Field theory of gravitational field sources

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It is shown that the analogy between gravitational and electromagnetic fields inherent in the equations of the theory of gravitation can be considerably extended by attaching a physical meaning to the geometric quantities encountered in the theory of embedding of Riemannian spaces.

1. INTRODUCTION

We write down the well-known Maxwell-Dirac system of equations, which describes interacting electromagnetic and electron-positron fields, in spinor form

$$\Box A^{\alpha\dot\beta} = s^{\alpha\dot\beta} = \xi^{\alpha}\xi^{\dot\beta} + \eta^{\alpha}\eta^{\dot\beta} \quad \text{(Maxwell)}; \tag{1}$$

$$\left(\partial^{\alpha\dot\beta} - A^{\alpha\dot\beta}\right)\eta_{\dot\beta} - m\xi^{\alpha} = 0,$$
$$\left(\partial_{\dot\beta\alpha} - A_{\dot\beta\alpha}\right)\xi^{\alpha} - m\eta_{\dot\beta} = 0 \quad \text{(Dirac)}, \tag{2}$$

where the spinor indices run through the values $\alpha = 1$ and 2 and $\dot\beta = 1$ and 2.

These equations can be regarded as a field theory of sources of an electromagnetic field. The spin tensor $s^{\alpha\dot\beta} = \xi^{\alpha}\xi^{\dot\beta} + \eta^{\alpha}\eta^{\dot\beta}$ is a Hermitian form made up of two spinors ξ and η describing the sources of the electromagnetic field. The divergence of the current vanishes, on one hand, by virtue of the structure of Maxwell's equations, and on the other hand by virtue of the Dirac equations. The Maxwell-Dirac equations are not universal in the sense that the sources of the electromagnetic field in them are only electrons and positrons, while other forms of charged matter, which also produce an electromagnetic field, are not taken into account.

We shall show that by making use of additional geometrical quantities that enter in the theory of embedding of Riemannian spaces, the equations of the gravitational field can be recast in the form (1) and (2).

2. EMBEDDING THEORY AND THE GAUSS-CODAZZI-RICCI EQUATIONS

It is shown in embedding theory[1] that the curvature tensor $R_{\alpha\beta\gamma\delta}$ ($\alpha, \beta, \gamma, \delta = 1, 2, 3, 4$) Riemannian space can be represented as a quadratic form made up (in the general case) of the components of six symmetrical tensors $H_{\alpha\beta}[A]$, where the "large" index is A = 1, 2, ..., 6:

$$R_{\alpha\beta\gamma\delta}(g) = e[A](H_{\alpha\gamma}[A]H_{\beta\delta}[A] - H_{\alpha\delta}[A]H_{\beta\gamma}[A]) \text{ (Gauss)}, \tag{3}$$

where $e[A] = \pm 1$. Here and throughout we sum over repeated "large" indices from 1 to p.

For Riemannian subspaces of special type, the number of tensors $H_{\alpha\beta}[A]$ in terms of which the curvature tensor is expressed decreases. The minimum number of tensors $H_{\alpha\beta}[A]$ in terms of which one can express the components of the curvature tensor is denoted by p($0 \le p \le 6$) and is called the class of the space.

The Gauss formula (3) is analogous to the Maxwell equation (1). The analog of the metric tensor $g_{\alpha\beta}$ is the electromagnetic potential $A^{\alpha\dot\beta}$, and the analog of the curvature tensor $R_{\alpha\beta\gamma\delta}$ is the current spin tensor $s^{\alpha\dot\beta}$.

The tensors $A_{\alpha\beta}[A]$ satisfy the equations of Codazzi and Ricci

$$\nabla_{\gamma}H_{\sigma\tau}[A] - \nabla_{\tau}H_{\sigma\gamma}[A] + e[B](T_{\gamma}[A, B]H_{\sigma\tau}[B] - T_{\tau}[A, B]H_{\sigma\gamma}[B]) = 0 \text{ (Codazzi} \tag{4}$$

