Kinetics of stimulated scattering of Langmuir waves by plasma ions

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The structure of stationary Langmuir turbulence spectra is investigated by taking into account generation of oscillations and their induced scattering by plasma ions. It is shown that under conditions of the "diffusion" approximation for scattering the spectrum has the form of thin jets; the oscillation spectral distribution n_k does not vanish on lines or on surfaces in k-space. The results are illustrated by the problems of relaxation of an electron beam in a plasma and heating of a plasma by an intense electromagnetic wave. The process of establishment of a stationary spectrum is modelled numerically.

INTRODUCTION

A typical situation in a plasma is one in which the presence of instabilities excites oscillations that grow to a level determined by nonlinear effects, so that a stationary spectrum of turbulent pulsations is established in the plasma. We consider the problem of determining this spectrum for Langmuir waves, assuming the principal nonlinear process to be induced scattering of oscillations by ions (see [1,2]). The scattering by ions limits, in particular, the development of the instability in two possible methods of heating the plasma—by a relativistic electron beam and by a powerful electromagnetic wave. Owing to the scattering, energy of the Langmuir oscillations is shifted over the spectrum from the region of instability into the region of long waves, where dissipation takes place. One of the methods of dissipation of long-wave oscillations can be the collapse of Langmuir waves, which was considered by one of the authors[3]. To answer the question of the efficiency of heating of the plasma, it is necessary to know the structure of the turbulence spectrum, and this reason alone makes our problem quite timely.

Up to now, the spectra of Langmuir waves established as a result of scattering have been investigated on the assumption that these waves are isotropic[2,4]. It was assumed in fact that the isotropization takes place even in the case of anisotropic excitation of the oscillations. It is shown in the present paper that the situation is quite different, and that even a small angular asymmetry of the instability increment makes the stationary spectrum of the Langmuir waves essentially anisotropic, and the oscillations turn out to be concentrated on lines or surfaces ("jets") in k-space. This result was derived analytically and was confirmed by simulation of Langmuir turbulence with a computer.

The main results of the present paper are given in Sec. 2. It is prefaced by Sec. 1, in which the induced-scattering matrix element is derived on the basis of the simplified dynamic description proposed in[3] for a plasma. In Secs. 3 and 4 we consider the applications of the general results to problems on relaxation of a relativistic electron beam and the nonlinear stage of parametric instability of a plasma in a homogeneous high-frequency (hf) field. The results of numerical experiments are given in Sec. 5.

1. FUNDAMENTAL EQUATIONS

To find the matrix element of the induced scattering of Langmuir oscillations by ions, we obtain an equation for slow (ionic) motions of the plasma, in which Langmuir oscillations are excited. In the derivation of this equation we average over the "fast" time $1/\omega_p$ ($\omega_p = (4\pi e^2 n_0/m)^{1/2}$ is the plasma frequency and n_0 is the unperturbed value of the plasma concentration), in analogy with the procedure used in[3].

We write down the hf part of the electrostatic potential in the form

$$\varphi = (\psi e^{i\omega_p t} + \psi^* e^{-i\omega_p t}),$$

where $\psi(\mathbf{r}, t)$ is a slowly varying function of the time. The equation for ψ can be obtained by linearizing the hydrodynamic equations for the electrons (see[3]):

$$\Delta(i\psi_t + {}^3/_2\omega_p r_D^2 \Delta\psi) = (\omega_p/2n_0)\,\mathrm{div}\,\delta n \nabla\psi. \tag{1}$$

Here $r_D = (T_e/4\pi e^2 n_0)^{1/2}$ is the Debye radius and δn is the perturbation of the plasma concentration under the influence of the hf oscillations. This action can be described with the aid of the hf potential

$$U = (e^2/4m\omega_p^2)\,|\nabla\psi|^2. \tag{2}$$

Then

$$\delta n_{\mathbf{k}\omega} = (n_0/T_e) G_{\mathbf{k}\omega} U_{\mathbf{k}\omega}. \tag{3}$$

In the approximation in which the plasma is quasi-neutral in slow motions, the Green's function $G_{\mathbf{k}\omega}$ takes the form

$$G_{\mathbf{k}\omega} = \frac{T_e}{Mn_0} L_{\mathbf{k}\omega} \Big/ \left(1 - \frac{4\pi e^2 r_D^2}{M} L_{\mathbf{k}\omega}\right),$$
$$L_{\mathbf{k}\omega} = \int \frac{(\mathbf{k}\,\partial f/\partial \mathbf{v})}{\mathbf{kv} - \omega} d\mathbf{v} \tag{4}$$

($f(\mathbf{v})$ is the unperturbed ion distribution function). The Green's function $G_{\mathbf{k}\omega}$ in an isotropic plasma depends only on $|\mathbf{k}|$ and has the obvious symmetry properties:

$$G_{\mathbf{k}\omega} = G^*_{-\mathbf{k},-\omega} = G_{-\mathbf{k},\omega}. \tag{5}$$

At $T_i/T_e \ll 1$, it has a pole corresponding to the ion-acoustic waves:

$$G_{\mathbf{k}\omega} \approx \frac{k^2 T_e}{M(\omega^2 + 2i\gamma_s\omega - c_s^2 k^2)} \tag{6}$$

($c_s = (T_e/M)^{1/2}$ is the velocity of the ion sound and γ_s is the damping decrement). Formula (4) takes direct account of only the Landau damping by the ions, but one can include in the pole part of the Green's function also the collision damping and the Landau damping by the electrons.

The system (1)—(4) can be conveniently reduced to a single equation by introducing the variable

$$a_{\mathbf{k}} = \frac{k}{8\pi e}\left(\frac{n_0}{2m\omega_p}\right)^{-1/2} \psi_{\mathbf{k}},$$

defined in such a way that the quantity

$$\varepsilon=\int \omega_p(1+{}^3/_2k^2r_D{}^2)\,|a_{\mathbf{k}}|^2\,d\mathbf{k}$$

coincides with the total energy of the Langmuir oscillations. This equation takes the form

$$i\frac{\partial a_{\mathbf{k}}}{\partial t}+(\omega_{\mathbf{k}}-i\gamma_{\mathbf{k}})a_{\mathbf{k}} \tag{7}$$

$$=\int T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}a_{\mathbf{k}_1}{}^*a_{\mathbf{k}_2}a_{\mathbf{k}_3}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3,$$

where $\omega_{\mathbf{k}}=(3/2)\omega_p(kr_D)^2$,

$$T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}=\frac{(\mathbf{k}\mathbf{k}_2)(\mathbf{k}_1\mathbf{k}_3)G_{\mathbf{k}_1-\mathbf{k}_3,\,\omega_{\mathbf{k}1}-\omega_{\mathbf{k}3}}+(\mathbf{k}_1\mathbf{k}_2)(\mathbf{k}\mathbf{k}_3)G_{\mathbf{k}_1-\mathbf{k}_2,\,\omega_{\mathbf{k}1}-\omega_{\mathbf{k}2}}}{kk_1k_2k_3}.$$

The right-hand side of (7) is assumed here to be sufficiently small and is taken into account by perturbation theory. The left-hand side includes the term $-i\gamma_{\mathbf{k}}a_{\mathbf{k}}$, where $\gamma_{\mathbf{k}}$ is the Langmuir-oscillation instability increment.

