Self-action of radio waves in the vicinity of plasma resonance

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(Submitted October 18, 1972)
Zh. Eksp. Teor. Fiz. **64**, 1272-1282 (April 1973)

Self-action of a transverse electromagnetic wave on the appearance of parametric instability in a plasma is considered. The variation of the wave field structure in the region of its reflection is investigated. It is shown that in the initial period of development of the instability, self-action results in deep amplitude and phase modulation of the reflected wave.

1. INTRODUCTION

The interaction of transverse electromagnetic waves with the natural oscillations of a plasma increases sharply in resonant regions, where the wave frequency is close to one of the natural frequencies of the plasma. In particular, at plasma resonance ($\omega \to \omega_0$), the real part of the dielectric constant tends to zero:

$$\varepsilon_0 = 1 - \omega_0^2/\omega^2 \to 0, \quad \omega_0^2 = 4\pi e^2 N/m. \tag{1}$$

In this case, parametric excitation of Langmuir waves and of the ion-acoustic oscillations in the field E of the transverse wave is possible. The threshold amplitude E_{thr} of the field, at which parametric excitation sets in, decreases resonantly under conditions (1)[1,2]. It is important that in a weakly inhomogeneous plasma the condition $\epsilon_0 \to 0$ also determines the point of reflection of the transverse wave z_0.[1)] In the vicinity of the reflection point, the intensity of the wave field increases in proportion to $\epsilon_0^{-1/2}$. Thus, as double resonance takes place $\epsilon_0 \to 0$: the threshold field E_{thr} decreases, and the amplitude of the radio-wave field increases simultaneously. It should therefore be particularly easy to excite parametric instability in this case, as is indeed observed in experiment[3,4].

When the instability is excited, the conditions under which the perturbation-inducing radio wave propagates in the plasma are altered. This induces self-action in it. It is important that in the vicinity of the reflection point, the self-action effects are also resonantly amplified. Indeed, in the region where ϵ_0 tends to zero, even small changes of the plasma lead to an appreciable change in the value of ϵ, and consequently also to a change in the structure of the field of the radio wave. This, in turn, strongly affects the development of the instability.

The perturbation ϵ in the field of the wave is connected firstly with the overall variation of the plasma concentration due to its being pushed out of the region heated by the wave, or to distortion of the ionization-recombination balance. These processes also take place in the absence of the instability. They lead to a shift of the radio-wave reflection point[5], to the appearance of moving inhomogeneities in the reflection region, and to a buildup of slow relaxation oscillations of the plasma[6]. It is important, however, that the processes connected with the change of the average plasma concentration are usually characterized by considerable times τ_N on the order of the lifetime of the electron or the time of the plasma diffusion in the dimensions of the entire perturbed region.

Another type of perturbation ϵ is connected directly with the excitation of the plasma oscillations. It is usually characterized by the "fast" time $\tau \sim 1/\gamma$ (where γ is the increment of the instability) and plays a decisive role during the initial stages of the instability development[2)], as a result of which it can be considered separately, regardless of the perturbations of the average concentration. The present paper is devoted to an investigation of "fast" self-action effects in parametric buildup of oscillations in the region of radio-wave reflection.

2. STRUCTURE OF WAVE FIELD IN THE REFLECTION REGION

The field of an ordinary radio wave incident normally on an inhomogeneous plane plasma layer is described by the wave equation[7]

$$\frac{d^2E}{dz^2} + \frac{\omega^2}{c^2}\frac{\varepsilon_0 + \Delta\varepsilon}{\sin^2\alpha}E = 0, \quad \varepsilon_0 \ll 1, \quad \mathbf{E}\parallel\mathbf{H}. \tag{2}$$

where ϵ_0 is the dielectric constant of the plasma unperturbed by the wave (its imaginary part is neglected), and α is the angle between the external magnetic field **H** and the wave propagation direction. In the vicinity of the reflection point z_0 it can be assumed that ϵ_0 varies as a function of z linearly:

$$\varepsilon_0(z) = \mu(z - z_0), \quad \mu = \left(\frac{1}{N}\frac{dN}{dz}\right)_{z_0}$$

Here N(z) is the plasma concentration. Introducing the dimensionless coordinate

$$\xi = \left(\frac{c^2}{\omega^2\mu}\sin^2\alpha\right)^{-1/3}(z - z_0),$$

we rewrite (2) in the form

$$\frac{d^2E}{d\xi^2} + \left[\xi + \left(\frac{\omega}{c\mu\sin\alpha}\right)^{2/3}\Delta\varepsilon\right]E = 0. \tag{3}$$

Further, $\Delta\epsilon(\xi, t)$ is the perturbation of the dielectric constant in the plasma, which occurs when parametric instability is excited[3)]. It is important here that the unstable region is a system of narrow layers, the characteristic width of which is less than the wavelength in the plasma[8]. The reason for this is that the field of the standing wave in the vicinity of the reflection point forms a system of sharp maxima, near which the oscillations are excited. The noted singularity makes it possible to integrate Eq. (2) but without specifying more concretely the form of the function $\Delta\epsilon(\xi, t)$.