$$\nabla_{\gamma}T_{\tau}[A, B] - \nabla_{\tau}T_{\gamma}[A, B] + e[C](T_{\gamma}[A, C]T_{\tau}[C, B] - T_{\tau}[A, C]T_{\gamma}[C, B]) + g^{\sigma\rho}(H_{\sigma\tau}[A]H_{\rho\gamma}[B] - H_{\sigma\gamma}[A]H_{\rho\tau}[B]) = 0 \text{ (Ricci)}, \tag{5}$$

where ∇_{γ} denotes covariant differentiation, $\sigma, \gamma, \tau, \rho = 1, 2, 3, 4$; A, B, C = 1, ..., p; $e[A] = \pm 1$. The vectors $T_{\gamma}[A, B]$ are antisymmetrical in the "large" indices A and B: $T_{\gamma}[A, B] = -T_{\gamma}[B, A]$.

The Gauss-Codazzi-Ricci (GCR) conditions (3)–(5) are the conditions for the embedding of 4-dimensional Riemannian space in the pseudo-Euclidian space E_{4+p} ($0 \le p \le 6$).

The Codazzi and Ricci equations are analogous to the Dirac equations (2). The tensors $H_{\sigma\tau}[A]$ are the analog of the spinor fields ξ and η, and the analog of the mass are the vectors $T_{\gamma}[A, B]$ which, unlike the mass m (an experimentally determined constant) are determined together with $H_{\sigma\tau}[A]$ from the GCR equations.

The Bianchi conditions

$$\nabla_{\rho}R_{\alpha\beta\gamma\delta} + \nabla_{\delta}R_{\alpha\beta\rho\gamma} + \nabla_{\gamma}R_{\alpha\beta\delta\rho} = 0 \tag{6}$$

are satisfied for two reasons: on the one hand, by virtue of the structure of the curvature tensor as a functional of the metric tensor, and on the other hand by virtue of the equations of Codazzi and Ricci.

In the presented theory, the geometrical quantities $H_{\alpha\beta}[A]$ and $T_{\gamma}[A, B]$ are interpreted as physical fields describing the sources of the gravitational field. The field equations describing the interacting gravitational field (metric tensor $g_{\alpha\beta}$) and the fields of the sources $H_{\alpha\beta}[A]$ and $T_{\gamma}[A, B]$ are the GCR equations. Unlike the Maxwell-Dirac equations, the GCR equations are universal in the sense that they describe all the gravitational-field sources without exception.

In gravitational theory, the geometrical tensor $R_{\alpha\beta} - (1/2)g_{\alpha\beta}R$ acquires, by virtue of Einstein's equations, the physical meaning of the energy-momentum tensor of the gravitational field sources. In our theory, a physical meaning is assumed by the fields $H_{\alpha\beta}[A]$ and $T_{\gamma}[A, B]$, and consequently by the curvature-tensor components $R_{\alpha\beta\gamma\delta}$, which are expressed in their terms.

The energy momentum tensor $T_{\alpha\beta}$ is constructed in quadratic form from the fields $H_{\alpha\beta}[A]$:

$$T_{\alpha\beta} = e[A]g_{\lambda\rho}H_{\alpha\beta}[A]H_{\lambda\rho}[A] - e[A]g_{\sigma\tau}H_{\alpha\sigma}[A]H_{\tau\beta}[A] - {}^{1}/_{2}e[A]g_{\alpha\beta}(g_{\mu\sigma}g_{\lambda\rho}H_{\mu\sigma}[A]H_{\lambda\rho}[A] - g_{\mu\sigma}g_{\lambda\tau}H_{\mu\lambda}[A]H_{\tau\sigma}[A]). \tag{7}$$

The covariant divergence of this tensor vanishes by virtue of the GCR equations. We see that the equations and quantities contained in gravitational theory are obtained as a result of a contraction of the corresponding equations of the developed theory. Thus, the GCR equations describe in greater detail the sources of the gravitational field than the Einstein equations.

3. SPHERICALLY SYMMETRICAL FIELDS

By way of illustration, we consider spherically symmetrical solutions of the GCR equations.

