Interaction between impurity excitations and negative ions in liquid helium

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A semi-phenomenological description of some properties of surface impurity excitations on a small-radius free spherical surface in liquid helium is presented. The results of the analysis are used to interpret the experimental data on mobility of anions in weak He^3—He^4 solutions.

Negative ions (or anions) in helium have the structure of a vacuum bubble of radius $R_0 \approx 20$ Å with an electron inside the cavity. In helium solutions there should occur on such a bubble, as on a flat surface of a solution (see[1]), surface impurity levels occupied to one degree of another by s-impurities (unlike the v-impurities that occupy the volume of the solvent). The purpose of the present paper is to describe certain specific properties of such s-impurities, due to the large curvature of the anion surface. In addition, we discuss the interaction of anions with v-impurities. The solution of the latter problem is all important for the understanding of the features of anion mobility in helium solutions.

s-IMPURITIES ON A SPHEREICAL SURFACE

1. Just as in the case of a plane surface, the question of the real existence of surface levels near strongly bent boundaries cannot be solved theoretically. To introduce the concept of such levels it is therefore necessary to carry out at least qualitative experiments that confirm the existence of adsorption phenomena on spherical small-radius surfaces. As one such experiment one can cite the observations by Rayfield[2], who has shown that the critical velocities of negative ions prior to formation of the vortex ring around the moving ion depend noticeably on introduction of small concentrations of He_3 in the He_4. At the same time, the critical velocity of the positive ions reveals no such dependence. And since the surface levels for impurities near the anion surface are the only possible detail that distinguishes the anions from cations in this case, the indicated difference can be regarded as an indirect confirmation of the existence of such levels. Unfortunately, it is hardly possible to present any quantitative estimates in this case (although some attempts have been made[3]), since the very mechanism of vortex formation remains practically unknown.

Another possible but still unrealized experiment admits of a quantitative description. We write down the total excess energy $\delta\Phi$ connected with formation of an anion in helium:

$$\delta\Phi = \pi^2\hbar^2 / 2m_eR_0^2 + 4\pi\alpha R_0^2, \tag{1}$$

where m_e is the electron mass and α is the coefficient of surface tension on the anion surface. Minimizing $\delta\Phi$ with respect to R_0, so that $R_0^4 = \pi\hbar^2/8m_e\alpha$, we rewrite (1) in the form

$$\delta\Phi = 4\pi\hbar(\pi\alpha / 2m_e)^{1/2}. \tag{2}$$

We assume now that the anion passes through a region of the solution with a steep gradient of He_3 concentration (for example, through a stratification boundary). If the surface tension coefficient α in (2) depends on the local value of the concentration n_3, then the anion will be acted upon by an effective force F

$$F = -\frac{\partial\delta\Phi}{\partial z} = -2\pi\hbar\left(\frac{\pi}{2m_e\alpha}\right)^{1/2}\frac{\partial\alpha}{\partial n_3}\frac{\partial n_3}{\partial z}$$

that either helps or hinders the passage of the anion through the region with ∇n_3. Actually, in the case of stratification of helium solutions, the component in the upper part of the path (medium I) is richer in He_3, and the component in the lower part (medium II) has a smaller He_3 content. Therefore, if surface impurity levels do exist on the anion, i.e., if α increases with decreasing n_3, then to move the anion through the stratification boundary downward it is necessary to overcome the barrier

$$\Delta\Phi = \delta\Phi_{II} - \delta\Phi_I = 4\pi\hbar\left(\frac{\pi}{2m_e}\right)^{1/2}(\alpha_{II}^{1/2} - \alpha_I^{1/2}). \tag{3}$$

On the other hand, if there are no levels then, according to Andreev[1], the dependence of α on n_3 is reversed and no barrier is produced.

In practice, the presence of any barrier in the path of the ions becomes manifest in an activation dependence of the current through the barrier on the temperature. Such activation dependences were observed by a number of workers both for the liquid-vapor boundary and for the stratification boundary. In the latter case, however, Kushner et al.[4] investigated only the singularities of the downward (and not upward) flow of the anion current. The barrier produced in this variant is of electrostatic origin, is due to the discontinuity of the dielectric constant on the stratification boundary, and has no bearing on the question of interest to us, that of surface impurity levels on the anion. Attempts to trace the motion of ions in the opposite direction have not yet been made, although they should encounter no difficulties.

