

Within the framework of the S theory, the collective oscillations of the system of parametric spin waves are spatially-homogeneous. When the spatial dispersion is taken into account, each normal mode corresponds to an entire branch of the spectrum $\Omega_m(\kappa)$; formula (13) determines its gap: $\Omega_m = \Omega_m(0)$. We shall confine ourselves henceforth to oscillations that are homogeneous in space and that interact most strongly with the electromagnetic pump field.

3. CONDITIONS FOR THE ONSET OF INSTABILITIES

Let us consider in detail the conditions for the instability of the spectrum of collective excitations. We ascertain first the effect exerted on this spectrum by the spin wave damping, all the more since the damping $\gamma_{\mathbf{k}}$ can be of the same order as the frequency of the collective oscillations (13). The damping is taken into account here within the framework of the canonical equations of motion

$$\frac{da_{\mathbf{k}}}{dt} + \gamma_{\mathbf{k}} a_{\mathbf{k}} = -i \frac{\delta \mathcal{H}}{\delta a_{\mathbf{k}}^*}. \tag{14}$$

Taking the Hamiltonian $\mathcal{H}$ in the form (1), we obtain

$$\frac{1}{2}\frac{dn_{\mathbf{k}}}{dt} + \gamma_{\mathbf{k}} n_{\mathbf{k}} + \operatorname{Im}(P_{\mathbf{k}}^* \sigma_{\mathbf{k}}) = 0,$$
$$\frac{1}{2}\frac{d\sigma_{\mathbf{k}}}{dt} + (\gamma_{\mathbf{k}} + i(\tilde{\omega}_{\mathbf{k}} - \omega_p/2))\sigma_{\mathbf{k}} + iP_{\mathbf{k}} n_{\mathbf{k}} = 0, \tag{15}$$

where $P_{\mathbf{k}}$ and $\tilde{\omega}_{\mathbf{k}}$ are defined by (2) and (3).

Linearizing (15) against the background of the ground state (6) relative to the perturbations (8) and putting $\alpha, \alpha^* \sim e^{-i\Omega t}$, we obtain a system of algebraic equations that are homogeneous in α and α^*; the condition for the solvability of these equations determines the spectrum and the damping of the collective oscillations:

$$\Omega_m = -i\gamma \pm [4S_m(2T_m + S_m)N_0^2 - \gamma^2]^{1/2}. \tag{16}$$

This dispersion relation solves completely the problem of the stability of the collective oscillations. It follows from it that when

$$S_m(2T_m + S_m) \geqslant 0$$

the oscillations become damped. This fact is also confirmed by experiments[9] on resonant excitation of the homogeneous mode m = 0. Namely, the measured width of the resonance band coincided with the value of γ determined from the threshold of the parametric excitation $hV = \gamma$.

If the inverse inequality

$$S_m(2T_m + S_m) < 0 \tag{17}$$

holds, an instability sets in with respect to the growth of the mode m, and this instability, as in the case of $\gamma = 0$, has no amplitude threshold.

To find the conditions under which the inequality (17) can hold, it is necessary to calculate the coefficients S_m and T_m for the investigated ferromagnet. The procedure of calculating these coefficients was described earlier[12]. In cubic ferromagnets such as YIG, taking into account the energy of the dipole-dipole and exchange interactions, the Zeeman energy, and the energy of the crystallographic anisotropy, we obtain the following expressions:

$$T_0 = \frac{g}{2M_0} \frac{\omega_p^2 + \omega_M^2}{\omega_p^2} \left[\omega_M(N_z - 1) + \frac{\omega_M^2}{(\omega_p^2 + \omega_M^2)^{1/2}} - \beta\omega_a \right],$$

$$S_0 = \frac{g}{2M_0} \left(\frac{\omega_M}{\omega_p}\right)^2 [\omega_M(N_z - 1) + (\omega_p^2 + \omega_M^2)^{1/2} - \beta\omega_a],$$