The nonzero components of the fields $g_{\alpha\beta}$, $H_{\alpha\beta}[A]$, and $T_\gamma[A, B]$ are written in the following form:

$$g_{\alpha\beta}=\begin{pmatrix} (1-e^{\nu})n_s n_t-\delta_{st} & 0 \\ 0 & e^{\mu} \end{pmatrix}, \tag{8}$$

$$H_{\alpha\beta}[A]=\begin{pmatrix} c[A]n_s n_t-d[A]\delta_{st} & 0 \\ 0 & f[A] \end{pmatrix}, \tag{9}$$

$$T_\gamma[A, B]=(0, 0, 0, T_4[A, B]), \tag{10}$$

where

$\alpha, \beta, \gamma = 1, 2, 3, 4$; $s, t = 1, 2, 3$; $A, B = 1, \ldots, p$; $n_s = x_s / r$, $r=\sqrt{x_s x_s}$.

After long manipulations, the GCR equations take the form (see Appendix II)

$$\frac{1}{r^2}(e^{-\nu}-1)+\frac{1}{2r}\nu'=e[A]c[A]d[A],$$

$$\frac{1}{r^2}(e^{-\nu}-1)=e[A]d[A]d[A],$$

$$-\frac{1}{2}\mu''e^{\mu}-\frac{1}{4}\mu'^2e^{\mu}+\frac{1}{2}\mu'\frac{1}{r}e^{\mu-\nu}+\frac{1}{4}\mu'\nu'e^{\mu}=e[A]c[A]f[A],$$

$$\frac{1}{2r}\mu'e^{\mu-\nu}=e[A]d[A]f[A], \tag{11}$$

$$d'[A]+\frac{1}{r}d[A]+(c[A]-d[A])\frac{1}{r}e^{-\nu}=0,$$

$$e[B]T_4[A, B]c[B]=0, \qquad e[B]T_4[A, B]d[B]=0,$$

$$f'[A]-\frac{1}{2}\mu'e^{\mu-\nu}(c[A]-d[A])-\frac{1}{2}\mu'f[A]=0,$$

$$T_4'[A, B]=0,$$

where the prime denotes differentiation with respect to r.

It is interesting to note that for spherically symmetrical spaces the vector fields $T_\gamma[A, B]$ are constants.

A solution of this system yields a spherically-symmetrical metric $g_{\alpha\beta}$ of a space of class p, and also the fields $H_{\alpha\beta}[A]$ and $T_\gamma[A, B]$ that produce this metric.

For spaces of second class, the GCR equations are of the following form:

$$\frac{1}{r^2}(e^{-\nu}-1)+\frac{1}{2r}\nu'=e[1]c[1]d[1]+e[2]c[2]d[2],$$

$$\frac{1}{r^2}(e^{-\nu}-1)=e[1]d^2[1]+e[2]d^2[2], \tag{12}$$

$$-\frac{1}{2}\mu''e^{\mu}-\frac{1}{4}\mu'^2e^{\mu}+\frac{1}{2r}\mu'e^{\mu-\nu}+\frac{1}{4}\mu'\nu'e^{\mu}=e[1]c[1]f[1]+e[2]c[2]f[2],$$

$$\frac{1}{2}\mu'\frac{1}{r}e^{\mu-\nu}=e[1]d[1]f[1]+e[2]d[2]f[2];$$

$$d'[A]+\frac{1}{r}d[A]+(c[A]-d[A])\frac{1}{r}e^{-\nu}=0 \qquad (A=1,2),$$

$$f'[A]-\frac{1}{2}\mu'e^{\mu-\nu}(c[A]-d[A])-\frac{1}{2}\mu'f[A]=0 \qquad (A=1,2),$$

$$T[1, 2]c[2]=0, \quad T[2, 1]c[1]=0,$$

$$T[1, 2]d[2]=0, \quad T[2, 1]d[1]=0, \tag{13}$$

$$T[1, 2]=-T[2, 1]=\text{const}.$$

This system has two different solutions:

1) $T \neq 0$. Then $c[1] = c[2] = 0$, $d[1] = d[2] = 0$, $\mu' = 0$, $\nu = 0$, $f[1] = f[2] =$ const. This solution corresponds to a Euclidean metric.

2) $T = 0$. We obtain a system of eight equations for eight unknown functions c[A], d[A], f[A], μ, and ν. The solution of this system gives a spherically-symmetric metric for a space of second class.