It is important that at not too large amplitudes of the incident wave, the oscillations that increase most intensely are those at the maximum that is closest to the reflection point, namely the principal maximum located at the point $\xi = \xi_1 = 1.019$. We therefore consider first the case when only excitation in the principal maximum is important, i.e., $\Delta\epsilon$ differs from zero only in one narrow layer near ξ_1. Then

$$\mathcal{R}_1^{2/3}\Delta\varepsilon = I\delta(\xi - \xi_1), \tag{4}$$

where we have introduced the notation

$$\mathcal{R}_1^{2/3}\int \Delta\varepsilon\, d\xi = I = I_1 + iI_2, \quad \mathcal{R}_m = \frac{\omega}{c\mu \sin^m \alpha}. \tag{5}$$

Substituting (4) in (3) and integrating it in the vicinity of ξ_1, we find that the derivative $dE/d\xi$ experiences a discontinuity at the point ξ_1:

$$\left(\frac{dE}{d\xi}\right)_{\xi_1+0} - \left(\frac{dE}{d\xi}\right)_{\xi_1-0} = -IE(\xi_1).$$

Since the point ξ_1 is located in our case at the first maximum, the position of which is not altered by the presence of an absorbing layer, we get $(dE/d\xi)_{\xi_1-0} = 0$. Consequently,

$$\frac{1}{E(\xi_1)}\left(\frac{dE}{d\xi}\right)_{\xi_1+0} = -I. \tag{6}$$

The solution of (3) in the region $\xi > \xi_1$ is given by

$$E(\xi) = E_{\rm inc} y_1(\xi) + (E_{\rm ref} - E_{\rm inc}) y_2(\xi), \tag{7}$$

where $y_1(\xi)$ and $y_2(\xi)$ are the solutions of the unperturbed equation (3) with $\Delta\epsilon = 0$, satisfying the relations

$$\begin{gathered} y_1(\xi \to -\infty) = 0, \\ y_1(\xi \gg 1) \approx (\sqrt{n}\sin\alpha)^{-1} 2\cos(2/3\,\xi^{3/2} - \varphi), \\ y_2(\xi \gg 1) \approx (\sqrt{n}\sin\alpha)^{-1} \exp\{i(2/3\,\xi^{3/2} - \varphi)\}. \end{gathered} \tag{8}$$

Here $n = \mathcal{R}_4^{-1/3}\sqrt{\xi}/\sin^2\alpha$ is the refractive index of the ordinary wave. The function $y_1(\xi)$ is a standing wave, i.e., a superposition of incident and reflected radio waves of equal intensity. Its normalization corresponds to a wave of unit amplitude normally incident from the region with n = 1 onto a layer of weakly-inhomogeneous plasma. The function $y_2(\xi)$ coincides with one of the waves making up $y_1(\xi)$, namely, with the reflected wave. Accordingly, the parameters $E_{\rm inc}$ and $E_{\rm ref}$, which enter in (7), denote the amplitudes of the plane waves incident on and reflected from the plasma layer. It follows from (7) and (6) that

$$\left.\frac{E_{\rm inc} y_1' + (E_{\rm ref} - E_{\rm inc}) y_2'}{E_{\rm inc} y_1 + (E_{\rm ref} - E_{\rm inc}) y_2}\right|_{\xi=\xi_1} = -I,$$

where the primes denote derivatives with respect to ξ. From this, recognizing that $y_1'(\xi_1) = 0$, we obtain the coefficient of reflection D of the radio wave from the layer and the amplitude of the field $E = E(\xi_1)$ in the principal maximum:

$$D = \frac{E_{\rm ref}}{E_{\rm inc}} = \frac{1+\eta^* I}{1+\eta I}, \tag{9}$$

$$\frac{E(\xi_1)}{E_{\rm inc} y_1(\xi_1)} = \frac{1}{1+\eta I}, \tag{10}$$

where

$$\eta = \eta_1 + i\eta_2 = \frac{y_2}{y_2'} = \frac{\operatorname{Im} y_2}{\operatorname{Im} y_2'} - i\frac{y_1^2}{4\mathcal{R}_4^{1/3}}. \tag{11}$$

In Eq. (11), the argument of all the functions is $\xi = \xi_1$. In the derivation of (9)—(11) we used the relations

$$\begin{gathered} y_1(\xi) = 2\operatorname{Re} y_2(\xi), \\ y_1(\xi) y_2'(\xi) - y_1'(\xi) y_2(\xi) = 2i\mathcal{R}_4^{1/3}, \end{gathered}$$

which follow from the asymptotic properties of the functions $y_{1,2}(\xi)$ (see (8)). Separating the real and imaginary parts in (9), we obtain the following expressions for the amplitude and phase of the reflected wave:

$$\begin{gathered} D = |D|e^{i\psi}, \quad |D|^2 = 1 + 4\eta_2 I_2 |1+\eta I|^{-2}, \\ \operatorname{tg}\psi = \frac{\operatorname{Im} D}{\operatorname{Re} D} = -2\eta_2 \frac{I_1 + \eta_1 |I|^2}{1 + 2\eta_1 I_1 + |I|^2(\eta_1^2 - \eta_2^2)}. \end{gathered} \tag{12}$$