$$
\begin{gathered}
S_{\pm 1}=0,\ S_m=T_m=0 \text{ for } |m|>2,\\
T_{\pm 2}=\frac{g}{2M_0}\left(\frac{\omega_M}{2\omega_p}\right)^2[\omega_{ex}(lk)^2-\omega_M+(\omega_p^2+\omega_M^2)^{1/2}-\beta\omega_a],\\
S_{\pm 2}=\frac{g}{2M_0}\left\{[\omega_{ex}(lk)^2-\omega_M-\beta\omega_a]u_\pm^2+\frac{\omega_M^2}{2\omega_p}u_\pm\right\},\\
u_\pm=[(\omega_p^2+\omega_M^2)^{1/2}\mp\omega_p]/2\omega_p.
\end{gathered}
\tag{18}
$$

Here ω_p is the pump frequency, $\omega_M = 4\pi g M_0$, $\omega_a = 2g|\mathbf{k}|/M_0$ is the anisotropy field in the difficult direction [100], and

$$
\beta=\begin{cases} 8 \text{ for } \mathbf{M}_0 \parallel [111] \\ -9 \text{ for } \mathbf{M}_0 \parallel [100] \end{cases}.
$$

The remaining symbols are clear from the expression for the wave dispersion law:

$$
\alpha=\begin{cases} 2/3 \text{ for } \mathbf{M}_0 \parallel [111] \\ -1 \text{ for } \mathbf{M}_0 \parallel [100] \end{cases}.
\tag{19}
$$

Formulas (18) explain the strong anisotropy of the spin waves beyond the threshold of parametric excitation, namely, the contribution of the anisotropy energy to the coefficients of the nonlinear interaction of the T and S waves turns out to be larger by one order of magnitude than its contribution to the natural frequencies of the waves $\omega_{\mathbf{k}}$. To illustrate this important circumstance, we present a table of the coefficients T and S for YIG in a typical experimental situation: $N_z = 1/3$ (sphere), $\omega_p = 9.4$ GHz, $\omega_{ex}(lk)^2 = 0$ ($H = H_c$), $\omega_M = 4.9$ GHz, and $\omega_a = 0.23$ GHz (room temperature). All the coefficients are given in units of $2\pi g^2$.

Substituting the tabulated data in the criterion (17), we can compare the predictions of theory with onset of instability on the one hand and the experimental results in YIG on the other. It is easy to verify with the aid of the table that in the easy direction ($\mathbf{M}_0 \parallel [111]$) there is an instability with respect to the zeroth mode ($m = 0$), and that in the difficult direction ($\mathbf{M}_0 \parallel [100]$) all the modes of the collective oscillations are stable. In experiment at $H \approx H_c$, there are actually no self-oscillations up to excesses of 6–7 dB (second threshold[13]) in the difficult direction; in the easy direction, on the other hand, intensive self-oscillations are observed almost immediately beyond the parametric-excitation threshold. An analysis of expressions (18) shows that when the wave vector of the spin waves increases (with decreasing external magnetic field H), an instability of the mode $m = -2$ sets in in the difficult direction. The onset of self-oscillations of small amplitude in fields $H \leq H^* < H_c$, as is clearly seen from Fig. 4.4 of Monosov's monograph[2], is indeed observed in experiment[1]. The agreement between the theoretical predictions concerning the onset of the self-oscillations and experiment[12] can be traced also for YIG samples in the form of disks magnetized parallel ($N_z = 0$) and perpendicular ($N_z = 1$) to the plane. It is of interest to note that turning off the anisotropy field ($\omega_a < 0.022$ GHz) leads to an unstable zeroth mode. Indeed, intensive self-oscillations are observed in a YIG sphere with a scandium impurity that decreases the anisotropy to $\omega_a \approx 0.016$ GHz.