4. THE SCHWARZSCHILD PROBLEM

Let us examine in greater detail a Schwarzschild metric, for which the functions $\mu(r)$ and $\nu(r)$ take the following form:

$$\mu(r)=-\nu(r)=-\ln(1-\alpha/r) \qquad (\alpha=2\varkappa m).$$

The system of the GCR equations is in this case a system of equations for the determination of six unknown functions c[A], d[A], and f[A] (A = 1, 2). Since the Schwarzschild space is Euclidean at infinity, it follows that the functions c[A], d[A], and f[A] should vanish as $r \to \infty$. We determine the signature e[A] (A = 1, 2) for the Schwarzschild space.

As is well known [1, 2], the Schwarzschild space for which the quadratic form ds^2 is represented in the form

$$ds^2=\left(1-\frac{\alpha}{r}\right)dt^2+\left[\left(1-\frac{1}{1-\alpha/r}\right)n_s n_t-\delta_{st}\right]dx^s dx^t.$$

has a metric of a second-class space. Indeed, if we put (at $r > \alpha$)

$$y^1=\alpha\sqrt{1-\frac{\alpha}{r}}\cos\frac{t}{\alpha}, \quad y^2=\alpha\sqrt{1-\frac{\alpha}{r}}\sin\frac{t}{\alpha}, \quad y^3=f(r),$$

$$y^4=x^1, \quad y^5=x^2, \quad y^6=x^3,$$

where the function f(r) is such that

$$\left(\frac{df}{dr}\right)^2=\frac{1}{r-\alpha}\left(\frac{\alpha^4}{4r^3}+\alpha\right),$$

then ds^2 takes the form

$$ds^2=(dy^1)^2+(dy^2)^2-(dy^3)^2-(dy^4)^2-(dy^5)^2-(dy^6)^2.$$

Thus, the signature e[A] for the Schwarzschild space has the form e[1] = 1, e[2] = − 1.

We turn to the GCR equations for a space of second class with signature e[1] = 1, e[2] = − 1. We introduce the six unknown functions c(r), d(r), f(r) and $\varphi_1(r)$, $\varphi_2(r)$, and $\varphi_3(r)$, which are connected with c[A], d[A], and f[A] by the following formula:

$$c[1]=c\,\text{sh}\,\varphi_1, \quad d[1]=d\,\text{sh}\,\varphi_2, \quad f[1]=f\,\text{ch}\,\varphi_3,$$
$$c[2]=c\,\text{ch}\,\varphi_1, \quad d[2]=d\,\text{ch}\,\varphi_2, \quad f[2]=f\,\text{sh}\,\varphi_3. \tag{14}$$

Equations (12) and (13) become

$$\alpha/r^3=d^2,$$

$$\frac{\alpha}{r^3}\left(1+\frac{r}{2(r-\alpha)}\right)=cd\,\text{ch}(\varphi_2-\varphi_1),$$

$$\frac{\alpha}{r^3}\left(1+\frac{r-\alpha}{2r}\right)=cf\,\text{sh}(\varphi_1-\varphi_3), \tag{15}$$

$$\frac{\alpha}{r^3}\frac{(r-\alpha)}{2r}=df\,\text{sh}(\varphi_2-\varphi_3) \qquad \text{(Gauss)};$$

$$d'\,\text{sh}\,\varphi_2+d\varphi_2'\,\text{ch}\,\varphi_2+\frac{1}{r}d\,\text{sh}\,\varphi_2+\frac{r-\alpha}{r^2}(c\,\text{sh}\,\varphi_1-d\,\text{sh}\,\varphi_2)=0,$$

$$d'\,\text{ch}\,\varphi_2+d\varphi_2'\,\text{sh}\,\varphi_2+\frac{1}{r}d\,\text{ch}\,\varphi_2+\frac{r-\alpha}{r^2}(c\,\text{ch}\,\varphi_1-d\,\text{ch}\,\varphi_2)=0, \tag{16}$$

$$f'\,\text{ch}\,\varphi_3+f\varphi_3'\,\text{sh}\,\varphi_3+\frac{\alpha}{2}\frac{(r-\alpha)}{r^3}(c\,\text{sh}\,\varphi_1-d\,\text{sh}\,\varphi_2)-\frac{\alpha}{2r(r-\alpha)}f\,\text{ch}\,\varphi_3=0,$$

$$f'\,\text{sh}\,\varphi_3+f\varphi_3'\,\text{ch}\,\varphi_3+\frac{\alpha}{2}\frac{(r-\alpha)}{r^3}(c\,\text{ch}\,\varphi_1-d\,\text{ch}\,\varphi_2)-\frac{\alpha}{2r(r-\alpha)}f\,\text{sh}\,\varphi_3=0$$