With the aid of (10) and (11) we can easily verify that the first equation of (12) represents the law of energy conservation

$$\frac{c}{8\pi}[|E_{\rm inc}|^2 - |E_{\rm ref}|^2] = \int \langle \mathbf{Ej}\rangle dz = \frac{\omega}{8\pi}|E|^2 \int \operatorname{Im}\Delta\varepsilon\, dz.$$

Let us see how the reflection coefficient |D| varies with increasing perturbation $I \sim \int \Delta\epsilon\, dz$. It is seen from (12) that |D| first decreases (inasmuch as $\eta_2 < 0$) and reaches a minimum at $-I_2\eta_2 \sim 1$. Subsequently, the intensity of the reflected wave increases and at $|I\eta| \gg 1$ it again approaches the intensity of the incident wave. Simultaneously, the amplitude of the field in the principal maximum $E(\xi_1)$ tends to zero (see (10)). Thus, in the case $|I\eta| \gg 1$, the incident wave is reflected from a thin layer with $\Delta\epsilon \neq 0$, located at the principal maximum of the unperturbed wave $E(\xi) = E_{\rm inc} y_1(\xi)$, i.e., at $\xi = \xi_1$. The wave reflected from the layer ξ_1 forms a new first maximum. In the vicinity of this maximum there also appears a narrow layer of unperturbed plasma, and the entire process can repeat. Calculation of the behavior of the amplitude and of the phase of the wave in the n-th reflection is carried out with the aid of the same expressions (11) and (12), in which ξ_1 must be replaced by the coordinate ξ_n of the n-th layer, and by way of $y_1(\xi)$ it is necessary to take the amplitude of the standing wave produced after the (n − 1)-st reflection $y_1^{(n)}(\xi)$. It is easy to verify that in the geometrical-optics approximation ($\xi_n \gg 1$) we have

$$\eta = -i/\sqrt{\xi_n}, \quad [y_1^{(n)}(\xi_n)]^2 = 4\mathcal{R}_4^{1/3}/\sqrt{\xi_n}. \tag{13}$$

To calculate the coefficient η in the principal maximum $\xi = \xi_1$, it is necessary to use the explicit form of the functions $y_1(\xi)$ and $y_2(\xi)$. We have

$$y_1(\xi) = 2\mathcal{R}_4^{1/6}\Phi(-\xi),$$

$$y_2(\xi) = \mathcal{R}_4^{1/6}\frac{\sqrt{\pi}}{3}\sqrt{\xi}\left\{J_{1/3}\left(\frac{2}{3}\xi^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}\xi^{3/2}\right)\right] + i\sqrt{3}\left[J_{1/3}\left(\frac{2}{3}\xi^{3/2}\right) - J_{-1/3}\left(\frac{2}{3}\xi^{3/2}\right)\right]\right\}, \tag{14}$$

where $\Phi(-\xi)$ is the Airy function and $J_{\pm 1/3}(2/3\,\xi^{3/2})$ are Bessel functions. Then

$$\eta = -\Phi^2(-\xi_1)[\sqrt{3}+i] + \frac{1}{\sqrt{\xi_1}}\frac{J_{1/3}(2/3\,\xi_1^{3/2})}{J_{2/3}(2/3\,\xi_1^{3/2})} = -[0.16 + i\cdot 0.9], \quad \xi_1 = 1.019. \tag{15}$$

Comparison of expressions (13) and (15) shows that geometrical optics gives a good approximation even for the principal maximum ($\xi = \xi_1$). We note that the inequality $|\eta_1| \ll |\eta_2|$ is always satisfied. Making use of this fact, we can easily see from the last equation of (12) that as a result of each reflection the phase of the wave ψ changes approximately by $\pm\pi$ (depending on the sign of the numerator at the instant when the denominator vanishes).

We have considered above the successive excitation of a plasma in the vicinity of the first maximum of the field of an incident transverse wave. In a number of cases, simultaneous excitation of a large number of maxima produced by the standing wave, $E(\xi) = E_{\rm inc} y_1(\xi)$, becomes significant[4)]. In this case, using the geometrical-optics approximation, we can easily show that

$$E(\xi) = \mathcal{R}_4^{1/6}\frac{2E_{\rm inc}}{1 - i\Sigma^{(n)}}\frac{1}{\xi^{1/4}}\left\{\cos\left(\frac{2}{3}\xi^{3/2} - \frac{\pi}{4}\right) - \Sigma^{(j)}\sin\left(\frac{2}{3}\xi^{3/2} - \frac{\pi}{4}\right)\right\}, \quad \xi_j \leqslant \xi \leqslant \xi_{j+1},\ j = 1, 2, \ldots, n;\ D = (1 + i\Sigma^{(n)})/(1 - i\Sigma^{(n)}), \tag{16}$$

where

$$\Sigma^{(j)}=\sum_{k=1}^{j}I^{(k)}/\sqrt{\xi_k}.$$

Here $I^{(k)}$ is an integral of the type (5) calculated in a small vicinity $(\Delta\xi)_k$ of the k-th maximum ξ_k of the Airy function. The only condition for the applicability of (16) is the inequality $(\Delta\xi)_k|I^{(k)}|\ll 1$.

