On the regularizing role of a gravitational field

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(Submitted June 30, 1972)
Zh. Eksp. Teor. Fiz. **64**, 1105-1110 (April 1973)

It is shown that a limiting mass in the form (9) or a corresponding cut-off length in the form (10) arises in intermediate states when gravitational interactions are taken into account in quantum electrodynamics. It is asserted that the electron mass cannot be of an electromagnetic nature since the electron-mass corrections due to the electromagnetic field yields are smaller than the experimental mass by several orders of magnitude. The possibility of the appearance of states characterized by a half-closed metric in intermediate states is discussed.

As is well known, the total mass of a classical charged particle, distributed over a region r_0, is given by the expression (see[1])

$$m_{tot} = m_0 + \frac{e^2}{2r_0c^2} - \varkappa\frac{m_{tot}^2}{2r_0c^2}, \tag{1}$$

where m_0 is the "bare" mass of the particle, e is its electric charge, and κ is the gravitational constant. It follows therefore that

$$m_{tot} = -\frac{r_0c^2}{\varkappa} + \left[\frac{r_0^2c^4}{\varkappa^2} + \frac{e^2}{\varkappa} + \frac{m_0^2r_0c^2}{\varkappa}\right]^{1/2}.$$

Relation (1) takes into account the equality of the inertial and gravitational masses.

When the particle dimensions tend to zero ($r_0 \to 0$) we have

$$m_{tot} \to e/\sqrt{\varkappa}. \tag{2}$$

The same result is obtained rigorously within the framework of general relativity theory[1-3]. In this rigorous description, however, the particle is in fact not pointlike. More accurately, in general relativity theory the particle turns out to be pointlike when described in isotropic coordinates. But isotropic coordinates do not give a complete spatial description of the given object. This circumstance can be explained directly when the problem is solved consistently and rigorously. The consistency and rigor in the investigation of this problem lie in the fact that one obtains not only an external solution, but also an internal solution describing the given extended system. From the condition that the internal and external solutions be mutually continuous, it is established[2, that the system cannot have pointlike dimensions at the mass given by (2). The minimum possible dimensions of the system are given by the expression

$$r^c = \varkappa m_{tot}/c^2 = e\sqrt{\varkappa}/c^2. \tag{3}$$

In other words, this expression for r^c is exactly half the Schwarzschild radius. Indeed, in this case, as shown in[2,3], the external metric is given by the element

$$ds^2 = \Phi(r)\,dt^2 - dr^2/\Phi(r) - r^2d\Omega^2,$$

where $\Phi(r) = (1 - \kappa m_{tot}/c^2r)^2$, and $m_{tot} = e/\sqrt{\kappa}$. A continuation of the given external metric to the region occupied by the matter (internal solution) is ambiguous: (i) a case of monotonic variation of r is possible and is realized by the Papapetrou model (the static model of charged dust $e/\kappa m = 1$), and (ii) a case is possible when the external and internal solutions are joined together through an orifice ("mole hole"), in this case the internal solution may not be static. In the present example, the external solution is described by a Friedmann metric, and r, as a function of χ[2], has a minimum ($r' = 0$ is the orifice). This case is the limiting case of a semi-closed world $m_{tot} > e/\sqrt{\kappa}$ as $m_{tot} \to e\sqrt{\kappa}$. As $e \to 0$, the external metric becomes Euclidean, and the external metric becomes the metric of Friedmann's closed world. The case of a system (ii) was called in[4] a fridmon, and the corresponding metric of semi-closed world was called a fridmon metric.

On the other hand, all the known attempts to take into account the regularizing role of the gravitational field within the framework of quantum theory lead to the appearance of another fundamental mass and accordingly to another fundamental length[5-8]:

$$m^q = \sqrt{\hbar c/\varkappa}, \tag{4}$$

or

$$r^q = \sqrt{\hbar\varkappa/c^3}. \tag{5}$$

According to these results, an impression may be gained that the regularizing ability of the gravitational field has a pure quantum nature. According to the classical analysis of the problem, on the other hand, gravitation plays the role of the regularizer even without making use of quantum theory.

The purpose of the present note is to indicate that a consistent theory should account for both lines (r^c and r^q), and that this result can be obtained from simple considerations based on the equivalence principle and on the Heisenberg uncertainty relation.