We note that Eq. (7) contains amplitudes of Langmuir waves only, since the ion-sound oscillations are assumed to be "static." This means that the characteristic sound-damping time is smaller than the time of the nonlinear process. The criterion for the applicability of Eq. (7), expressed in terms of the energy density of the Langmuir oscillations W, is

$$\frac{W}{n_0T_e}<\begin{cases}(kr_D)^2 & \text{for } kr_D<\left(\frac{m}{M}\right)^{1/2}\\[2ex] \left(\frac{\gamma_s(k)}{c_sk}\right)^2(kr_D)\left(\frac{m}{M}\right)^{1/2} & \text{for } kr_D>\left(\frac{m}{M}\right)^{1/2}\end{cases} \tag{8}$$

(k is the characteristic value of the wave vector of the Langmuir waves).

The kernel of the integral in the right-hand side of (7) has the following symmetry properties, which follow from the symmetry of the Green's function (see (5)):

$$T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}=T_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2\mathbf{k}_3},\qquad \operatorname{Re}T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}=\operatorname{Re}T_{\mathbf{k}_2\mathbf{k}_3\mathbf{k}\mathbf{k}_1}$$

$$\text{at}\quad \omega_{\mathbf{k}}+\omega_{\mathbf{k}_1}=\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3}. \tag{9}$$

In addition, we have the obvious relation

$$T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}=T_{\mathbf{k}_1\mathbf{k}\mathbf{k}_3\mathbf{k}_2}.$$

Assuming the phases of the quantities $a_{\mathbf{k}}$ to be random, we can carry out the averaging

$$n_{\mathbf{k}}\delta(\mathbf{k}-\mathbf{k}')=\langle a_{\mathbf{k}}a^*_{\mathbf{k}'}\rangle$$

and obtain from (7) a kinetic equation for the Langmuir waves, describing the generation of oscillations and their induced scattering by ions:

$$\frac{\partial n_{\mathbf{k}}}{\partial t}=2n_{\mathbf{k}}\left(\gamma_{\mathbf{k}}+\int T_{\mathbf{k}\mathbf{k}'}n_{\mathbf{k}'}\,d\mathbf{k}'\right)$$

$$T_{\mathbf{k}\mathbf{k}'}=-T_{\mathbf{k}'\mathbf{k}}=2\operatorname{Im}T_{\mathbf{k}\mathbf{k}',\mathbf{k}\mathbf{k}'}=\frac{\omega_p{}^2}{2n_0T_e}\frac{(\mathbf{k}\mathbf{k}_1)^2}{k^2k_1{}^2}\operatorname{Im}G_{\mathbf{k}-\mathbf{k}',\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}}. \tag{10}$$

At $\gamma_{\mathbf{k}}\equiv 0$, Eq. (10) conserves the quantity

$$I=\int |a_{\mathbf{k}}|^2\,d\mathbf{k},$$

which has the meaning of the total number of Langmuir quanta.

From the energy and momentum conservation laws it follows that at $kr_D>(m/M)^{1/2}$ the change of the modulus of the wave vector in one scattering act is small in comparison with the modulus itself:

$$\frac{\Delta k}{k}\sim\left(\frac{m}{M}\right)^{1/2}\frac{1}{kr_D}\ll 1.$$

It is then possible to change over the so-called differential approximation for the kernel $T_{\mathbf{k}\mathbf{k}'}$ (see [1,2]). Taking the anti-symmetry of $T_{\mathbf{k}\mathbf{k}'}$ into account, we have

$$T_{\mathbf{k}\mathbf{k}'}\approx\frac{2}{9}\frac{m}{M}\frac{1}{r_D{}^2}\frac{\omega_p{}^2}{n_0T_e}\cos^2\Omega(1-\cos\Omega)\alpha\delta'(|\mathbf{k}'|-|\mathbf{k}|). \tag{11}$$

Here Ω is the angle between the vectors **k** and **k'**, the prime on the δ function denotes differentiation with respect to the argument, and the dimensionless quantity α is given by the following formulas:

$$\alpha=\int z\frac{\operatorname{Im}(F(z))\,dz}{|1-(T_e/T_i)F(z)|^2}, \tag{11a}$$

$$F(z)=-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{u\exp(-u^2/2)}{z-u+i0}du. \tag{11b}$$

The integral (11a) can easily be calculated[1)], (see, for example,[5]). It is equal to $-\pi$.

Equation (11) has a somewhat simpler form in the presence of axial symmetry. We denote by θ and θ' the angles between the chosen direction and **k** and **k'**, and obtain ultimately

$$\frac{\partial N(k,x)}{\partial t}=N(k,x)\left\{\gamma(k,x)+\frac{\partial}{\partial k}\int_{-1}^{1}T(x,y)N(k,y)\,dy\right\}. \tag{12}$$

We have introduced here the notation $x=\cos\theta$, $y=\cos\theta'$,

$$N(k,x)=k^2n(k,x),$$

$$T(x,y)=\frac{\pi^2}{9}\frac{m}{M}\frac{1}{r_D{}^2}\frac{\omega_p{}^2}{n_0T_e}\{1-x^2-y^2+3x^2y^2-3xy+3xy^3+3x^3y-5x^3y^3\}$$

We note that the kernel T(x, y) is of definite sign

$$T(x,y)\geqslant 0$$

and is symmetrical:

$$T(x,y)=T(y,x),\quad T(-x,-y)=T(x,y).$$

In addition

$$T(1,1)=T(-1,-1)=0.$$

2. JETS IN k-SPACE

Let us consider the stationary spectra of Langmuir turbulence. The stationary spectra $n_{\mathbf{k}}$ satisfy the equation

$$n_{\mathbf{k}}\left(\gamma_{\mathbf{k}}+\int T_{\mathbf{k}\mathbf{k}'}n_{\mathbf{k}'}\,d\mathbf{k}'\right)=0. \tag{13}$$

At first glance it may seem that this equation determines $n_{\mathbf{k}}$ with a great degree of leeway; one can assume, for example, $n_{\mathbf{k}}\equiv 0$ in any prescribed region of **k**-space. This leeway is eliminated by requiring that the stationary states be stable with respect to wave excitation in those **k**-space regions where $n_{\mathbf{k}}=0$. The stability condition, together with the stationarity condition (13), leads to the relations

$$\begin{aligned}\gamma_{\mathbf{k}}&=\tilde\gamma_{\mathbf{k}}\quad\text{if}\quad n_{\mathbf{k}}\neq 0,\\ \gamma_{\mathbf{k}}&<\tilde\gamma_{\mathbf{k}}\quad\text{if}\quad n_{\mathbf{k}}=0,\end{aligned} \tag{14}$$

where $\tilde\gamma_{\mathbf{k}}=-\int T_{\mathbf{k}\mathbf{k}'}n_{\mathbf{k}'}d_{\mathbf{k}'}$.