3. PERTURBATION OF THE DIELECTRIC CONSTANT

We determine now the form of the perturbation of the dielectric constant $\Delta\epsilon$ in (5) and (12). We consider the case of not too strong a wave

$$\frac{|E_{\rm inc}|^2}{8\pi}<\sqrt{\frac{2}{3}}\,1.75\frac{\sqrt{\xi_1}}{|\eta_2|}\mathscr{R}_1^{2/3}\sin\alpha\frac{v_{Ti}}{v_{Te}}NT_i. \tag{17}$$

Here $v_{Te,i}=\sqrt{T_{e,i}/m_{e,i}}$ is the thermal velocity of the electrons (ions), and it is assumed that the ion temperature T_i is close to the electron temperature ($T_e/T_i\sim 1$). Under similar conditions there are excited in the plasma Langmuir oscillations whose wavelength $\lambda=2\pi/k$ is much larger than the Debye radius of the electrons $D_e=\sqrt{T_e/4\pi e^2N}$ (condition for the smallness of the Landau damping)[8]. The increment γ of the excited instability is much less than the beat frequency $\Omega\sim kv_{Ti}$ between the incident pump wave and the Langmuir oscillations of the plasma. This makes it possible, in the calculation of the intensity of the oscillations, to use the formulas of the weak coupling between the waves:

$$\begin{gathered}\varepsilon^{(0)}\varphi_{\mathbf k}^{(0)}+\frac{a_e}{2}\delta\varepsilon_e^{(0)}\varphi_{\mathbf k}^{(-1)}=\frac{4\pi}{k^2}[\delta\rho_{\rm nl}^{(0)}+\delta\rho_{\rm sp}^{(0)}],\\ \frac{a_e^*}{2}\delta\varepsilon_e^{(0)}\varphi_{\mathbf k}^{(0)}+\left[\hat\varepsilon^{(-1)}+\frac{|a_e|^2}{4}\delta\varepsilon_e^{(0)}\right]\varphi_{\mathbf k}^{(-1)}=\frac{4\pi}{k^2}[\delta\rho_{\rm nl}^{(-1)}+\delta\rho_{\rm sp}^{(-1)}];\\ a_e=\frac{e(\mathbf{kE})}{m\omega^2},\quad \hat\varepsilon^{(-1)}=\varepsilon^{(-1)}+i\frac{\partial\varepsilon^{(-1)}}{\partial\Omega}\frac{\partial}{\partial t},\quad \Omega>0,\end{gathered} \tag{18}$$

where $\mathbf E$ is the amplitude of the transverse wave $[\mathbf E^*\exp(i\omega t)/2i+\text{c.c.}]$, $\varphi_{\mathbf k}^{(0)}$ and $\varphi_{\mathbf k}^{(-1)}$ are the harmonics of the potential $\varphi_{\mathbf k}$, corresponding to the frequencies Ω and $\Omega-\omega$. The longitudinal dielectric constant $\epsilon^{(0),(-1)}$ is equal in this case to

$$\varepsilon^{(-1)}=\varepsilon(\Omega-\omega,\mathbf k)=\frac{2}{\omega_0}\left[\delta-\Omega-\frac{i\nu_e}{2}\right],\quad \delta=\omega-\omega_k;$$

$$\varepsilon^{(0)}=\varepsilon(\Omega,\mathbf k)=1+\delta\varepsilon_e^{(0)}+\delta\varepsilon_i^{(0)},\quad \delta\varepsilon_e^{(0)}=(kD_e)^{-2},$$

$$\delta\varepsilon_i^{(0)}=(kD_i)^{-2}\tilde\alpha,\quad \tilde\alpha=\tilde\alpha_1+i\tilde\alpha_2=1+ix\left[\sqrt{\pi}+2i\int_0^x e^{t^2}dt\right],$$

$$x=\Omega/\sqrt{2}kv_{Ti}.$$

Here $\omega_{\mathbf k}=\omega_0[1+(3/2)(kD_e)^2]$ is the frequency of the Langmuir wave and ν_e is the frequency of the collisions of the electrons with the ions and the neutral particles. In the right-hand side of (18) are included the nonlinear charge density $\delta\rho_{\rm nl}$, due to the interaction of the harmonics of the Langmuir oscillations, and the spontaneous term $\delta\rho_{\rm sp}$.