The third possible source of information on s-impurities on the anion surface are data on the anion mobility in weak He_3—He_4 solutions and offer practically unequivocal evidence of the presence of surface states. This question will be discussed in detail below.

2. Having verified that s-impurities actually exist on the anion surface, let us consider the properties of such states. It is necessary to note in this connection, first, that a similar problem was already investigated by a number of workers. Thus, Lekner[5] has shown that in a spherical surface, just as in a plane one, one can construct a single-particle Schrödinger equation for the s-impurities. In addition, mention should be made of the work of Dahm[3] and Kramer[6], aimed at calculating different concrete effects due to the presence of surface impurity levels on the anion; the starting point was the use of a "plane" dispersion law for the s-impurities. Such an approximation for a real anion, as will be seen below, turns out to be very crude.

The formal problem of the properties of s-impurities on a spherical surface begin with the determination of the excitation spectrum. Just as in the planar case, this question has a simple solution for low-lying energy levels

$$\varepsilon_l = -\varepsilon_0 + \gamma l(l+1),$$

$$\gamma = \hbar^2 / 2m_s R_0^2, \quad \gamma l(l+1) < \varepsilon_0, \quad l = 0, 1, 2, \ldots, \quad (4)$$

$\varepsilon_0 > 0$, m_S is the mass of the s-impurity, and the energy ε_l is reckoned from the bottom of the end of the v-impurities. The condition for applicability of the definition (4) is the requirement $R_0 \gg \lambda$, where λ is the characteristic damping length of the wave functions of the s-impurities. In our case, when $R_0 \approx 20$ Å and $\lambda \approx \hbar(2m_S\varepsilon_0)^{-1/2}$ ($\varepsilon_0 \approx 2°$ K, $m_S \approx 10^{-23}$ g, i.e., $\lambda \approx 2-3$ Å), the condition $R_0 \gg \lambda$ is indeed satisfied.

It should be noted that in the general case the spectrum of the s-impurities on a sphere, as well as on a flat surface, has a more complicated form

$$\varepsilon_{nl} = -\varepsilon_n + \gamma l(l+1).$$

This circumstance is illustrated by the model example considered in the Appendix. For the concrete values of ε_0 and m_S known for the plane case, the probability of existence of levels with $n > 0$ does not exceed one-half. For this reason there are no special grounds for introducing into the theory the additional quantum number n in either the plane or the spherical problems.

One more fact, which will be needed later on, should be mentioned. As seen from the model example in the Appendix, the damping length λ of the wave functions of the s-impurities in the interior of the liquid phase depends only on the quantum number n, and does not depend on l. For this reason, all the s-impurities at the level ε_0, regardless of the energy ε_l they possess, have a damping depth λ. This situation is not universal for surface problems. Thus, surface electrons on the free surface of liquid helium have, like the s-impurities, a damping length that does not depend on the wave number k of the electron along the surface[7]. But surface phonons, for example, have a damping length that depends on k[8]. For real s-impurities, it is still more reasonable to assume that λ depends on l.

3. Having an excitation spectrum at our disposal, we can easily establish the connection between the total number N_S of the s-impurities and the chemical potential μ_S of the given system:

$$N_s = \sum_{l=0}^{L} 2(2l+1)\left[\exp\left(\frac{\varepsilon_l - \mu_s}{T}\right)+1\right]^{-1}. \quad (5)$$

Here ε_l is taken from (4), and $L \equiv l_{\max}$ is defined by the requirement

$$\varepsilon_0 - \gamma L(L+1) \geqslant 0. \quad (5a)$$

The chemical potential μ_S, as follows from general thermodynamic considerations[1,9], should coincide with the chemical potential μ_V of the impurities that fill the volume of the solvent:

$$\mu_s = \mu_v, \quad \mu_v = -T\ln\left[\frac{2m_4}{c\rho}\left(\frac{m_3 T}{2\pi\hbar^2}\right)^{1/2}\right], \quad (6)$$

where ρ is the density of the solvent, c is the relative volume concentration of the solutions, and m_3 and m_4 are the masses of the He$_3$ and He$_4$ atoms under volume conditions.