In quantum theory, the problem is considered within the framework of perturbation theory. In other words, we consider a charged particle of mass m_p, which in the intermediate state emits an energy quantum whose mass M_0, according to the Heisenberg uncertainty relation, is localized in the region r_0, so that[1)]

$$M_0 \approx \hbar/2r_0c. \tag{6}$$

Alternately, writing down in analogy with (1) an expression for the total mass of intermediate state, we obtain

$$m_{tot} = m_p + \frac{\hbar}{2r_0c} + \frac{e^2}{2r_0c^2} - \varkappa\frac{m_{tot}}{2r_0c^2}, \tag{7}$$

whence

$$m_{tot} = -\frac{r_0c^2}{\varkappa} + \left[\frac{r_0^2c^4}{\varkappa^2} + \frac{\hbar c}{\varkappa} + \frac{e^2}{\varkappa} + \frac{2r_0c^2m_p}{\varkappa}\right]^{1/2}. \tag{8}$$

As $r_0 \to 0$ we have

$$m_{tot} \to \left(\frac{\hbar c}{\varkappa} + \frac{e^2}{\varkappa}\right)^{1/2}. \tag{9}$$

Expression (9) is the maximum value of the intermediate-state mass in quantum perturbation theory.

The role of the gravitational field in the intermediate states is usually not taken into account, and this leads

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to infinite values of the energy in these states. Allowance for the gravitational mass defect limits the upper limit of the energy (mass) of the intermediate states to expression (9). Now the gravitational radius $r_{gr} = 2M_{tot}\kappa/c^2$ contains a mass M_{tot}, given by expression (9), and on the basis of (3) the corresponding length is given by the expression

$$r^q = \left(\frac{\hbar c + e^2}{c^4}\varkappa\right)^{1/2}. \qquad (10)$$

Both expression (9) for the mass expression and (10) for the length go over into the corresponding classical formulas (2) and (3) as $h \to 0$. In the case of electrodynamics, $e^2/\hbar c \ll 1$ and the quantity e^2 in (9) and (10) can be neglected. But another situation can arise in mesodynamics, where the specific charge of a massive vector field g can no longer be regarded as small: $g^2/\hbar c \geq 1$.

Thus, taking into account the foregoing, the known expression for the electromagnetic self-energy of the electron takes the form

$$m_{el} \approx \frac{3}{2\pi}\frac{e^2}{\hbar c} m_e \ln\frac{m_{max}}{m_e} \sim \frac{m_e}{6}. \qquad (11)$$

It should be noted, however, that the logarithmically divergent expression for the electromagnetic self-energy of the electron, obtained in Dirac's theory many years ago, raises many questions, some of which could be satisfactorily answered only lately. Indeed, expression (11) for the electrostatic self-energy of the electron does not go over to the corresponding classical expression as $\hbar \to 0$. Moreover, the quantum expression (11) depends strongly on the bare mass of the electron. On the other hand, a linearly diverging classical analog of the electrostatic self-energy of the electron has in general no dependence whatever on the bare mass of the particle.

Recently, however, searches for the corresponding solutions outside the framework of perturbation theory had led[9] to an expression different from (11) for the electromagnetic self-mass of the electron:

$$m_{el} = m_{max}\exp\left\{-\frac{3\pi}{2}\frac{\hbar c}{e^2}\right\} \sim \sqrt{\frac{\hbar c}{\varkappa}}\exp\left\{-\frac{3\pi}{2}\frac{\hbar c}{e^2}\right\}. \qquad (12)$$

This "superconducting-type" solution cannot be expanded in terms of the fine-structure constant and cannot be obtained within the framework of perturbation theory.

Expression (12), unlike (11), in accordance with the classical expression, does not depend on any bare mass of the electron at all. It diverges linearly with increasing m_{max}, like the classical one, and goes over at $\hbar \to 0$ into the corresponding classical expression. The exponential factor $\sim e^{-650}$, which takes into account the quantum corrections (polarization of vacuum), makes the contribution of the electromagnetic field to the self-energy of the electron negligible, in spite of the seemingly tremendous value of the limiting mass of the intermediate state. According to this analysis, the mass of the electron cannot be of electromagnetic origin.

Although the earlier estimates took the equivalence principle into account in the form (2) and the quantum character of this phenomenon is recognized (which is important), a shortcoming of such an analysis is that the metric in this problem remains Euclidean, although its conditions are such that they can lead to the occurrence of a closed or almost-closed metric. Let us examine in greater detail these conditions from the point of view of the possible occurrence of similar fridmon[2-4] formations in intermediate states.

If the density ϵ/c^2 of electrically neutral matter is such that a system with a metric of the close Friedmann world is indeed produced, then the corresponding Einstein equation takes the form

$$\frac{8\pi\varkappa}{3c^4}\varepsilon = \frac{1}{a^2}\left[\left(\frac{da}{d\eta}\right)^2\frac{1}{a^2} + 1\right], \qquad (13)$$

where $a(\eta)$ is the radius of the given system at the instant t, and $cdt = a(\eta)d$.