(Codazzi).

It can be verified in the usual manner that only three of the four Codazzi equations are independent and are of the form

$$d\varphi_2'+\frac{(r-\alpha)}{r^2}c\,\text{sh}(\varphi_1-\varphi_2)=0,$$

$$f\varphi_3' + \frac{\alpha(r-\alpha)}{2r^3}[c\,\mathrm{ch}(\varphi_1-\varphi_3) - d\,\mathrm{ch}(\varphi_2-\varphi_3)] = 0,$$
$$f' + \frac{\alpha(r-\alpha)}{2r^3}[c\,\mathrm{sh}(\varphi_1-\varphi_3) - d\,\mathrm{sh}(\varphi_2-\varphi_3)] - \frac{\alpha}{2r(r-\alpha)}f = 0. \tag{16a}$$

Solving the last of the Codazzi equations (16a) with respect to f and taking into account the third and fourth Gauss equations (15), we obtain, allowing for the boundary conditions,

$$f = \frac{\alpha}{2r^2}\sqrt{1-\frac{\alpha}{r}}.$$

Thus, the Gauss-Codazzi system of equations reduces to a system of five equations for four unknown functions $c(r)$, $\varphi_1(r)$, $\varphi_2(r)$ and $\varphi_3(r)$:

$$c\,\mathrm{ch}(\varphi_2-\varphi_1) = \sqrt{\frac{\alpha}{r^3}}\frac{(3r-2\alpha)}{2(r-\alpha)},$$
$$c\,\mathrm{sh}(\varphi_1-\varphi_3) = \frac{(3r-\alpha)}{r^2}\sqrt{\frac{r}{r-\alpha}},$$
$$\mathrm{sh}(\varphi_2-\varphi_3) = \sqrt{\frac{r-\alpha}{\alpha}}, \tag{17}$$
$$\varphi_2' + \frac{(r-\alpha)}{\sqrt{\alpha r}}c\,\mathrm{sh}(\varphi_1-\varphi_2) = 0,$$
$$\varphi_3' + \sqrt{1-\frac{\alpha}{r}}\left[c\,\mathrm{ch}(\varphi_1-\varphi_3) - \sqrt{\frac{\alpha}{r^3}}\mathrm{ch}(\varphi_2-\varphi_3)\right] = 0.$$

The solution of this system is

$$c^2(r) = -\frac{9\alpha}{2r^2(r-\alpha)} - \frac{\alpha^2}{4r^2(r-\alpha)^2} - \frac{2\alpha(r-\alpha)}{r^4} + \frac{\alpha^2}{r^3(r-\alpha)}\Big[302\frac{1}{2} - 289\frac{3}{4}\frac{\alpha}{r} + 160\frac{\alpha^2}{r^2} - 50\frac{\alpha^3}{r^3} + 4\frac{\alpha^4}{r^4} - 147\frac{3}{8}\frac{r}{\alpha} + 45\frac{r^2}{\alpha^2} - \frac{17}{8}\frac{r^2}{\alpha(r-\alpha)}\Big]^{1/2};$$

$$\varphi_1(r) = \int_\infty^r dr'\left(\frac{\sqrt{r'(r'-\alpha)}}{r'^2} - \frac{\mathcal{R}}{r'^2}\right) + \int_\infty^r \frac{dr'}{\mathcal{R}}\left\{\frac{2\alpha-3r'}{r'} - \frac{(3r'-\alpha)\alpha}{4r'(r'-\alpha)} - \frac{c'(r')(3r'-\alpha)}{c(r')}\right\}, \tag{18}$$