Combining (5) and (6) we have

$$N_s(c,T) = \sum_{l=0}^{L} 2(2l+1)\left[\frac{2m_4}{c\rho}\left(\frac{m_3 T}{2\pi\hbar^2}\right)^{1/2}\exp\left(-\frac{|\varepsilon_l|}{T}\right)+1\right]^{-1} \quad (7)$$

This relation connects N_S with c, i.e., it solves the problem of the dependence of $N_S(c, T)$ on the volume characteristics of the solution.

Let us discuss the content of formula (7). Let the concentration c and the temperature T be such that the following inequality is satisfied

$$\Lambda(c,T)\exp\left(-\frac{|\varepsilon_0|}{T}\right) \gg 1, \quad \Lambda(c,T) = \frac{2m_4}{c\rho}\left(\frac{m_3 T}{2\pi\hbar^2}\right)^{1/2}. \quad (8)$$

In this limit, all the remaining terms from $N_S(c, T)$ with $l > 0$ will be exponentially small in comparison with the first term $l = 0$, and expression (7) simplifies to[1)]

$$N_s(c, T) \approx 2\Lambda^{-1}(c, T)\exp\left(|\varepsilon_0|/T\right) \ll 1, \quad (7a)$$

recalling in this form the definition given by Andreev[1] for n_S of the "plane" case. However, if we fix c in (8) and start to lower the temperature, then very soon this inequality is violated for the level $l = 0$, corresponding to saturation of the first possible level in the well for the s-impurities. Under similar conditions, when $\Lambda(c, T)\exp(-|\varepsilon_0|/T) < 1$, but $\Lambda(c, T)\exp(-|\varepsilon_1|/T) > 1$, the occupation of the level $l = 1$ begins, and the quantity $N_S(c, T)$ follows the following behavior:

$$N_s(c, T) \approx 2 + 6\Lambda^{-1}(c, T)\exp\left(|\varepsilon_1|/T\right).$$

When the temperature is reduced further, higher and higher levels are included, up to the degeneracy temperature T_F^S, defined for a given concentration c by the inequality

$$\Lambda(c, T_{F'})\exp\left(-|\varepsilon_L|/T_{F'}\right) \leqslant 1. \quad (8a)$$

In the region $T < T_F^S$ we have

$$N_s(c,T) \approx N_s(c,0) \equiv N_s^0 = \sum_{0}^{L} 2(2l+1). $$

Thus, unlike the "plane" case, the temperature dependence of $N_S(c, T)$ cannot be characterized by any one activational exponential, with the exception of the vicinity of the parameters satisfying the inequality (8).

The contribution $\Delta\alpha$ of the impurity surface excitations to the coefficient of surface tension on a spherical surface is determined in complete analogy with $N_S(c, T)$[9]:

$$\alpha = \alpha_0 + \Omega_s / 4\pi R_0^2,$$

$$\Omega_s = -T\sum_{l=0}^{L} 2(2l+1)\ln\left[1+\exp\left(\frac{\mu_s - \varepsilon_l}{T}\right)\right], \quad (9)$$

or, taking into account the relation $\mu_S = \mu_V$,

$$\Omega_s = -2T\sum_{l=0}^{L}(2l+1)\ln\left[1+\Lambda^{-1}(c,T)\exp\left(\frac{|\varepsilon_l|}{T}\right)\right]. \quad (9a)$$

Just as $N_S(c, T)$, the quantity $\Delta\alpha = \alpha_0 - \alpha$ at finite c and as $T \to 0$ ($T \to T_F^S$) tends to a finite limit but does not depend on the temperature:

$$\Delta\alpha = \frac{1}{2\pi R_0^2}\left\{\sum_{l=0}^{L}(2l+1)|\varepsilon_l| - TN_s^0\ln[\Lambda(c,T)]\right\}. \quad (10)$$

However, whereas in the case of $N_S(c, T)$ the approach to a constant value with changing temperature is at an exponential rate, in the case of $\alpha(c, T)$ the rate of this approach is given by a power law.