It must be emphasized that in the theory considered here, the threshold of the self-oscillations is equal to zero (it coincides with the threshold of the parametric excitation), whereas in experiments one observes a small threshold ~0.1–0.5 dB in YIG. We believe that this is connected with the influence of the inhomogeneities of the YIG crystal. The magnetic inhomogeneities lead to a "smearing" of the functions $n_{\mathbf{k}}^{(0)}$ and $\sigma_{\mathbf{k}}^{(0)}$ in the ground state (6) and to a certain renormalization of the damping and of the coefficients S and T[14]. One should therefore expect formulas (17) and (18) to describe at best qualitatively the real situation in crystals of poor quality.

TABLE.

Orientation	T_0	S_0	$T_2 = T_{-2}$	S_2	S_{-2}
[100]	0.28	0.52	0.11	0.01	−0.36
[111]	−0.75	0.30	0.05	0.01	−1.27

4. SELF-OSCILLATIONS

The problem of the nonlinear stage of development of the instability of collective excitations can be solved only with a computer. Calculations for a concrete ferromagnet (for example YIG) call for a large amount of computer time and are hardly advantageous. From the physical point of view, it is much more interesting to consider some simple model that accounts qualitatively for the main properties of the self-oscillations.

We consider a system of pairs of spin waves that interact in like fashion with the pump field and fill two beams $\mathbf{k}_1$ and $\mathbf{k}_2$ in k-space. In the initial state (prior to application of the pump) the distribution of the pairs on the beams $n_{1\mathbf{k}}$ and $n_{2\mathbf{k}}$ is, generally speaking, arbitrary. For concreteness we assume

$$
n_{1\mathbf{k}}=n_{2\mathbf{k}}=n_0,\quad \sigma_{1\mathbf{k}}=\sigma_{2\mathbf{k}}=0.
\tag{20}
$$

The quantity n_0 can be regarded as the level of the thermal noise in the system.

Parametric excitation of the spin waves as a function of the level of the thermal noise is described by equations of the type

$$
\begin{gathered}
\frac{1}{2}\frac{dn_{1\mathbf{k}}}{dt}+\gamma(n_{1\mathbf{k}}-n_0)+\operatorname{Im}(P_1^*\sigma_{1\mathbf{k}})=0,\\
\frac{1}{2}\frac{d\sigma_{1\mathbf{k}}}{dt}+\left[\gamma+i\left(\tilde{\omega}_{1\mathbf{k}}+\frac{\omega_p}{2}\right)\right]\sigma_{1\mathbf{k}}+iP_{1\mathbf{k}}n_{1\mathbf{k}}=0
\end{gathered}
\tag{21}
$$

with analogous equations for the second beam. Here

$$
\begin{gathered}
P_1=hV+S_{11}\sum_{\mathbf{k}}\sigma_{1\mathbf{k}}+S_{12}\sum_{\mathbf{k}}\sigma_{2\mathbf{k}},\\
\tilde{\omega}_{1\mathbf{k}}=\omega_1+2T_{11}\sum_{\mathbf{k}}n_{1\mathbf{k}}+2T_{12}\sum_{\mathbf{k}}n_{2\mathbf{k}},\\
S_{12}=S_{21}^*,\quad T_{12}=T_{21}.
\end{gathered}
$$

The change of the sum of the amplitudes on the beams

$$
\sum_{\mathbf{k}}(n_{1\mathbf{k}}+n_{2\mathbf{k}})\equiv N_1+N_2
$$

simulates the zeroth collective mode ($m = 0$), and the change of the difference $N_1 - N_2$ simulates the higher modes ($m \neq 0$). For a number of reasons, which will become clear later on, the investigation of a mode with $m \neq 0$ is a simpler problem, and will therefore be considered first. To decrease the number of parameters of the problem, we put

$$
S_{11}=S_{22}\equiv S_1,\quad S_{12}=S_{21}\equiv S_2,\quad T_{11}=T_{22}=-T_{12}\equiv T
$$

($m = 0$ is stable in this case).