Relations (14) determine the spectral distribution together with the region where this distribution is different from zero. In the particular case when the Fredholm equation of the first kind $\gamma_{\mathbf{k}}=\tilde\gamma_{\mathbf{k}}$ has a regular positive solution, this region can be all of **k**-space. Relations (14) are similar to the "external stability" conditions in nonlinear theory of parametric excitation (see[6]).

The conditions (14) can be interpreted geometrically. To this end, we consider them at a fixed modulus of the wave vector. Then the quantities $\gamma_{\mathbf{k}}$, and $\tilde{\gamma}_{\mathbf{k}}$ become functions of the solid angle; their plots can be represented in the form of two surfaces. Formulas (14) signify that the surface $\gamma_{\mathbf{k}}$ is located inside the surface $\tilde{\gamma}_{\mathbf{k}}$. Tangency between these surfaces is possible; the aggregate of all the tangency points constitutes the "carrier" of the function $n_{\mathbf{k}}$—the set of points at which $n_{\mathbf{k}} \neq 0$.

We assume the differential approximation (11) for the kernel $T_{\mathbf{kk}'}$. Then the angular dependence of the function $\tilde{\gamma}_{\mathbf{k}}$ has a perfectly defined meaning, and represents a third-degree polynomial of the sines and cosines of the angles. The function $\gamma_{\mathbf{k}}$, generally speaking, is arbitrary. Therefore tangency of the surfaces is possible only at a discrete set of points, or else, if $\gamma_{\mathbf{k}}$ has axial symmetry, on a discrete set of circles[2)].

We note further that in the differential approximation, the relations (14) with different $|\mathbf{k}|$ are independent and can be regarded as a set of equations for $n_{\mathbf{k}}$. This means that the set of points at which the surfaces $\gamma_{\mathbf{k}}$ and $\tilde{\gamma}_{\mathbf{k}}$ are tangent has a continuous dependence on $|\mathbf{k}|$. In other words, each element of this set (point or circle) generates a line or surface of revolution in **k**-space. These lines (or surfaces), on which the spectral distribution is concentrated, will be called one-dimensional or two-dimensional "jets," respectively. We confine ourselves henceforth to the axially-symmetrical situation. The conditions (14) now take the form

$$\begin{aligned} \gamma(k, x) &= \tilde{\gamma}(k, x) \quad \text{if} \quad N(k, x) \neq 0, \\ \gamma(k, x) &< \tilde{\gamma}(k, x) \quad \text{if} \quad N(k, x) = 0. \end{aligned} \tag{15}$$

Here

$$\tilde{\gamma}(k, x) = -\frac{\partial}{\partial k} \int_{-1}^{1} T(x, y) N(k, y)\, dy.$$

In accordance with all the foregoing, we must seek the spectral density N(k, x) of the oscillations in the form

$$N(k, x) = \sum_i N_i(k)\, \delta(x - x_i(k)). \tag{16}$$

Here $x_i(k)$ is the shape of the jet and $N_i(k)$ is the intensity distribution along the jet. In the axially-symmetrical situation, the jets are two-dimensional; the only possible type of one-dimensional jet is $x = \pm 1$, when the surfaces $\gamma_{\mathbf{k}}$ and $\tilde{\gamma}_{\mathbf{k}}$ are tangent at their poles.

Let us assume that we know the number of jets r and their shape $x_i(k)$, i = 1, ..., r. Then, substituting (16) in (15), we obtain a system of ordinary differential equations for the determination of the intensities:

$$\gamma(k, x_i(k)) + \sum_j T(x_i(k), x_j(k)) \frac{dN_j}{dk} - \sum_j \frac{\partial}{\partial x_j} T(x_i(k), x_j(k)) \frac{dx_j(k)}{dk} N_j(k) = 0. \tag{17}$$

The number of jets and the fact of existence or absence of one-dimensional jets on the poles should be determined from geometrical considerations. To determine the shape of the i-th two-dimensional jets it is necessary to use the obvious relation

$$\frac{d}{dx}[\gamma(k, x) - \tilde{\gamma}(k, x)]\big|_{x = x_i(k)} = 0. \tag{18}$$

Substituting N(k, x) in (18), we obtain an additional set of equations which makes the system (17) closed.

The jets transport the flux of Langmuir quanta over the spectrum into the region of small wave numbers. Let us determine the value of this flux $P_{\mathbf{k}}$. To this end, we integrate (12) over the angles and introduce the symbol

$$\overline{N} = \int_{-1}^{1} N(k, x)\, dx.$$

We have

$$\frac{\partial \overline{N}}{\partial t} = \int_{-1}^{1} \gamma(k, x) N(k, x)\, dx + \frac{\partial P_k}{\partial k}, \tag{19}$$

$$P_k = \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} T(x, y) N(k, x) N(k, y)\, dx\, dy > 0.$$

Substituting (16) in (19), we express $P_{\mathbf{k}}$ in terms of the intensities of the jets:

$$P_k = \frac{1}{2} \sum_{i,j} T(x_i(k), x_j(k)) N_i(k) N_j(k). \tag{20}$$

It is seen from (20) that the spectrum cannot consist of merely one one-dimensional jet, since a single one-dimensional jet $N_i(k, x) = N\delta(x \pm 1)$ would lead to a zero flux, by virtue of the conditions T(1, 1) = T(−1, −1) = 0.

We consider now several examples of the determination of the shape of the jets.

1. Assume that the condition $\gamma(k, x) \equiv 0$ is satisfied in a region of **k**-space, $k_1 < |\mathbf{k}| < k_2$. What is realized in this region is the Kolmogorov situation, which corresponds to constancy of the flux of the Langmuir quanta. The Kolmogorov solution of (12) is obviously of the form

$$N(k, x) = f(x),$$

where f is an arbitrary function of x. We see therefore that the trajectories of the jets on the (k, x) plane should be straight lines parallel to the k axis. The position of these lines is determined by the condition that they be joined together at $|\mathbf{k}| = k_2$.