The solutions of the dispersion equation corresponding to the system (18) yields the frequency and increment of the excited oscillations:

$$\Omega+i\gamma=\delta-i\frac{\nu_e}{2}+\frac{\omega_0}{2}\frac{|a_e|^2}{4}\frac{\delta\varepsilon_e^{(0)}\delta\varepsilon_i^{(0)}}{\varepsilon^{(0)}},$$

$$|a_e|^2=\frac{|E|^2\cos^2\theta_0}{4\pi NT_e}(kD_e)^2,$$

where θ_0 is the angle between the vector $\mathbf k$ of the Langmuir wave and the electric field $\mathbf E$ of the transverse pump wave. Carrying out the usual averaging of (18), we obtain an equation for the spectral density of the energy of the Langmuir oscillations $W_{\mathbf k}=(k^2/2\pi)\langle|\varphi^{(-1)}|^2\rangle_{\mathbf k}$:

$$\frac{dW_{\mathbf k}}{dt}+[\nu_e-2\gamma_{\rm nl}+2\Delta\gamma_{\rm nl}]W_{\mathbf k}=(2\pi)^{-3}\left[T_e\nu_e+\frac{\omega_0}{\Omega}T_i\cdot 2\gamma_{\rm nl}\right]. \tag{19}$$

The nonlinear increment $\gamma_{\rm nl}$ in this equation describes the growth of the oscillations in the field E of the transverse wave, and is equal to

$$2\gamma_{\rm nl}=\frac{|E|^2\cos^2\theta_0}{8\pi NT_i}\frac{\omega_0}{2F(x)}. \tag{20}$$

The correction $\Delta\gamma_{\rm nl}$ to the nonlinear increment, brought about by the induced scattering of the Langmuir waves by the ions, is given by

$$2\Delta\gamma_{\rm nl}=\omega_0\int\frac{(\mathbf{kk}_1)^2}{k^2k_1^2}\frac{1}{2F(\Delta x)}\frac{W_{\mathbf k_1}d^3k_1}{NT_i},$$

$$\Delta x=\frac{\omega_k-\omega_{k_1}}{\sqrt{2}|\mathbf k-\mathbf k_1|v_{Ti}}.$$

Formula (19) should be supplemented by an equation expressing the perturbation of the dielectric constant in terms of the spectral density of the noise $W_{\mathbf k}$:

$$\Delta\varepsilon=\int\frac{d^3kW_{\mathbf k}}{NT_i}\frac{\cos^2\theta_0}{2F(x)}[i+L(x)]. \tag{21}$$

We have used here the notation

$$\frac{\delta\varepsilon_e^{(0)}\delta\varepsilon_i^{(0)}}{\varepsilon^{(0)}}=\frac{T_e}{T_i}(kD_e)^{-2}\frac{i+L(x)}{F(x)},$$

$$1/F(x)=\sqrt{\pi}xe^{-x^2}\left|1+\frac{T_e}{T_i}\tilde\alpha(x)\right|^{-2},\quad L(x)=\frac{\tilde\alpha_1(x)+|\tilde\alpha(x)|^2T_e/T_i}{\tilde\alpha_2(x)},$$

$$x=\Omega/\sqrt{2}kv_{Ti},\quad \Omega=\omega-\omega_0[1+{}^3/_2(kD_e)^2].$$

It is easy to see that the system of equations (19)–(21) satisfies the energy conservation law

$$\langle\mathbf{Ej}\rangle=\frac{\omega}{8\pi}|E|^2\,\mathrm{Im}\,\Delta\varepsilon=\int[\nu_e+2\gamma(\mathbf k)]W_{\mathbf k}d^3k.$$

In a one-temperature plasma with $T_e\approx T_i$, the minimum value of the function $F(x)$ (corresponding to the maximum increment) turns out to be almost constant: $\min F(x)=F(x_1)=1.75$; the parameter $x_1\sim 1$, $|L(x_1)|<0.1$ (thus, at $T_e=T_i$ we have $x_1=1.24$, $L(x_1)=0.09$)[8].

The equation for the waves (19) was used by other authors[9,10] to calculate the level of the steady-state noise in parametric excitation of plasma. We note that in[9,10] they used an approximate equation for the nonlinear increment $\gamma_{\rm nl}$. When averaging the equations in (18), it must be borne in mind that the nonlinear charge density $\delta\rho_{\rm nl}$ is due to the interaction of the harmonics $\varphi_{\mathbf k}^{(1)}$ (and therefore the correction $\Delta\gamma_{\rm nl}$ to the increment has the same form as in the absence of a pump field), whereas the spontaneous source in (19), to the contrary, is generated mainly by the low-frequency component $\delta\rho_{\rm sp}^{(0)}$ of the spontaneous charge density. The condition for the existence of the considered kinetic instability $\nu_e/2<\gamma_{\rm nl}<\sqrt{2}kv_{Ti}\sim\Omega$ leads to a limitation on the intensity of the pump wave (17):

$$\begin{gathered}E_{\rm thr.}^2<|E|^2<E_{\rm thr\,1}^2,\\ E_{\rm thr}^2=16\pi F(x_1)\frac{\nu_e}{\omega_0}NT_i,\\ E_{\rm thr\,1}^2=32\sqrt{2}\pi F(x_1)\frac{v_{Ti}}{v_{Te}}(k_nD_e)NT_i.\end{gathered} \tag{22}$$