The equations (20) have a stationary solution that possesses external stability (see the condition (5) and is of the form (as $n_0 \to 0$)

$$
N_1=N_2=[(hV)^2-\gamma^2]^{1/2}/|S_1+S_2|\equiv N_0.
\tag{22}
$$

The nonstationary solutions were obtained with a computer for a system of 120 pairs under the initial conditions (20).

Figure 1 shows the calculated plots of $N_{1,2}(t)$ at a weak supercriticality $(hV - \gamma < \gamma)$ and a relatively low noise level $(\sum_k n_0/N_0 \sim 10^{-4})$. At first, at $t = t_1$, the packets n_{1k} and n_{2k} have a Gaussian form with a maximum at the point $\omega_k = \omega_p/2$ and increase with an identical increment $hV - \gamma$. When N_1 and N_2 become comparable in order of magnitude with the characteristic amplitude (22), internal instability with respect to the difference of the amplitudes $N_1 - N_2$ comes into play, and the growth of one of them, say N_2, lags the other $(t = t_2)$ and subsequently N_2 decreases. The appearance of the difference $N_1 - N_2$ shifts the packets in opposite directions away from the point $\omega_k = \omega_p/2$.

This process continues until it turns out that $N_2 \ll N_1 \sim N_1^{(0)}$ at $t = t_3$, and accordingly the maximum of the packet n_{1k} is in the position

$$\omega_k - \omega_p/2 = -2TN_1^{(0)}, \quad N_1^{(0)} = [(hV)^2 - \gamma^2]^{1/2}/|S_1|,$$

where $N_1^{(0)}$ is the stationary amplitude of the S theory for one beam. Such a state, which is outwardly stable on its own beam, is unstable against pair production on the foreign beam, and the instability increment ν is maximal at $\omega_k - \omega_p/2 = 2TN_1^{(0)}$, and is equal to

$$\nu_{max} = |P_2| - \gamma \approx \left(\frac{S_1 - S_2}{S_1}\right)^2 [(hV)^2 - \gamma^2]^{1/2}.$$

During the expectation time

$$(t_4 - t_3) \sim \nu_{max}^{-1} \ln\left(\sum_k n_0/N_0\right)$$

The packet n_{2k} grows to amplitudes on the order of $N_2^{(0)}$, while the difference $N_1 - N_2$ decreases. This leads to a reverse motion of the packets toward the center: $\omega_k = \omega_p/2$ $(t = t_5)$. However, the state at which $N_1 \approx N_2 \approx N_0$ exhibits no internal stability, and the motion continues. At $t = t_7$ the picture coincides exactly with $t = t_3$, except that the packets have exchanged places. Everything is then repeated. The swing of the motion of the packets in k-space is determined by the supercriticality and by the coefficients T and S:

$$\Delta k = \frac{4T}{S_1}[(hV)^2 - \gamma^2]^{1/2} \Big/ \frac{\partial \omega_k}{\partial k}.$$

The period and the waveform of the self-oscillations depends strongly on the supercriticality and on the noise level n_0. If n_0 is small, the period is determined mainly by the expectation time and depends logarithmically on n_0, while the depth of modulation of the amplitude is close to 100%. With increasing n_0, the expectation time decreases and at $\sum_k n_0/N_0 \sim 10^{-3}$ it becomes comparable with the "time of motion" $(t_4 - t_6$ in Fig. 1). With further increase of n_0, the dependence of the period on n_0 becomes even weaker, the depth of modulation decreases, and the self-oscillations assume a smoother form that is close to harmonic at certain values of S and T (for example at $S_1 = 0$ and $T = 3S_2/4$). In a real situation we have $\sum n_0/N_0 \sim 10^{-2}$–$10^{-3}$, i.e., we are at the plateau of the $\tau(n_0)$ plot shown in Fig. 2. The period and the other properties of the self-oscillations are then not critical to the noise level, which subsequently is taken to equal 10^{-2}.