SCATTERING OF v-IMPURITIES BY AN ANION

1. The description of the interaction of v-impurities with anions is best started with a discussion of the ex-
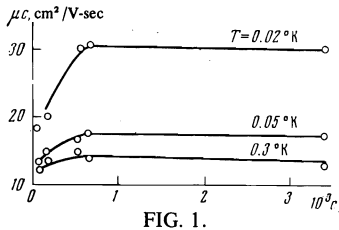
FIG. 1.

$10^4 c$	T, °K	ϵ_L, °K
3.430	0.15 ± 0.01	0.27 ± 0.03
0.726	0.10 ± 0.01	0.23 ± 0.04
0.578	0.09 ± 0.01	0.26 ± 0.04
0.170	0.07 ± 0.01	0.26 ± 0.05
0.070	0.06 ± 0.01	0.26 ± 0.06

perimental data on the mobility of the anions in He₃—He₄ solutions. Figure 1 of the article by Ketterson et al.[10] gives the presently most complete results of measurements of μ in weak He₃—He₄ solutions for a broad temperature interval. From these data we plotted in Fig. 1 the product μc against c at different temperatures. Comparison of these plots with Fig. 1 of[10] leads to the following conclusion:

At low temperatures $T \ll T_F^S$ and at solution concentrations that are not small (c $\gtrsim 10^{-3}$), there is a well pronounced temperature dependence of the mobility, $\mu \sim T^{-1/2}$, and μc is independent of c. These circumstances enable to use in the indicated region of c and T the well known expression for the mobility of an impermeable sphere of radius R in a Boltzmann gas of impurity excitation with volume density n₃:

$$\mu = 3e / 8n_3 R^2 (2\pi m_3 T)^{1/2}.\quad (11)$$

On the basis of (11) and the data of Ketterson et al.[10], we obtain for R the value R = (2.6 ± 0.1) × 10⁻⁷ cm, which is much larger than the value $R_0 \approx 20$ Å in pure He₄.

With increasing temperature, the s-impurities begin to leave the s-levels. The start of this process corresponds to the vicinity of the minima on the plots of μ against T in Fig. 1 of[10]. Thus, the position of the maxima of μ(c, T) characterizes qualitatively the degeneracy temperature T_F^S for the s-impurities on the surface levels. Using the connection (9a) between c, T_F^S, and ϵ_L, which has the following explicit form:

$$\frac{2m_4}{c\rho}\left(\frac{m_3 T_{F'}}{2\pi\hbar^2}\right)^{1/2}\exp\left(-\frac{|\epsilon_L|}{T_{F'}}\right) \approx 1,$$

and the experimental values of c and T_F^S, which determine the positions of the minima of μ(c, T), we can estimate the value of ϵ_L. The corresponding ϵ_L calculated for different concentrations c are gathered in the table. The approximate constancy of ϵ_L with change of c in a wide interval indicates that the definition of T_F^S is reasonable.

The transition region of mobility, which follows the temperature minimum, corresponds to a gradual spreading away of the s-impurity cloud, up to the vicinity of the maximum, where it is natural to assume the surface of the anion becomes completely rid of the s-impurities.

2. The observed increase ΔR of the anion radius at $T \ll T_F^S$, together with the determination of the temperature-independent limits (7) and (10) for N_S^0 and $\Delta \alpha$, respectively, the presence of which qualitatively justifies the appearance of ΔR, is the most reliable (of all published) proofs of the existence of s-impurities on the anion surface. Moreover, the available experimental data on the anion mobility in helium solutions make it possible to estimate with a certain degree of accuracy the parameters ϵ_0 and m_S of the s-impurity spectrum. To obtain these estimates it is necessary to calculate, in terms of ϵ_0 and m_S, the total cross section for the scattering of v-impurities by an anion. The anion mobil-

ity corresponding to this cross section, which contains the arbitrary parameters ϵ_0 and m_S, should be compared with the experimental data, from which ϵ_0 and m_S are indeed obtained.

One of the main causes for the change of R in helium solution is the decrease of the value of the surface tension on the liquid boundary of the anion. It is easy to take this influence of the impurities on R into account in the entire temperature interval:

$$R^4 = \pi\hbar^2 / 8m_e \alpha, \quad \alpha = \alpha_0 + \Omega_s / 4\pi R_0^2.\quad (12)$$

with Ω_S from (9).