The threshold value of the wave number k_n, which enters in (22), corresponds simultaneously to the maximum of the nonlinear increment. It is obtained from the equation

$$\frac{\Omega(k_n)}{\sqrt{2}\,k_n v_{Ti}}=x_1,\quad F(x_1)=\min F(x),\quad \Omega=\omega-\omega_k. \tag{23}$$

The solution of (23) in the widely encountered case

$$\omega-\omega_0\gg {}^1/_3 x_1^2(v_{Ti}^2/v_{Te}^2)\omega_0,$$

or, equivalently

$$\beta=\frac{3}{\sqrt{2}}\frac{v_{Te}}{v_{Ti}}(k_nD_e)\gg x_1, \tag{24}$$

is given by

$$(k_nD_e)^2=\frac{2}{3}\frac{\omega-\omega_0}{\omega_0}. \tag{25}$$

4. SELF-MODULATION OF THE WAVE

Equations (10), (12), (19)—(21), and (5) form a complete system that makes it possible to calculate the behavior of the amplitude of the wave reflected from the plasma layer with allowance for the self-action. They have a simple analytic solution in the case when the time t_0, during which the developed oscillations do not exert a noticeable influence on the initial amplitude of the incident pump wave, $E(\xi)=E_{inc}y_1(\xi)$, greatly exceeds $1/\gamma_0$, where γ_0 is the maximum instability increment (see (20) and (23))[6)]

$$2\gamma_0 t_0\gg 1,\quad 2\gamma_0=\frac{|E_{inc}y_1(\xi_1)|^2}{8\pi NT_i}\frac{\omega}{2F(x_1)}. \tag{26}$$

We have taken into account here the fact that the instability develops most strongly at the principal maximum $\xi=\xi_1$ of the function $y_1(\xi)$, the function proportional to the Airy function (see (14)). Under the conditions (26), the spectral noise density $W_{\mathbf{k}}$ has a sharp maximum not only in coordinate space, but also in wave-number space. We assume furthermore that the amplitude $E_{inc}y_1(\xi_1)$ greatly exceeds the threshold value (22) (i.e., $2\gamma_0\gg\nu_e$), and consider the initial stage of the process, where the nonlinear interaction of the Langmuir waves in Eq. (19) is negligible ($|\Delta\gamma_{nl}|\ll\gamma_{nl}$). Then, using the dimensionless variables

$$\tau=2\gamma_0 t,\quad a(\tau)=\left|\frac{E(\xi_1,\tau)}{E_{inc}y_1(\xi_1)}\right|^2,$$

we can easily find from (5) and (19)—(21) that in the case $|\tau-\tau_0|\ll\tau_0$ and $\nu_e|\tau-\tau_0|/2\gamma_0\ll 1$ the integral I in (5) is equal to

$$I=I_0[i+L]\exp\left\{\int_{\tau_0}^{\tau}a(\tau')\,d\tau'\right\},$$
$$I_1/I_2=L(x_1)\equiv L, \tag{27}$$

where

$$I_0=\mathcal{R}_1^{2/3}\frac{(k_nD_e)^3}{ND_e^3}\frac{1}{\beta}\left[\frac{\omega}{\Omega(k_n)}+(2\pi)^3\left(\frac{W_{\mathbf{k}_n}}{T_i}\right)_0\right]$$
$$\times\frac{\exp\{-\nu_e\tau_0/2\gamma_0\}}{F(x_1)d}(8\pi)^{-1}\left[\frac{2F(x_1)}{F''(x_1)}\right]^{1/2}e^{\tau_0}/\tau_0^2. \tag{28}$$

Here $\mathcal{R}_1$ is defined in (5), and $(W_{\mathbf{k}_n})_0$ is the initial value of the spectral density. The parameters $F''(x_1)=d^2F(x)/dx^2|_{x=x_1}$ and $d=\sqrt{\xi_1}$, which enter in (28), are the result of the expansion of the function

$$\frac{y_1^2(\xi)}{F(x)}\cos^2\theta_0\tau_0\approx\frac{y_1^2(\xi_1)}{F(x_1)}[1-d^2(\xi-\xi_1)^2]$$
$$\times\left[1-\theta_0^2-\frac{F''(x_1)}{2F(x_1)}\beta^2\left(\frac{k-k_n}{k_n}\right)^2\right]\tau_0$$

in the vicinity of its maximum (the spectral noise density $W_{\mathbf{k}}(\xi)$ is proportional to an exponential of this function). The parameter $\beta\gg 1$ is defined in accordance with (24). The effective dimension of the region $\Delta\xi$, in which $\Delta\epsilon\neq 0$, turns out to be equal to $\Delta\xi=\sqrt{\pi/d^2\tau_0}$.