Figure 3 shows the dependence of the period of the self-oscillations on the excess above threshold. At small and large excesses, the period is proportional to the characteristic time of the problem $[(hV)^2 - \gamma^2]^{1/2}$. The period of the oscillations also depends on the coefficients T and S, albeit in a rather complicated manner; for example, at $hV \sim 2\gamma$ we can suggest the empirical formula

$$\tau \approx \frac{2(48T - 7S_-)}{S_-^2 + 4TS_-}[(hV)^2 - \gamma^2]^{-1/2}, \quad S_- = S_1 - S_2.$$

At small excesses, the distributions $n_{1,2k}$ take the form of a packet with a single maximum. This inspires the natural desire to describe the motion in k-space within the framework of the equations for the moments of the distributions of n_{1k} and n_{2k}. An analysis has shown, however, that even following very small transformations the description with the aid of three moments (the summary amplitude, center of gravity and width of the packet) is insufficient, since the period, waveform, and even the very existence of the self-oscillations depend on the method of splitting up the infinite chain of moments. For a correct description of the self-oscillations it is necessary to take the higher moments into account, and it turns out that there is not even a numerical splitting parameter.

In the case of larger supercriticality, substantial differences are observed in the properties of the self-oscillations. With increasing pump amplitude, a change takes place in the character of motion in k-space, namely, the distribution functions $n_{1,2k}$ reveal new maxima, and the number of degrees of freedom that participate effectively in the motion increases. This leads to violation of the "long-range order" in the time dependence of N(t). As seen from Fig. 4, at $hV = 10\gamma$ the motion is completely randomized; the characteristic time of variation of N(t) is of the order of $1/hV$. The distribution function n(k) has the shape of a "picket

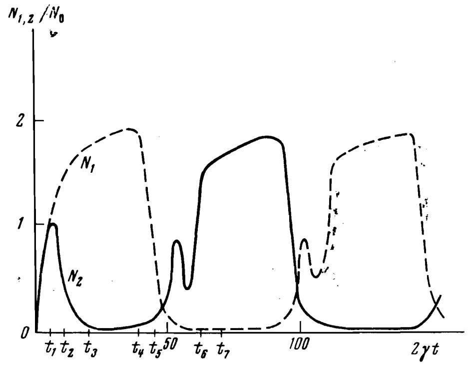

FIG. 1. Summary amplitudes of the pairs on each beam as a function of the time ($hV = 1.4\gamma$, $\Sigma n_0/N_0 \approx 10^{-2}$, $S_- = -S_0$, $T = 3S_0/4$).

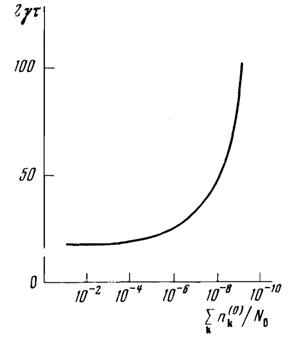

FIG. 2.

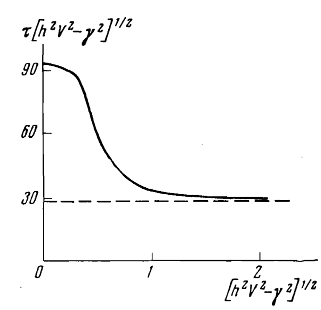

FIG. 3.

FIG. 2. Dependence of the self-oscillation period on the noise level n_0 ($hV = 2\gamma$, $S_- = -S_0$, $T = 3S_0/4$).

FIG. 3. Dependence of the self-oscillation period on the excess above the instability threshold ($\Sigma n_0/N_0 = 10^{-2}$, $S_- = -S_0$, $T = 3S_0/4$).

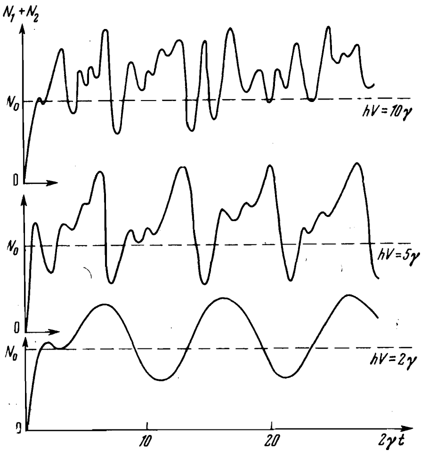

FIG. 4. Time dependence of the summary amplitude of the pairs at different excesses ($\Sigma n_0/N_0 = 10^{-2}$, $S_1 = 0$, $T = 3S_0/4$).