2. Let $\gamma(k, x)$ have a sharply pronounced maximum at $x = \pm 1$. In this case there are two one-dimensional jets

$$N(k, x) = N_1 \delta(x - 1) + N_2 \delta(x + 1),$$

with

$$\frac{dN_1}{dk} = -\frac{\gamma(k, -1)}{T(-1, 1)}, \quad \frac{dN_2}{dk} = -\frac{\gamma(k, 1)}{T(1, -1)}.$$

The condition of "external stability" (15) yields the necessary and sufficient criterion for the existence of two one-dimensional jets:

$$\gamma(k, x) < {}^{1}\!/_{2}\{(x^2 + x^3)\gamma(k, 1) + (x^2 - x^3)\gamma(k, -1)\}, \quad -1 < x < 1. \tag{21}$$

This criterion takes on a particularly simple form in the symmetrical situation, when $\gamma(k, x) = \gamma(k, -x)$. We then have

$$\gamma(k, x) < x^2 \gamma(k, 1), \quad |x| < 1. \tag{22}$$

3. Let $\gamma(k, x)$ be a symmetrical function of x and let it have a sharply pronounced maximum at x = 0. We

consider the possible existence of one two-dimensional jet at the point x = 0. Putting $N(k, x) = N(k)\,\delta(x)$, we obtain

$$\gamma(k, 0) = -T(0, 0)\, dN(k)\,/\,dk. \tag{23}$$

The external-stability condition (15) yields

$$\gamma(k, x) < \gamma(k, 0)\,(1 - x^2).$$

The situation with one two-dimensional jet is also characteristic of the case when $\gamma(k, x)$ has a sharp maximum at sufficiently small x.

In the general case, the problem of determining the number of jets and their shapes is quite complicated; nor is the question of the uniqueness of such a distribution trivial.

We now raise the question of the real thickness of the jets. To this end we include in the kinetic equation (10) small terms connected with the thermal noise and four-plasmon processes. Then Eq. (10) takes the form

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = 2n_{\mathbf{k}}\left(\gamma_{\mathbf{k}} + \int T_{\mathbf{k}\mathbf{k}'} n_{\mathbf{k}'}\, d\mathbf{k}'\right) + 2\pi \int |T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}|^2 \delta(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3})$$
$$\times\, \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\,(n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} + n_{\mathbf{k}} n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_3} - n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2})\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3 + F_{\mathbf{k}}. \tag{24}$$

Here $F_{\mathbf{k}} \approx \nu_{ei} T_e / \omega_p$ is the contribution from the thermal noises, and ν_{ei} is the frequency of the electron-ion collisions. The four-plasmon collision term becomes much simpler in the diffusion approximation.

We note first that when calculating the quantity $|T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}|^2$ we can neglect the crossing terms and put

$$|T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}|^2 \approx \left(\frac{\omega_p}{8n_0 T_e}\right)^2 \left[\frac{(\mathbf{k}\mathbf{k}_2)^2(\mathbf{k}_1\mathbf{k}_3)^2}{k^2 k_1^2 k_2^2 k_3^2} |G_{\mathbf{k}_1-\mathbf{k}_3,\,\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_3}}|^2 + \frac{(\mathbf{k}\mathbf{k}_3)^2(\mathbf{k}_1\mathbf{k}_2)^2}{k^2 k_1^2 k_2^2 k_3^2} |G_{\mathbf{k}_1-\mathbf{k}_2,\,\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}}|^2\right].$$

For the squares of the moduli of the Green's functions we have in the differential approximation an expansion in terms of the even derivatives of the δ function:

$$|G_{\mathbf{k}_1-\mathbf{k}_2,\,\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}}|^2 \approx \sqrt{\frac{2m}{9M}}\frac{|\mathbf{k}_1-\mathbf{k}_2|}{k_1 r_D}\,\alpha_1\!\left(\frac{T_i}{T_e}\right)\delta(|\mathbf{k}_1| - |\mathbf{k}_2|)$$
$$+ \frac{2\sqrt{2}}{27}\left(\frac{m}{M}\right)^{3/2}\frac{|\mathbf{k}_1-\mathbf{k}_2|^3}{k_1^3}\,\alpha_2\!\left(\frac{T_i}{T_e}\right)\delta''(|\mathbf{k}_1| - |\mathbf{k}_2|) + \ldots \tag{25}$$

Here

$$\alpha_1\!\left(\frac{T_i}{T_e}\right) = \left(\frac{T_e}{T_i}\right)^{3/2}\int_{-\infty}^{\infty}\frac{|F(z)|^2\,dz}{|1-(T_e/T_i)F(z)|^2},$$

$$\alpha_2\!\left(\frac{T_i}{T_e}\right) = \left(\frac{T_e}{T_i}\right)^{1/2}\int_{-\infty}^{\infty}\frac{z^2|F(z)|^2\,dz}{|1-(T_e/T_i)F(z)|^2}.$$

The function F(z) is defined by the formula (11b). The coefficients α_1 and α_2 show a strong dependence on T_i/T_e. At $T_e \ll T_i$ we have

$$\alpha_1 \approx \frac{1}{\sqrt{2\pi}}\left(\frac{T_e}{T_i}\right)^{3/2}, \quad \alpha_2 \approx \frac{1}{2\sqrt{2\pi}}\left(\frac{T_e}{T_i}\right)^{1/2}.$$

At $T_i \ll T_e$

$$\alpha_1 = \alpha_2 = {}^1\!/_2\pi\omega_s/\gamma_s.$$

Here ω_s is the ion-sound frequency and γ_s is the decrement of the Landau damping by ions. It is easy to show that if the damping by the ions is small, then γ_s should be replaced by the decrement for the sound damping by the electrons.

The expansion (25) enables us to replace the four-plasmon collision term in (24) by the model expression

$$(\partial n_{\mathbf{k}}/\partial t)_{st} \approx \varepsilon_1 + \varepsilon_2 \partial^2 n_{\mathbf{k}}/\partial k^2. \tag{26}$$

Here

$$\varepsilon_1 \approx \omega_p\left(\frac{m}{M}\right)^{1/2}\frac{1}{(kr_D)^3}\left(\frac{W}{n_0 T_e}\right)^2 \alpha_1 n_{\mathbf{k}},$$

$$\varepsilon_2 \approx \omega_p\left(\frac{m}{M}\right)^{3/2}\frac{1}{(kr_D)^3 r_D^2}\left(\frac{W}{n_0 T_e}\right)^2 \alpha_2, \tag{27}$$

W is the energy density of the Langmuir oscillations.

The first term in (26) can be interpreted as an effective increment to the thermal noise, resulting from the four-plasmon processes (intrinsic noise), and the second term as "diffusion" in **k**-space as a result of the four-plasmon processes.