Equation (28) at fixed I_0 determines the dimensionless time $\tau_0=2\gamma_0 t_0$ during which the Langmuir oscillations exert no influence on the amplitude (and phase) of the pump wave. It follows from (10) that $I_0\ll 1$. Combining (10) and (27) we can easily obtain an equation that expresses the dimensionless time τ in terms of the dimensionless oscillation energy $I_2=\operatorname{Im} I$:

$$\ln\frac{I_2}{I_0}+2(I_2-I_0)(\eta_1L-\eta_2)+\frac{1}{2}(I_2^2-I_0^2)(1+L^2)|\eta|^2=\tau-\tau_0. \tag{29}$$

We now take into consideration the smallness of the parameters $|L|<0.1$ and $\eta_1/\eta_2=0.18$. Then, accurate to small terms quadratic in L and η_1/η_2, Eqs. (10), (12), and (29) can be rewritten in the form

$$a=|E(\xi_1)/E_{inc}y_1(\xi_1)|^2=(1+\tilde{I})^{-2}, \tag{30a}$$
$$|D|^2=1-4\tilde{I}/(1+\tilde{I})^2, \tag{30b}$$
$$\operatorname{tg}\psi=2\tilde{I}(L-\tilde{I}\eta_1/\eta_2)/(1-\tilde{I}^2), \tag{30c}$$
$$\ln(\tilde{I}/\tilde{I}_0)+{}^1/_2(2+\tilde{I})^2-{}^1/_2(2+\tilde{I}_0)^2=\tau-\tau_0, \tag{30d}$$

where $\tilde{I}=|\eta_2|I_1$, and the quantity $\tilde{I}_0=|\eta_2|I_0\ll 1$ can be reduced with the aid of (23) and (24) to the form

$$\tilde{I}_0=\mathcal{R}_1^{1/3}(ND_e^3)^{-1}\frac{|\eta_2|\sqrt{\pi\xi_1}}{x_1d}\frac{\exp\{-\nu_e\tau_0/2\gamma_0\}}{(12\pi)^{3/2}F(x_1)}\left[\frac{2F(x_1)}{F''(x_1)}\right]^{1/2}\frac{e^{\tau_0}}{\tau_0^2};\quad d=\sqrt{\xi_1}. \tag{31}$$

We have taken into account here the fact that, in accordance with (25),

$$(k_nD_e)^2={}^1/_3\mathcal{R}_1^{-2/3}\xi_1$$

and have omitted the small term $(W_{\mathbf{k}_n})_0\Omega(k_n)/T_i\omega$. (In the absence of a pump field we have $W_{\mathbf{k}}=T_e/(2\pi)^3$.) According to (30b) and (30c), at $\tilde{I}=1$ the amplitude of the reflected wave vanishes, and $\tan\psi\to\infty$. A more rigorous allowance for the small corrections leads to the values

$$\min|D|^2={}^1/_4[L-\eta_1/\eta_2]^2,\quad \operatorname{tg}\psi=-\eta_2/\eta_1$$
$$\text{at}\quad |I_{min}|=|\eta|^{-1}.$$

The dependence of the reflection coefficient $|D|=|E_{ref}/E_{inc}|$, of the phase ψ, of the dimensionless time $\tau-\tau_0=2\gamma_0(t-t_0)$, and of the dimensionless intensity of the transverse wave at the point ξ_1

$$a=|E(\xi_1)/E_{inc}y_1(\xi_1)|^2$$

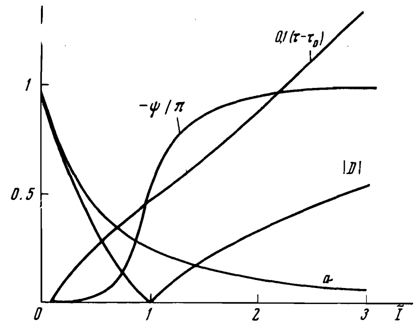
on the dimensionless noise energy

$$\tilde{I}=\frac{|\eta_2|}{2F(x_1)}\mathcal{R}_1^{1/3}\int\frac{W_{\mathbf{k}}\,d^3k}{NT_i}\frac{\omega}{c}dz$$

is shown in the figure (for $\tilde{I}_0=0.1$, $L=0$, and $\eta_1/\eta_2=0.18$). The noise energy $\tilde{I}$ at $t<t_0$, i.e., $\tau<\tau_0$, increases exponentially with time, and at $\tau>\tau_0$, as seen from the figure, its growth becomes much weaker due to the self-action of the transverse wave, $\tilde{I}\sim\sqrt{t-t_0}$. The amplitude of the reflected wave first decreases and then, at $t-t_0>1/\gamma_0$, increases somewhat more slowly; the variation of its phase is analogous. At

$$\tilde{I}\approx\sqrt{2(\tau-\tau_0)}\gg 1$$

there is a strong reflection of the incident wave from the thin perturbed layer located at $\xi=\xi_1$, i.e., a new standing wave is produced. As already indicated in Sec.