$$\varphi_2(r) = \int_\infty^r \frac{dr'}{2r'^2}\left[(3r'-2\alpha)^2 - \frac{4}{\alpha}r'^3(r'-\alpha)^2c^2(r')\right]^{1/2},$$

$$\varphi_3(r) = \int_\infty^r dr'\left\{\frac{\sqrt{r'(r'-\alpha)}}{r'^2} - \frac{\mathcal{R}}{r'^2}\right\},$$

where $\mathcal{R} = [(3r'-\alpha)^2 + r'^3(r'-\alpha)c^2(r')]^{1/2}$. From formulas (14) we get $c[A]$, $d[A]$, and $f[A]$.

We have thus obtained the fields $H_{\alpha\beta}[A]$ that produce a gravitational field with a Schwarzschild metric.

APPENDIX I

GCR EQUATIONS FOR n-DIMENSIONAL RIEMANNIAN SPACE

We denote by $Y_n(x_1, \ldots, x_n)$ a certain Riemannian space, and by $E_{n+p}(y_1, \ldots, y_{n+p})$ a pseudoeuclidean space into which the space $V_n(x)$ is embedded with the aid of the formulas $y^1 = x^1, \ldots, y^n = x^n, y^{n+1} = y^{n+1}(x_1, \ldots, x_n), \ldots, y^{n+p} = y^{n+p}(x_1, \ldots, x_n)$.

At the point P of the space $E_{n+p}(y)$ we can construct the following:

1) A tangent manifold over n vectors e_α with coordinates

$$e_\alpha = \{e_\alpha{}^k\} = \left\{\frac{\partial y^k}{\partial x^\alpha}\right\}, \quad k = 1, \ldots, n+p, \quad \alpha = 1, \ldots, n;$$

2) A normal manifold over p vectors $n_A = \{n_A^k\}$, forming an orthonormalized basis

$$(n_A, n_B) = e[A]\delta_{A,B} \quad (n_A, e_\alpha) = 0,$$
$$(e_\alpha, e_\beta) = g_{\alpha\beta} \quad (A, B = 1, \ldots, p).$$

Differentiating the preceding equations with respect to x^γ, we have

$$\frac{\partial e_\alpha}{\partial x^\gamma}e_\beta + e_\alpha\frac{\partial e_\beta}{\partial x^\gamma} = \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \neq 0,$$
$$\frac{\partial n_A}{\partial x^\gamma}e_\alpha + n_A\frac{\partial e_\alpha}{\partial x^\gamma} = 0,$$
$$\frac{\partial n_A}{\partial x^\gamma}n_B + n_A\frac{\partial n_B}{\partial x^\gamma} = 0.$$

Consequently

$$\Gamma_{\alpha,\beta\gamma} + \Gamma_{\beta,\alpha\gamma} \neq 0, \quad \Gamma_{A,\beta\gamma} + \Gamma_{\beta,A\gamma} = 0, \quad \Gamma_{A,B\gamma} + \Gamma_{B,A\gamma} = 0.$$

We introduce the quantities $H_{\alpha\beta}[A]$ and $T_\gamma[A, B]$ in the following manner:

$$\Gamma_{A,\beta\gamma} = -\Gamma_{\beta,A\gamma} = H_{\beta\gamma}[A] = H_{\gamma\beta}[A],$$
$$\Gamma_{A,B\gamma} = -\Gamma_{B,A\gamma} = T_\gamma[A, B] = -T_\gamma[B, A]. \tag{19}$$

In n- + p-dimensional Euclidean space, the Riemannian tensor vanishes:

$$R_{sklm}(y) = \frac{\partial\Gamma_{s,km}}{\partial y^l} - \frac{\partial\Gamma_{s,kl}}{\partial y^m} + g^{pq}(\Gamma_{p,kl}\Gamma_{q,sm} - \Gamma_{p,km}\Gamma_{q,sl}) = 0$$
$$(s, k, l, m, p, q = 1, \ldots, n+p). \tag{20}$$