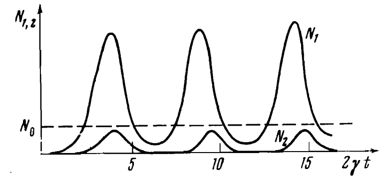

FIG. 5. Time dependence of the amplitude of waves with $\theta = \pi/2(N_1)$ and $\theta = \pi/4(N_2)$. $hV_{\pi/2} = 2\gamma$, $\Sigma n_0/N_0 = 10^{-2}$, $T_{22} = S_{22} = S_{11}/2$, $S_{12} = S_{21} = 3S_{11}/4$, $T_{11} = -0.7S_{11}$, $T_{12} = T_{21} = -S_{11}/2$.

fence" consisting of $\sim hV/\gamma$ maxima of different heights, the average distance between which is

$$\Delta k \approx \gamma \Big/ \frac{\partial \omega}{\partial k}.$$

The form of $n(\mathbf{k})$ changes completely after a time $\sim 1/hV$. This picture corresponds to strong turbulence of the correlation functions $n_{\mathbf{k}}$ and $\sigma_{\mathbf{k}}$.

We now briefly consider the self-oscillations produced in the case of instability with respect to the zeroth mode $m = 0$. It is shown in[15] that development of instability with respect to the zeroth mode in a medium that has spherical symmetry leads to an unlimited growth of the pair amplitude. In the axially-symmetric situation of interest to us, however, the growth of the amplitude N_θ with $\theta = \pi/2$ leads to an increase of the self-consistent pumping

$$P_{\theta'} = hV_{\theta'} + S_{\theta\theta'}N_\theta \exp(-i\psi_\theta)$$

into other pairs with $\theta' \neq \pi/2$, which are indeed excited as soon as $P_{\theta'}$ exceeds the damping $\gamma_{\theta'}$. The reaction of the new pairs blocks the growth of the pairs with $\theta = \pi/2$. On the other hand, as already noted above, in the stationary state (at not too large excesses $h/h_0 \lesssim 2.5$), there can be excited only pairs with $\theta = \pi/2$; they are, however, unstable with respect to the zeroth mode. As a result, self-oscillations are produced, the characteristic deviation of which from the self-oscillations considered above is the excitation of a packet of parametric waves, with extensive width relative to θ. In the study of these self-oscillations we used the "asymmetrical" model of two beams $\mathbf{k}_1$ and $\mathbf{k}_2$, representing spin waves with $\theta_{\mathbf{k}} = \pi/2$ and $\theta_{\mathbf{k}} = \pi/4$; $V_2 = V_1/2$. The values of the coefficients S and T were chosen close to the real ones in the YIG. The characteristic dependence of the summary amplitudes on the beams N_1 and N_2 is shown in Fig. 5 ($hV_1 = 2\gamma$).

We note in conclusion that the simple model of "two beams" conveys qualitatively the main self-oscillation properties observed in experiment, namely, the frequency, its dependence on the pump amplitude, the transition from the harmonic spectrum to a noise spectrum with increasing amplitude (Fig. 4), the large amplitude of the self-oscillations in the case of instability with respect to the zeroth mode in comparison with the amplitude when the higher modes are unstable (compare Figs. 4 and 5), and finally, the important circumstance that the self-oscillations occur near the ground state of the S theory, and their onset thus does not change essentially the level to which the amplitude of the parametric spin waves is limited.

[1)]The behavior of the instability-threshold curve in fields $H < 1300$ Oe may be connected with the excitation of spatially-inhomogeneous self-oscillations.

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