At the instant of maximum expansion of the "world" $(da/d\eta = 0)$, its maximum radius a_0 is given by

$$\varepsilon_0 = 3c^4/8\pi\varkappa a_0^2. \qquad (14)$$

The total "bare" mass of the system takes the form

$$M_0 = \frac{1}{c^2}\int \varepsilon_0 dv = \frac{3}{4}\pi c^2\frac{a_0}{\varkappa}. \qquad (15)$$

It must be emphasized that a closed world can exist in principle in the form of a world with very small dimensions (small a_0) and contain matter with very small masses (M_0). However, the necessary homogeneous density of matter (μ_0) at the instant of maximum expansion should only satisfy the condition

$$\mu_0 = \varepsilon_0/c^2 \sim c^6/\varkappa^3 M_0^2. \qquad (16)$$

If we take the liberty of setting M_0 in (15) equal to the limiting mass M_{max} of the intermediate states $M_0 \sim \sqrt{\hbar c/\kappa}$, then we obtain for the dimensions of the "world" the critical length

$$a_0 = \sqrt{\hbar\varkappa/c^3} \sim \hbar/M_0 c. \qquad (17)$$

If a similar situation were to arise in classical, non-quantum physics, we could say that in these intermediate states there can arise states with closed (in the case of an electrically neutral system) or semiclosed fridmon[2-4] metric in the case of charged systems. Moreover, attention is called to the fact that when the mass in the intermediate state is of the order of $M_0 \sim \sqrt{\hbar c/\kappa}$, then the gravitational radius of this mass

$$r_{gr} = 2\varkappa m/c^2 = 2\sqrt{\hbar\varkappa/c^3}$$

coincides with the region of localization of the given mass, admitted by the Heisenberg uncertainty relation

$$l \sim \hbar/mc \sim \sqrt{\hbar\varkappa/c^3}.$$

With further increase of the energy mc^2 of the intermediate states, the gravitational radius should increase accordingly. On the other hand, the region of localization of the energy of the intermediate states, according to the Heisenberg relation, should decrease correspondingly, and should become smaller than the gravitational radius at $m > \sqrt{\hbar c/\kappa}$. If such a situation were to arise in the region where classical physics applies, then we would say that we are dealing with a system whose mass is under the gravitational Schwarzschild sphere. In other words, we would deal with systems in the collapsed state. This would be either the state of a "black hole," or more readily the state of a system with semiclosed metric, if the "bare" mass of the intermediate state decreases strongly as a result of the gravitational defect. At the present time, we do not know the extent to which the metric concepts remain in force in this state, although in modern theory we stubbornly use a Euclidean metric in these cases. We know, however, that with increasing energy of the intermediate states the region of localization of the mass decreases in accordance with the Heisenberg relation. Consequently, owing to the high mass concentration, its gravitational defect should increase, and this decreases accordingly the total mass of the intermediate state. Evidently, only if we take into account the gravitational mass defect will the gravitational radius of an intermediate-state system not exceed the dimensions admitted by the Heisenberg relation, i.e., this may resolve the discussed contradiction.

Semiclosed-system states or black-hole states should seemingly be included in the complete set of states that can arise spontaneously in the discussed cases. Moreover, these states have the lowest energies, and this, as we have seen above, is important in the general picture of the intermediate states if gravitational interactions are taken into account. On the other hand, it would be a direct violation of elementary logic not to take gravitational interactions into account in the intermediate states, and to admit at the same time the possible occurrence of macroscopically ultralarge masses in these states.

Of course, an adequate quantum description of collapsing systems can reveal the need for appreciable corrections, but apparently mainly in their space-time description. The energy picture of these states can hardly change significantly. More accurately, this can hardly concern significantly such an effect of the gravitational defect of masses localized in a small region, and the equivalence principle. States of semiclosed systems can be characterized in the energy representation by a relation between the total mass (M_{tot}) and the "bare" mass (M_0). Namely[3], $M_{tot} = 4/3\pi\mu_0 a_0^3 \sin^3\chi$, where $a_0 = \kappa M_0/3\pi c^2$, and consequently $M_{tot} = 2/3\pi^{-1} M_0 \sin^3\chi$. If $\pi/2 < \chi < \pi$, then the relation $M_{tot}/M_0 \approx 2/3\pi^{-1}\sin^3\chi$ is the condition of semi-closedness[2,3] of a state in the energy representation.

1) In Fermi's terminology, a particle that emits in the intermediate state a mass quantum $M_0 = E/c^2$ makes a "loan" for a time $\Delta t \sim \hbar/E = \hbar/M_0c^2$. During the time Δt, the emitted quantum can "fly away" from the particle only to a distance $r \leqslant c\Delta t = \hbar/M_0 c$.

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Translated by J. G. Adashko
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