The contribution made to Eq. (24) by the four-plasmon processes exceeds the contribution from the thermal noise, provided the following inequality is satisfied:

$$\frac{W}{n_0 T_e} > (kr_D)^2\left(\frac{M}{m}\right)^{1/4}\left(\frac{\nu_{ei}}{\omega_p}\right)^{3/4}\alpha_1^{1/2}.$$

Under conditions when the diffusion approximation is valid, the second term in (26) is essentially smaller than the first. Finally, to determine the structure of the jets, we have the equation

$$n(k, x)\,[\gamma(k, x) - \tilde{\gamma}(k, x)] + \varepsilon_1 = 0.$$

Let $x = x_0(k)$ be the shape of the jet at $\epsilon_1 = 0$, i.e., the line where the functions $\gamma(k, x)$ and $\tilde{\gamma}(k, x)$ coincide. At $\epsilon_1 = 0$ we can put near $x = x_0(k)$

$$\gamma(k,x) - \tilde{\gamma}(k,x) \approx \beta_k[x - x_0(k)]^2, \quad \beta_k = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\gamma - \tilde{\gamma})\,|_{x=x_0(k)}.$$

Recognizing that ϵ_1 is finite, the functions $\gamma(k, x)$ and $\tilde{\gamma}(k, x)$ are equal only accurate to ϵ_1^2. With the same degree of accuracy we have

$$\gamma(k, x) - \tilde{\gamma}(k, x) = \alpha_k \varepsilon_1^2 + \beta_k[x - \hat{x}(k)]^2 + \ldots$$

Here $\hat{x}(k)$ is the jet shape "renormalized" on account of ϵ_1. For the jet structure we obtain the formula

$$n(k, x) = \frac{-\varepsilon_1}{\alpha_k \varepsilon_1^2 + \beta_k[x - \hat{x}(k)]^2}$$

To determine α_k, we note that in the zeroth approximation in ϵ_1 the integral intensity of the jet should remain unchanged:

$$\int_{-1}^{1} n(k, x)\,dx = n_k,$$

from which we get

$$\alpha_k = \pi^2/\beta_k n_k^2.$$

For the characteristic thickness of the jet we obtain

$$\Delta x \approx (\varepsilon_1/\gamma_k n_k)\,(\Delta x_0)^2. \tag{28}$$

Here Δx_0 is the characteristic angular width of the instability increment.

The arguments advanced above concerning the thickness of the jet are suitable only for two-dimensional jets. In the one-dimensional problem, the conservation laws $\omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3}$ and $k + k_1 = k_2 + k_3$ are satisfied only at $k_2 = k$, $k_3 = k_1$ or $k_2 = k_1$, $k_3 = k$; the collision term vanishes in this case. This means that to determine the thickness of the jet it is necessary to use higher orders of perturbation theory.

In concluding this section, we obtain an estimate of the effectiveness of plasma heating in the case when the **k**-space contains a region where oscillations are generated. Let us assume, for concreteness, that the problem is axially-symmetrical and that one two-dimensional jet is produced in the generation region. We denote by k_0 the characteristic value of the wave vector, by Δk_0 and Δx_0 the parameters of the region in question, and by γ the characteristic instability increment. Then, as follows from (12), the following estimate holds for the jet intensity N:

$$N \sim \gamma n_0 T_e \Delta k_0 r_D^2 M / m.$$

From this we easily obtain the total energy density of the oscillations in the jet

$$W \sim \frac{n_0 T_e \gamma}{\omega_p} \frac{\Delta k_0}{k_0} (k_0 r_D)^2 \frac{M}{m}.$$

The energy contained inside the instability region is

$$W_\gamma \sim \frac{\Delta k_0}{k_0} W \sim \frac{n_0 T_e \gamma}{\omega_p} (\Delta k_0 r_D)^2 \frac{M}{m}.$$

Consequently, the energy dissipated per unit time in the plasma is

$$\gamma W_\gamma \sim \frac{\gamma^2 n_0 T_e}{\omega_p} (\Delta k_0 r_D)^2 \frac{M}{m}.$$

We present also an estimate of the jet thickness Δx (see (27) and (28)):

$$\Delta x \sim \frac{W_\gamma}{n_0 T_e} \frac{1}{k r_D} \left(\frac{M}{m}\right)^{1/2} \alpha_1 (\Delta x_0)^2.$$

The condition for the validity of our theory includes the requirement (8):

$$\frac{W_\gamma}{n_0 T_e} < \frac{W}{n_0 T_e} < k r_D \left(\frac{m}{M}\right)^{1/2} \frac{1}{\alpha_1},$$

which means that even in the case of a "broad" increment, when $\Delta x_0 \sim 1$, the jets are narrow ($\Delta x \ll 1$).

3. INSTABILITY OF ELECTRON BEAM

We consider the question of the system of the Langmuir oscillations excited in a plasma by a beam of relativistic electrons. As noted in[7], under the conditions when the beam is used to heat a dense plasma to thermonuclear temperatures, the principal mechanism that limits the growth of the Langmuir oscillations is induced scattering by ions, which leads to a transfer of energy from the instability zone ($k > \omega_p/c$) into the region of smaller wave numbers. The characteristic time of establishment of the quasistationary spectrum of the oscillations then turns out to be significantly smaller than the time of the variation of the distribution function of the electron beam. Thus, the distribution function of the beam electrons can be regarded as specified in the problem of determining the spectrum.

We denote by E the energy of an individual electron, by $\Delta\theta$ and ΔE the angle and energy spreads of the particles, and by n' the beam concentration. If the angle spread $\Delta\theta$ is not too small,

$$\Delta\theta > \left(\frac{n'}{n_0}\frac{mc^2}{E}\right)^{1/3},$$

then the instability is kinetic, i.e., the beam does not influence the wave dispersion law, and determines only their growth increment. If in addition

$$\Delta\theta > \frac{mc^2}{E}\left(\frac{\Delta E}{E}\right)^{1/2},$$

then the spread of the beam electrons relative to the absolute value of the velocity can be neglected in the calculation of the increment, and we can put $\mathbf{v} = c\mathbf{p}/|\mathbf{p}|$, where **p** is the electron momentum. The only oscillations that can interact with the beam are those whose wave vectors satisfy the Cerenkov resonance condition

$$\omega_p - \mathbf{kv} = 0$$

or

$$|\omega_p / c - k_\parallel| \sim (\omega_p / c)(\Delta\theta)^2 + k_\perp \Delta\theta,$$

where $k_\parallel$ and $k_\perp$ denote the longitudinal and transverse (with respect to the beam axis) components of the vector **k**. The instability increment $\gamma(k, x)$ is given by the formula (see[8])

$$\gamma(k, x) = \pi\omega_p \frac{n'}{n_0}\left(\frac{\omega_p}{kc}\right)^3 \int_{x_1}^{x_2} \frac{dx'}{[(x'-x_1)(x_2-x')]^{1/2}} \left[-2g - \left(x' - \frac{kcx}{\omega_p}\right)\frac{\partial g}{\partial x'}\right],$$

$$x'_{1,2} = (\omega_p / kc)[x \pm (1-x^2)^{1/2}(k^2c^2/\omega_p^2 - 1)^{1/2}],$$

$$g(x') = mc\int_0^\infty f(p, x')\, p\, dp \qquad (29)$$

(f is the beam distribution function). For the maximal (at fixed k) value of the increment, which is reached at $x \sim \omega_p/kc$, the following estimate holds true

$$\gamma \sim \omega_p \frac{n'}{n_0}\frac{mc^2}{E}\frac{1}{(\Delta\theta)^2}\frac{\omega_p^2}{k^2c^2}. \qquad (30)$$

A plot of $\gamma(k, x)$ at fixed k is shown in Fig. 1. The function $\gamma(k, x)$ has a narrow maximum with a width on the order of $\Delta\theta$. This circumstance greatly simplifies the problem of finding the stationary spectrum of the oscillations in the case when the angle spread of the beam is small ($\Delta\theta \ll 1$).