Much less rigorous is the description of the direct interaction of the v-impurities with the cloud of the s-impurities. Taking into account the complicated character of this multiparticle problem and planning henceforth only qualitative estimates, we confine ourselves here to the approximation called the "optical model." Within the framework of this model, the interaction of the cloud of s-impurities with the v-impurities is described by introducing a certain potential barrier V_1 (which is generally complex), which additionally scatters the v-impurities. Recognizing the δ-function character of the impurity-impurity interaction

$$V_{s-v} = 4\pi\hbar^2 a_0 \frac{m_3 + m_s}{m_3 m_s}\delta(\mathbf{r}_3 - \mathbf{r}_s)$$

(a_0 is the v-impurity—s-impurity scattering length), the height of the barrier V_1 can be estimated from the expression

$$V_1 \approx 4\pi\hbar^2 a_0 \frac{m_3 + m_s}{m_3 m_s}n(r),\quad (13)$$

where n(r) is the average volume density of the s-impurities in a spherical layer of thickness λ around the anion:

$$n(r) = \frac{N_s(c, T)}{4\pi R^2 \lambda}\exp\left(\frac{R - r}{\lambda}\right), \quad r \gg R.\quad (13a)$$

When writing down (13a), we used the fact noted above, namely that all the wave functions of the s-impurities with n = 0 attenuate at one and the same depth λ which does not depend on l.

Thus, the question of scattering of v-impurities by s-impurities reduces to a solution of the problem of s scattering of v-impurities by the potential V(r),

$$V(r) = \begin{cases} U_0, & r \leqslant R(T) \\ V_0(r) + V_1(r), & R(T) \leqslant r < \infty \end{cases},\quad (14)$$

$V_0(r)$ is an attraction potential that acts on the v-impurities as well as on the s-impurities, $V_1(r)$ is given by (14), and U_0 is the height of the principal barrier[2], $U_0 \approx 6-7°$ K.

3. In this section we obtain simple estimates for m_S and ϵ_0, using the experimental data on the anion mobility only in the low-temperature region $T < T_F^S$. The lower bound for m_S is obvious beforehand: $m_S > m_S^0$, where m_S^0 is the mass of the individual He₃ atom. As to the upper bound, it can be determined from the following useful considerations: it is natural to assume that the last L-th level in the well should be located at a height

such that the (L + 1)-st level lies outside the well. Consequently, the distance between levels, $\Delta\epsilon_l = \epsilon_{l+1} - \epsilon_l$, should satisfy the inequality $\Delta\epsilon_L \gtrsim \epsilon_L$ in the region $l \sim L$.

Taking into account the explicit form $\Delta\epsilon_l = 2\gamma(l + 1)$, we obtain an estimate for L in terms of ϵ_L and m_s:

$$(L + 1) = \Delta\epsilon_L / 2\gamma \gtrsim \epsilon_L / 2\gamma, \quad \gamma = \hbar^2 / 2m_s R_0^2. \quad (15)$$

We now write down the total change ΔR in the region $T \ll T_F^S$:

$$\Delta R \approx {}^1/_4 R_0 \Delta\alpha / \alpha_0 + \lambda;$$
$$\Delta\alpha = \frac{1}{2\pi R_0^2} \sum_{l=0}^{L} (2l + 1)|\epsilon_l|_{L \gg l} \approx \frac{\gamma L^4}{4\pi R_0^2}, \quad \epsilon_0 \approx \gamma L^2 + \epsilon_L. \quad (16)$$

In expression (16) for ΔR, the influence of the scattering potential (14) was taken into account by simply adding to ΔR the thickness λ of the spherical layer occupied by the s-impurities. Such an approximation is valid if the height of the barrier (14) in the region $T \ll T_F^S$ greatly exceeds the formal energy of the incident v-impurities. A numerical check on the inequality $V_0 + V_1 \gg T$ in the region $T \sim 0.1°$K and $a_0 \approx 2-3$ Å shows that at reasonable m_s and ϵ_0 (taken from the plane problem) it is indeed satisfied with a large margin.