From (20) we have

$$R_{\alpha\beta\gamma\delta}(y) = \frac{\partial\Gamma_{\alpha,\beta\delta}}{\partial x^\gamma} - \frac{\partial\Gamma_{\alpha,\beta\gamma}}{\partial x^\delta} + g^{\sigma\tau}(\Gamma_{\delta,\beta\gamma}\Gamma_{\tau,\alpha\delta} - \Gamma_{\sigma,\beta\delta}\Gamma_{\tau,\alpha\gamma})$$
$$+ e[A](\Gamma_{A,\beta\gamma}\Gamma_{A,\alpha\delta} - \Gamma_{A,\beta\delta}\Gamma_{A,\alpha\gamma}) = 0$$
$$(e[A] = \pm 1), \quad (\alpha, \beta, \gamma, \delta = 1, \ldots, n).$$

Hence, taking (19) into account, we obtain

$$R_{\alpha\beta\gamma\delta}(x) = e[A](H_{\alpha\gamma}[A]H_{\beta\delta}[A] - H_{\alpha\delta}[A]H_{\beta\gamma}[A]) \qquad \text{(Gauss)} \tag{21}$$

From (21) we get

$$R_{\alpha\gamma} = e[A](g_{\beta\delta}H_{\alpha\gamma}[A]H_{\beta\delta}[A] - g_{\beta\delta}H_{\alpha\beta}[A]H_{\delta\gamma}[A]),$$
$$R = e[A](g_{\alpha\gamma}g_{\beta\delta}H_{\alpha\gamma}[A]H_{\beta\delta}[A] - g_{\alpha\gamma}g_{\beta\delta}H_{\alpha\beta}[A]H_{\delta\gamma}[A]).$$

From (20) we also have

$$R_{A\beta\delta\tau}(y) = \frac{\partial\Gamma_{A,\beta\tau}}{\partial x^\sigma} - \frac{\partial\Gamma_{A,\beta\sigma}}{\partial x^\tau} + g^{\gamma\delta}(\Gamma_{\gamma,\beta\delta}\Gamma_{\delta,A\tau} - \Gamma_{\gamma,\beta\tau}\Gamma_{\delta,A\sigma})$$
$$+ e[B](\Gamma_{B,\beta\sigma}\Gamma_{B,A\tau} - \Gamma_{B,\beta\tau}\Gamma_{B,A\sigma}) = 0.$$

Using (19), we obtain

$$\nabla_\sigma H_{\beta\tau}[A] - \nabla_\tau H_{\beta\sigma}[A] + e[B](T_\sigma[A, B]H_{\beta\tau}[B]$$
$$-T_\tau[A, B]H_{\beta\sigma}[B]) = 0 \quad \text{(Codazzi)}$$

Finally, since

$$R_{AB\sigma\tau}(y) = \frac{\partial\Gamma_{A,B\tau}}{\partial x^\sigma} - \frac{\partial\Gamma_{A,B\sigma}}{\partial x^\tau} + g^{\gamma\delta}(\Gamma_{\gamma,B\sigma}\Gamma_{\delta,A\tau} - \Gamma_{\gamma,B\tau}\Gamma_{\delta,A\sigma})$$
$$+ e[C](\Gamma_{C,B\sigma}\Gamma_{C,A\tau} - \Gamma_{C,B\tau}\Gamma_{C,A\sigma}) = 0,$$

we have

$$\nabla_\sigma T_\tau[A, B] - \nabla_\tau T_\sigma[A, B] + e[C](T_\sigma[A, C]T_\tau[C, B] -$$
$$-T_\tau[A, C]T_\sigma[C, B]) - g^{\gamma\delta}(H_{\sigma\gamma}[A]H_{\delta\tau}[B] - H_{\tau\gamma}[A]H_{\delta\sigma}[B]) = 0 \quad \text{(Ricci)}$$