When $k \gg \omega_p/c$, the maximum of the function $\gamma(k, x)$ lies close enough to the point $x = 0$. In accordance with the results of Sec. 2, in this case the spectrum should consist of one two-dimensional jet, the position of which coincides, accurate to $\Delta\theta$, with the position of the maximum of the increment. Therefore in the region $k \gg \omega_p/c$ the spectrum takes the form

$$N(k, x) = N(k)\,\delta(x - \omega_p / kc), \qquad (31)$$

where the intensity N(k) can be obtained from formula (17) by using (30):

$$N(k) = \{T(x_0(k), x_0(k))\}^{-1/2} \int_k^\infty \gamma(k')\{T(x_0(k'), x_0(k'))\}^{-1/2} dk',$$

where $x_0(k) \equiv \omega_p/kc$.

Calculations show that at $k < 1.52\,\omega_p/c$ the spectrum (31) does not have external stability. Consequently, formula (31) holds true only at $k > 1.52\,\omega_p/c$. From the fact that $\tilde{\gamma}(k, x)$ is a polynomial of third degree in x, we can see that in addition to the initial jet (31), there can appear in the region $k < 1.52\,\omega_p/c$ not more than two

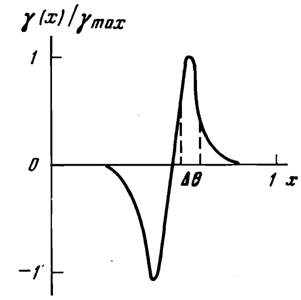

FIG. 1. Plot of the instability increment of a relativistic electron beam against the angle (x = cos θ) at a fixed modulus of the wave vector.

additional jets, and in the case of two jets, one of them must be one-dimensional.

All the foregoing calculations are based on the use of the diffusion approximation. Strictly speaking, this is possible only in the case of a sufficiently "broad" increment of the two-stream stability, i.e., for beams with not too small angle spreads:

$$(\Delta\theta)^2 > {}^1/_3 (m/M)^{1/2} c/v_{Te}.$$

It is obvious, however, that when the condition

$$(1/r_D)(m/M)^{1/2} \ll \omega_p/c \qquad (v_{Te}/c \gg (m/M)^{1/2})$$

is satisfied for sufficiently large wave numbers

$$k - \omega_p/c \gg (1/r_D)(m/M)^{1/2}$$

the conditions for the applicability of the diffusion approximation are satisfied independently of the angular width of the beam. Thus, a jet picture is always obtained in the region of large wave numbers. The question of the behavior of the spectrum in the region of small wave numbers for narrow beams still remains open, although apparently the spectrum is essentially anisotropic in this region.

4. PARAMETRIC EXCITATION OF WAVES

Let us examine the parametric instability of a plasma placed in a homogeneous oscillating electric field. Equation (7) can easily be generalized to include this case. To this end, it suffices to make the following change of variable in (7):

$$a_{\mathbf{k}} \to a_{\mathbf{k}} + \frac{k}{8\pi e}\left(\frac{2m\omega_p}{n_0}\right)^{1/2} (\mathbf{E}_0\nabla\delta(\mathbf{k}))e^{i\Omega t}.$$

It is assumed that the external electric field has a frequency $\omega_p + \Omega$ which is close to the plasma frequency ($\Omega \ll \omega_p$).

Assuming the amplitude of the external field to be small, we take it into account only in the approximation linear in $\mathbf{E}_0$. Then the equation for $a_{\mathbf{k}}$ takes the form

$$\frac{\partial a_{\mathbf{k}}}{\partial t} + i\tilde{\omega}_{\mathbf{k}} a_{\mathbf{k}} = V_{\mathbf{k}} a_{-\mathbf{k}}^* e^{2i\Omega t} + i\int T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \times a_{\mathbf{k}_1}^* a_{\mathbf{k}_2} a_{\mathbf{k}_3} \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3, \tag{32}$$

where

$$\tilde{\omega}_{\mathbf{k}} = \omega_{\mathbf{k}} + i\gamma_{\mathbf{k}} + \frac{\omega_p}{4k^2}\frac{(\mathbf{k}\mathbf{E}_0)^2}{8\pi n_0 T_e} G_{\mathbf{k},\Omega-\omega_{\mathbf{k}}},$$

$$V_{\mathbf{k}} = \frac{\omega_p}{4k^2}\frac{(\mathbf{k}\mathbf{E}_0)^2}{8\pi n_0 T_e} G^*_{\mathbf{k},\Omega-\omega_{\mathbf{k}}}.$$

Equation (32) is valid if

$$\frac{|\mathbf{E}_0|^2}{8\pi n_0 T_e} \ll \left(\frac{\Omega}{\omega_p}\right)^{1/2}\left(\frac{m}{M}\right)^{1/2}.$$

Linearizing Eq. (32) and putting $a_{\mathbf{k}} \sim \exp(i\Omega t + i\Omega_{\mathbf{k}} t)$, we obtain the dispersion equation

$$(-\Omega_{\mathbf{k}} + \tilde{\omega}_{\mathbf{k}} - \Omega)(\Omega_{\mathbf{k}} + \tilde{\omega}_{-\mathbf{k}} - \Omega) = |V_{\mathbf{k}}|^2,$$

whence

$$\Omega_{\mathbf{k}} = \operatorname{Im}\tilde{\omega}_{\mathbf{k}} \pm [{}^1/_4(\Omega - \operatorname{Re}\tilde{\omega}_{\mathbf{k}})^2 - |V_{\mathbf{k}}|^2]^{1/2}. \tag{33}$$

Formula (33) describes the parametric instability of the plasma. In a narrow interval of wave numbers

$$|\Omega - \operatorname{Re}\tilde{\omega}_{\mathbf{k}}|^2 < |V_{\mathbf{k}}|^2$$

we have two-stream oscillating instability[9]. At $T_i \ll T_e$, however, a larger increment and a lower threshold are possessed by decay instability of first order, for which

$$\Omega \approx \omega_{\mathbf{k}} + c_s k.$$

At $T_i \sim T_e$, the two instabilities have increments and thresholds that are of the same order of magnitude.