With the aid of (15), the value of L from (16) can be expressed in terms of ϵ_L and m_s. Replacement of L by the smaller value $\epsilon_L / 2\gamma$ increases the term λ in the expression (16) for ΔR, and decreases the term $(1/4)R_0\Delta\alpha/\alpha_0$. As shown by trial estimates, the decrease of the term $(1/4)R_0\Delta\alpha/\alpha_0$ exceeds the increase of λ, because in the former case L is raised to a higher power. As a result, relation (16) with allowance for (15) should be rewritten in the form
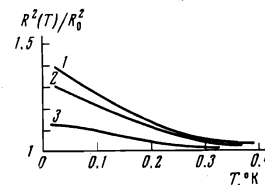
$$\Delta R \gtrsim \frac{\epsilon_L R_0^5 m_s^3}{32\pi\alpha_0\hbar^6} + \frac{\hbar}{[2m_s(\epsilon_L/\gamma + 1)\epsilon_L]^{1/2}}. \quad (17)$$

The inequality (17) connects the numerical values of ΔR, ϵ_L, and m_s. Substituting here $\Delta R \approx 6$ Å and $\epsilon_L \approx 0.26°$ K, we obtain for m_s the estimate $m_s \lesssim 1.2 \times 10^{-23}$ g. Accordingly, $L \gtrsim 17$ and $\epsilon_0 \lesssim 2.4-2.5°$K.

4. We proceed to a discussion of the singularities of the mobility μ (T) in the temperature region $T > T_F^S$. Experiment reveals clearly that μ (T) is nonmonotonic in this region. Theory, on the other hand, yields the following: We start with R(T). The quantity R(T) defined by (12) has the temperature dependence illustrated in Fig. 2. The curves in this figure were calculated for $c = 0.726 \times 10^{-3}$ and for different variants of m_s and ϵ_0 in the vicinity of $m_s \approx 10^{-23}$ and $\epsilon_0 \approx 2°$K. With the aid of the data of Fig. 2 it is easy to verify that the expression $f(T) = R\partial R/4T\partial T$ is positive in the entire transition region of the temperatures (this statement is valid in a wide range of values of c, m_s, and ϵ_0). However, the positiveness of f(T) means that the observed nonmonotonicity of μ (T) is not connected with the dependence of the anion radius R(T) on the temperature. In other words, the rate of change of R with temperature turns out to be insufficient for a change in the sign of $\partial\mu/\partial T$. The formal reason for the strong "smearing" of the transition region for R(T) is the already mentioned power-law character of the α(T) dependence at low temperatures (see (10)).

In view of these remarks, it becomes clear that the principal role in the formation of the temperature dependence of μ(T) in the transition region is played by the direct interaction of the v- and s-impurities. The amplitude of this interaction is proportional to N_s(c, T) and

FIG. 2. Plots of R^2(T)/R_0^2 against T at $c = 0.726 \times 10^{-3}$. The family of curves was constructed for the following values of the parameter m_s and ϵ_0($m_0 = 10^{-23}$ g): 1—$m_s = 1.3$ m_0, $\epsilon_0 = 2.1°$K; 2—$m_s = 1.1$ m_0, $\epsilon_0 = 2.4°$K; 3—$m_s = 0.8$ m_0, $\epsilon_0 = 1.8°$K.

consequently varies in the transition region at an exponential (and not power-law) rate, i.e., in principle, it can ensure a sufficiently rapid change, and of the required sign, in the total scattering cross section with increasing temperature. More detailed results, which confirm quantitatively the indicated point of view, are still unavailable in view of the complexity of the corresponding multiparticle problem of v-impurity scattering by a cloud of s-impurities. The simplified variant of the description of this interaction, used above to take s-v scattering in the region $T \ll T_F^S$ into account (see formulas (13) and (14)), was essentially based on the satisfaction of the inequality $V_0 + V_1 \gg T$. In the transition region of temperatures, this inequality is violated. The applicability of the optical model to the description of s-v scattering under similar conditions calls for a more detailed discussion than given above.