2, a new perturbed layer is produced at its first maximum, after a time $\sim t_0$, and leads to analogous changes of the amplitude and of the phase of the wave, etc. Thus, the reflected wave is amplitude- and phase-modulated, with a period on the order of $t_0 = \tau_0/2\gamma_0$.

We have neglected above the nonlinear interaction of the Langmuir waves. Using relations (19) and (22) we can show that they become significant if $\tilde{I}$ becomes larger than or approximately equal to the quantity $\tilde{I}_m$, defined by the relation

$$\frac{1}{\varkappa\sigma}\tilde{I}_m\left(1+\tilde{I}_m+\frac{1}{3}\tilde{I}_m^2\right)=\left|\frac{E_{\text{inc}}y_1(\xi_1)}{E_{\text{thr}\,1}}\right|^2<1, \qquad (32)$$

where, with logarithmic accuracy,

$$\varkappa=\frac{4|\eta_2|\xi_1}{\beta}\sqrt{\frac{\pi}{\tau_0 d^2}}, \quad \sigma=\tau_0+\frac{2}{3}\ln\mathscr{R}_1.$$

We consider therefore another limiting case, when the nonlinear interaction of the Langmuir waves becomes decisive, and $\tilde{I}_m > 1$. In this case, the noise density is redistributed energywise over the spectrum, owing to the interaction between the waves. In the steady state, according to[9-11], the dimensionless oscillation energy in the excitation region ($x = \Omega/\sqrt{2}kv_{Ti} \sim 1$, $\xi \approx \xi_1$) is equal to $\tilde{I}_{sat}$, where

$$\tilde{I}_{\text{sat}}\approx\frac{4|\eta_2|\xi_1}{\beta}\left|\frac{E(\xi_1)}{E_{\text{thr}\,1}}\right|^2. \qquad (33)$$

Thus, as a result of the nonlinear interaction of the Langmuir waves, the parameter $\tilde{I}$ in expressions (30a) and (30c) decreases from the value $\tilde{I}_m$ defined by (32) to a value $\tilde{I}_{sat} \ll 1$. (We note that the process of establishment of the noise density can be accompanied by rapid oscillations of the parameter $\tilde{I}$, with the characteristic time $T \sim \tilde{I}_{sat}\sqrt{\pi/\tau_0 d^2}/2\gamma_0\tilde{I}_m$.) Consequently, as seen from formula (33), the plasma perturbation in the region of the principal maximum ceases, after establishment of the oscillations, to exert a noticeable influence on the propagation of the transverse wave, i.e., $E(\xi) = E_{inc}y_1(\xi)$. The reflection coefficient increases in this case to unity. Subsequently, however, after a time on the order of $t_0 = \tau_0/2\gamma_0$, a plasma perturbation builds up in another narrow layer adjacent to the next, second maximum of the Airy function $y_1(\xi)$, and then in the third layer, etc. Calculation of the reflection coefficient upon excitation of the n-th maximum, located at the point $\xi = \xi_n$, is based on the same formulas (30) and (31), in which ξ_1 must be replaced by ξ_n and we must put $\eta \approx -i/\sqrt{\xi_n}$ (see (13) and (15)).

Thus, even when the influence of the nonlinear interaction of the Langmuir waves is taken into account, the reflected transverse wave turns out to be modulated at the initial stage of the perturbation, with a frequency on the order of $2\gamma_0/\tau_0$.

The authors are indebted to I. S. Shlyuger, who called their attention to the questions considered in this paper.

[1)]We are considering here a planar-stratified plasma, which is inhomogeneous in the direction of the radio-wave propagation. It is assumed that the characteristic dimension of the inhomogeneity is much larger than the wavelength. Such conditions are realized, for example, in the ionosphere.

[2)]For example, in the upper ionosphere $\tau \sim 10^{-2} - 10^{-4}$ sec, whereas $\tau_N \sim 10^2 - 10^3$ sec.

[3)]It is assumed in (2) that the temporal variation of $\Delta\epsilon$ are quasistationary, i.e., the change of $\Delta\epsilon$ during the time of wave propagation in the perturbed zone is small.

[4)]Excitation of a large number of maxima occurs at a sufficiently high intensity of the incident wave, and also upon saturation of the turbulence (see below).

[5)]Equation (19) was derived for an isotropic plasma. However, since the maximum of γ_{nl} corresponds to $\cos^2\theta_0 = 1$, i.e., $\mathbf{k} \parallel \mathbf{E} \parallel \mathbf{H}$, the influence of the external magnetic field **H** can indeed be neglected.

[6)]For example, under the conditions of the ionosphere, the parameter $2\gamma_0 t_0$ has an approximate value 15–20.

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Translated by J. G. Adashko
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