A consistent statistical description of parametric instability of a plasma entails certain difficulties, since in addition to the "normal" averages $\langle a_{\mathbf{k}} a_{\mathbf{k}}^* \rangle$ it is necessary to take into account also the "anomalous" averages $\langle a_{\mathbf{k}} a_{-\mathbf{k}} \rangle$ (see, for example, [6]). In addition, there are no grounds in the problem of parametric excitation of waves for changing over to the differential approximation in $|\mathbf{k}|$, since the instability increment and the kernel of integrand vary in $\mathbf{k}$-space in one and the same characteristic scale ($\sim (1/r_D)(m/M)^{1/2}$ at $T_i \sim T_e$). Nonetheless, following[10] and claiming only an order-of-magnitude estimate, we describe the nonlinear stage of the parametric instability by means of Eq. (12), in which we put

$$\gamma(k, x) = \frac{\omega_p |\mathbf{E}_0|^2}{8\pi n_0 T_e} x^2 \operatorname{Im} G_{\mathbf{k},\Omega-\omega_{\mathbf{k}}}. \tag{34}$$

It follows from (34) that $\gamma(k, x) = \gamma(k, -x)$ and, in addition,

$$\gamma(k, x) < x^2\gamma(k, 1).$$

From this, in accordance with the criterion (22), it follows that within the framework of the considered model the spectrum of the Langmuir turbulence excited by a homogeneous oscillating electric field consists of two one-dimensional jets at $x = \pm 1$. Putting $T_i \sim T_e$ and recalling that $\Delta k r_D \sim (m/M)^{1/2}$, we find that the total energy contained in the Langmuir oscillations is of the order of

$$\frac{W}{n_0 T_e} \sim \frac{|\mathbf{E}_0|^2}{8\pi n_0 T_e} k r_D \left(\frac{m}{M}\right)^{1/2}. \tag{35}$$

Formula (35) differs significantly from the result of[10] (see also[11]):

$$\frac{W}{n_0 T_e} \sim \left(\frac{|\mathbf{E}_0|^2}{8\pi n_0 T_e}\right)^2 \frac{\omega_p}{\nu_{ei}}. \tag{35a}$$

The point is that the conditions for the applicability of relations (35) and (35a) are different: the first is valid in the case of a low-frequency of Coulomb collisions ν_{ei}, and the second is valid for sufficiently large values of ν_{ei}, when

$$\nu_{ei} > \omega_p \frac{|\mathbf{E}_0|^2}{8\pi n_0 T_e}\frac{1}{k r_D}\left(\frac{m}{M}\right)^{1/2}.$$

Taking this circumstance into account, we can write for W the following interpolation formula, which gives a correct result in both limiting cases:

$$\frac{W}{n_0 T_e} \sim \left(\frac{|\mathbf{E}_0|^2}{8\pi n_0 T_e}\right)^2 \Big/ \left[\frac{\nu_{ei}}{\omega_p} + \left(\frac{m}{M}\right)^{1/2}\frac{|\mathbf{E}_0|^2}{8\pi n_0 T_e k r_D}\right].$$

5. NUMERICAL EXPERIMENT

To verify the ideas concerning the jet character of the spectrum and to investigate the kinetics of the jets, we solved Eq. (12) with a computer with a four-plasmon collision term in the form (26). Specifically, we considered the following equation:

$$\frac{\partial N'(k',x)}{\partial t'} = N'(k',x)\left(\gamma'(k',x) + \frac{\partial}{\partial k'}\int_{-1}^{1}\tilde{T}'(x,y)N'(k',y)\,dy\right) + \varepsilon_2' \frac{\partial^2 N'(k',x)}{\partial k'^2}, \tag{36}$$

where t', k', N', γ', $\tilde{T}'$, and ϵ_2' are dimensionless quantities determined by the relations

$$t' = \gamma_{max} t, \quad k' = (\omega_p/c)k, \quad \gamma' = \gamma_{max}^{-1}\gamma,$$
$$N'(k', x) = \gamma_{max}^{-1}\frac{\pi^2}{9}\frac{m}{M}\frac{1}{r_D^2}\frac{c\omega_p}{n_0T_e}N(k, x), \qquad (37)$$
$$T'(x, y) = 1 - x^2 - y^2 - 3xy + 3yx^3 + 3xy^3 + 3x^2y^2 - 5x^3y^3,$$
$$\varepsilon_2' = \frac{c^2}{\gamma_{max}\omega_p}\left(\frac{m}{M}\right)^{3/2}\frac{1}{(kr_D)^3}\frac{1}{r_D^2}\left(\frac{W}{n_0T_e}\right)^2\alpha_2\left(\frac{T_i}{T_e}\right).$$

γ_{max} denotes the maximum instability increment.

The term corresponding to thermal noise was not introduced in explicit form in (36), but it was assumed in the calculations that $N'(k', x)$ has a lower bound $N_0 \sim 10^{-3}-10^{-2}$; the diffusion coefficient ϵ_2' ranged from 2.5×10^{-4} to 5×10^{-3}. We applied to Eq. (36) a difference scheme of the Krank-Nicholson type of second order of accuracy in time (see, for example, [12]). To integrate with respect to the cosine of the angle x in (36) we use Gauss quadratures of suitable order of accuracy, with nodes that condense toward the points $x = \pm 1$, thus ensuring the best accuracy for solutions of the type of one-dimensional jets.

In typical variants, the number of points was 100 for the modulus of the wave vector and 30 for the angle. The initial conditions corresponded to the minimal level of the oscillations $N'(k', x) = N_0$. The instability zone was located at $2 > k' > 1$; at small $k' < 0.2$ we introduced linear damping that increased towards $k' = 0$ and ensured a "sink" for the energy. Thus, we expected realization of the Kolmogorov regime in the region $0.2 < k' < 1$. In addition to the usual methods of verifying the difference scheme, we monitored the conservation of the total number of quasiparticles at $\gamma'(k', x) \equiv 0$ up to $t' = 100$.

In the first run of experiments we substituted for $\gamma'(k', x)$ the real instability increment of a relativistic beam of electrons with total angle width $\Delta\theta \sim 15°$. The development of the instability for this case is illustrated in Fig. 2. We see that a stationary spectrum of the jet type develops in the course of time. In the inertial region ($k' < 1$) there are two one-dimensional jets, and in the region of large wave numbers ($k > 1.5$) there is one two-dimensional jet, as predicted by the theory. In the intermediate region ($1 < k' < 1.5$), two two-dimensional jets are formed, and they "stick" at $x = 1$ to the ends of the interval $|x| = 1$ and are transformed into one-dimensional ones.

We note that the point at which the second jet appears coincides with the value ($k' = 1.52$) obtained analytically in Sec. 3.