APPENDIX II

In the spherically-symmetrical case the GCR equations (3), (4), and (5) take the form

$$R_{stpq} = e[A](H_{sp}[A]H_{tq}[A] - H_{sq}[A]H_{tp}[A]),$$
$$R_{s4p4} = e[A]H_{sp}[A]H_{44}[A],$$
$$\frac{\partial H_{ps}[A]}{\partial x^t} - \frac{\partial H_{pt}[A]}{\partial x^s} + \Gamma_{ps}{}^qH_{qt}[A] - \Gamma_{pt}{}^qH_{qs}[A] = 0,$$

$$\frac{\partial H_{ps}[A]}{\partial x^4}+e[C]T_4[A,C]H_{ps}[C]=0, \tag{22}$$
$$\frac{\partial H_{44}[A]}{\partial x^p}+\Gamma_{44}{}^t H_{tp}[A]-\Gamma_{4p}{}^4 H_{44}[A]=0,$$
$$g^{rq}(H_{rs}[A]H_{qp}[B]-H_{rp}[A]H_{qs}[B])=0,$$
$$\partial T_4[A,B]/\partial x^p=0,$$

where p, s, t, r, q run through the values 1, 2, 3.

For the Christoffel symbols we have

$$\Gamma_{44}{}^s=\frac{1}{2}\mu' e^{\mu-\nu}n_s,\qquad \Gamma_{4s}{}^4=\frac{1}{2}\mu' n_s,$$
$$\Gamma_{rs}{}^t=n_t\left[\frac{1-e^{-\nu}}{r}(\delta_{rs}-n_r n_s)+\frac{1}{2}\nu' n_r n_s\right].$$

After simple but cumbersome manipulations we obtain an expression for the components of the Riemannian tensor

$$R_{stpq}=\left(\frac{1}{r^2}(e^{-\nu}-1)+\frac{1}{2r}\nu'\right)(\delta_{sq}n_t n_p+\delta_{tp}n_s n_q-\delta_{sp}n_t n_q-\delta_{tq}n_s n_p)$$
$$+\frac{1}{r^2}(e^{-\nu}-1)(\delta_{sp}\delta_{tq}-\delta_{sq}\delta_{tp}),$$
$$R_{s4p4}=\left(-\frac{1}{2}\mu'' e^{\mu}-\frac{1}{4}\mu'^2 e^{\mu}+\frac{1}{2}\mu'\frac{1}{r}e^{\mu-\nu}\right.$$
$$\left.+\frac{1}{4}\mu'\nu' e^{\mu}\right)n_s n_p-\frac{1}{2r}\mu' e^{\mu-\nu}\delta_{sp}.$$

Further, it is easy to show that Eqs. (22) take the form

$$\left(\frac{1}{r^2}(e^{-\nu}-1)+\frac{1}{2r}\nu'\right)(\delta_{sq}n_t n_p+\delta_{tp}n_s n_q-\delta_{sp}n_t n_q-\delta_{tq}n_s n_p)$$
$$+\frac{1}{r^2}(e^{-\nu}-1)(\delta_{sp}\delta_{tq}-\delta_{sq}\delta_{tp})=e[A]c[A]d[A](\delta_{sq}n_t n_p+\delta_{tp}n_s n_q$$
$$-\delta_{sp}n_t n_q-\delta_{tq}n_s n_p)+e[A]d[A]d[A](\delta_{sp}\delta_{tq}-\delta_{sq}\delta_{tp}),$$
$$\left[-\frac{1}{2}\mu'' e^{\mu}-\frac{1}{4}\mu'^2 e^{\mu}+\frac{1}{2r}\mu' e^{\mu-\nu}+\frac{1}{4}\mu'\nu' e^{\mu}\right]n_s n_p-\frac{1}{2r}\mu' e^{\mu-\nu}\delta_{sp}$$
$$=e[A]c[A]f[A]n_s n_p-e[A]d[A]f[A]\delta_{sp},$$
$$\left[\frac{1}{r}c[A]+d'[A]+(d[A]-c[A])\frac{1-e^{-\nu}}{r}\right](n_s\delta_{pt}-n_t\delta_{ps})=0,$$
$$e[B]T_4[A,B](c[B]n_p n_s-d[B]\delta_{ps})=0,$$
$$\left(f'[A]+\frac{1}{2}\mu' e^{\mu-\nu}(c[A]-d[A])-\frac{1}{2}\mu' f[A]\right)n_p=0,$$
$$\frac{\partial T_4[A,B]}{\partial r}n_p=0.$$

As a result of obvious simplifications we obtain Eqs. (11).

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