The development of the nonlinear instability picture proceeds as follows. At first the oscillations grow exponentially in the region where the increment is positive, and the first two-dimensional jet is formed. Then, at $x = -1$, a "germ" of the second two-dimensional jet and of the one-dimensional jets is produced.

The development of the one-dimensional jet recalls the propagation of shock waves in **k**-space (c.f. [13]) in the region of small wave numbers. The thickness of these shock waves increased with increasing diffusion coefficient.

The complete steady-state picture is established within a time on the order of 30—40 reciprocal increments, and a stationary flux of the number of quasiparticles is produced in this case in the inertial region. The

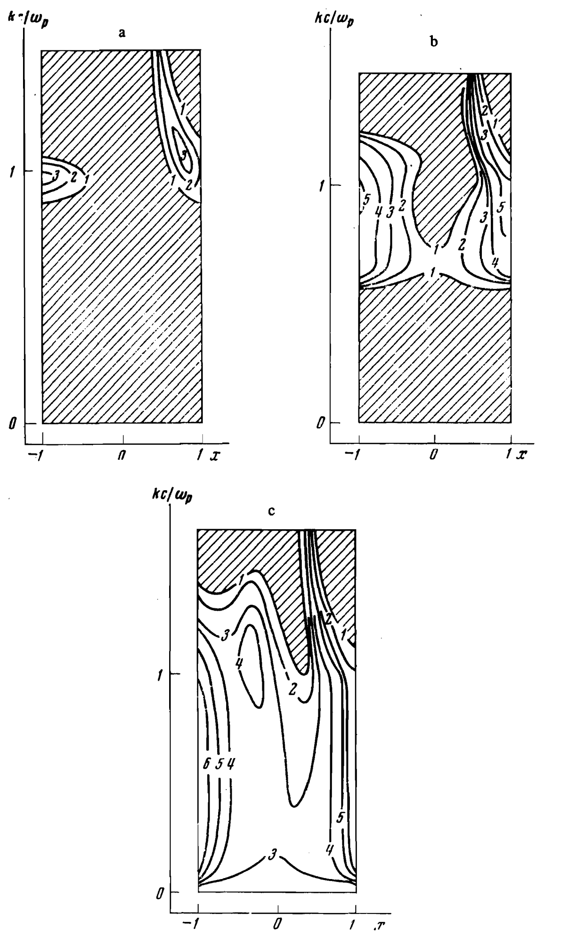

FIG. 2. Level lines of the function $\ln[N'(k, x)/N_0]$ at different instants of time: a – $t' = 8$, b – $t' = 20$; c – $t' = 100$. The lines are marked with the values of the functions. The oscillations are excited by a relativistic electron beam. In the shaded region we have $N'(k, x) = N_0$.

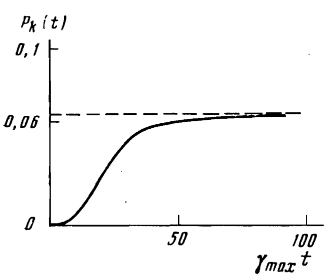

FIG. 3.

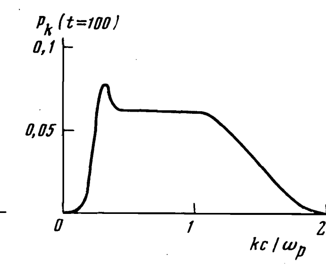

FIG. 4.

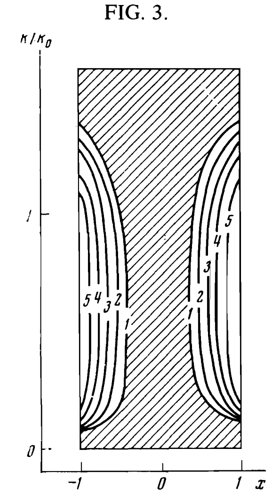

FIG. 5.

FIG. 3. Time dependence of the flux of Langmuir plasmons at $k' = 0.7$.
FIG. 4. Plot of the flux of Langmuir plasmons against the modulus of the wave vector $t' = 100$.
FIG. 5. Level lines of the function $\ln[N'(k, x)/N_0]$ in the case of parametric instability ($t' = 15$). The shaded region has $N'(k, x) = N_0$. The point $k = k_0$, $x = 1$ corresponds to the maximum increment of the parametric instability.

character of the establishment of the flux at the point $k' = 0.70$ is shown in Fig. 3 (it shows the instant $t' = 10$ of shock-wave arrival), and the dependence of the steady-state flux on k is shown in Fig. 4. The value of the flux in the region of small wave numbers, $P_k = 0.06$, which is expressed in terms of dimensionless variables (37), agrees with the estimate $P_k \sim (\Delta k'/k')^2$ (see (19)) at an effective width of the increment $\Delta k'/k' \sim 0.25$.

The second series of experiments was carried out to study the nonlinear stage of the parametric instability. Taking into account the model character of this problem, we chose for the increment not the exact value (34), but an approximate value taken from[10]:

$$\tilde{\gamma}(k, x) = \alpha \frac{x^2}{1 + [(k - k_0)/\delta k]^2} - 1.$$

Here α is the dimensionless excess over the instability threshold, δk is the width of the instability region, and k_0 is the point of the maximum increment. In typical variants it was assumed that $\alpha = 10$ and $\delta k = 3r_D^{-1}(m/M)^{1/2}$ (strictly speaking, one should put $\delta k = r_D^{-1}(m/M)^{1/2}$, but this would make it impossible to use the difference scheme). The experiment confirmed the existence of a quasi-one-dimensional spectrum in the form of two jets localized at $x = \pm 1$. The development of one-dimensional jets also had the character of the propagation of shock waves; the steady-state picture is shown in Fig. 5.

CONCLUSION

As shown in Sec. 5, numerical experiments confirm the concept of jet-type spectra. Let us now dwell on the questions that call for further research. First among them is the question of the stability of the jets. One cannot exclude the possibility that the jets may turn out to be unstable, for example, against self-modulation. In this case one should expect the spectrum to retain its jet character, but the jets to become "turbulent," and their width to be determined by the instability characteristics.

Another important problem concerns the character of the spectra for increments that are "narrow" in **k**-space, when the conditions for the applicability of the differential approximation no longer hold. In this case one should expect the appearance of spectra that are even more singular than jets, for example, spectra in the form of discrete sets of monochromatic waves.

In conclusion, the authors thank D. D. Ryutov for a discussion of the work.

1)We are grateful to A. A. Galeev for calling our attention to this circumstance.

2)In the particular case when γ_k is also a polynomial of third degree, partial or complete coincidence of the surfaces γ_k and $\tilde{\gamma}_k$ is possible. However, even a small change of γ_k will lead to "stratification" of these surfaces.

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