CONCLUSION

The aggregate of the available experimental data on the properties of anions in weak He3—He4 solutions offers practically unequivocal evidence of the existence of s-impurity excitations on the surface of the anion. The spectrum of these excitations, unlike in the plane case, turns out to be discrete, and the occupation numbers are finite and sufficiently small. The calculation of different thermodynamic characteristics for an s-impurity gas on a spherical surface can be easily carried out by standard means and leads to a number of observable consequences. One such consequence is relatively high fermization temperature of the surface-impurity gas (in comparison with the volume temperature degeneracy for the given volume concentration of the solution). In practice this circumstance becomes manifest, for example, in the fact that the temperature dependence of μ (T) takes the form μ (T) $\sim T^{-1/2}$ in a rather wide range of low temperatures, with a scattering length appreciably increased in comparison with R_0. Under a similar situation, the s-impurities fill completely the existing levels, so that the scattering cross section, increasing to the maximum possible value, ceases to depend on T, and the v-impurity gas retains its Boltzmann characteristics, as evidenced by the temperature dependence μ(T) $\sim T^{-1/2}$.

The discrete character of the s-impurity spectrum introduces an entire set of activation exponentials into the temperature dependences of various thermodynamic quantities (unlike the plane case, when there is only a single exponential $\exp(\epsilon_0/T)$). The smoothest exponential $\exp(\epsilon_L/T)$ with $\epsilon_L \approx 0.26°$ K, which determines the degree of occupation of the last and shallowest L-level in the well for the s-impurities, has nothing in common numerically with $\exp(\epsilon_0/T)$, $\epsilon_0 \approx 2-2.5°$ K, which characterizes the degree of occupation of the ground level. Therefore experimental observation of $\exp(\epsilon_L/T)$ with $\epsilon_L \ll \epsilon_0$ (these data are gathered in the table), can be understood only within the framework of a theory that makes use of a discrete dispersion law for the s-impurity excitations. Finally, the increase of the mobility in the transition region of temperatures and the tendency

of R(T) to approach R_0 (which can be verified by direct estimates of the mobility in the vicinity of the temperatures preceding the roton fall-off) is also qualitatively in good agreement with the general notions concerning the properties of s-impurities on the anion surface.

However, not all the problems that have arisen in connection with the interpretation of the observed properties of the anions in weak He_3—He_4 solutions can be regarded as solved even in principle. Thus, the question of the correct description of the interaction of s- and v-impurities in the transition region of temperatures remains open. The appearance of a dependence of μc on c at low concentrations (Fig. 1) remains unexplained, and this question can be regarded as a general one, having a bearing on the mobility of both the anions and the cations. In fact, plots of μc against c for cations, constructed by Neeper and Meyer[11], have a form similar to Fig. 1 at low concentrations.

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APPENDIX

We obtain the spectrum of the surface excitations on a sphere R with an attractive potential

$$V(r) = -V_0 \exp\left(\frac{R-r}{\lambda}\right), \quad V_0 > 0$$

under the conditions $\lambda/R \ll 1$ and $\psi(R) = 0$ ($\psi(r)$ is the wave function of the s-impurity; it is assumed that a single-particle Schrödinger equation exists for the s-impurities). With the aid of the substitutions

$$x = (r - R)/R, \quad \varphi = r\psi \tag{A.1}$$

we rewrite the Schrödinger equation in the following dimensionless form

$$\varphi'' - \left[x^2 + \frac{l(l+1)}{(x+1)^2} - \tilde{V}_0 e^{-\delta x}\right]\varphi = 0,$$
$$\varphi(x)\big|_{0,\infty} = 0, \quad \varphi'' \equiv \partial^2\varphi/\partial x^2, \tag{A.2}$$
$$\tilde{V}_0 = V_0 2m_s R^2/\hbar^2, \quad \delta = R/\lambda \gg 1.$$

Recognizing that at $\delta \gg 1$ and not too large $|\tilde{V}_0|$ (corresponding estimates for $|\tilde{V}_0|$ are given below), the major role in (A.2) is played, for all admissible l, by small $x \ll 1$, we replace (A.2) by the approximate equation

$$\varphi'' + (\tilde{V}_0 e^{-\delta x} - \tilde{\varkappa}^2)\varphi = 0,$$
$$\varphi(0) = \varphi(\infty) = 0, \quad \tilde{\varkappa}^2 = \varkappa^2 + l(l+1). \tag{A.3}$$

Equation (A.3) can be solved exactly:

$$\varphi(x) = J_\nu(\xi), \tag{A.4}$$
$$\xi = \alpha e^{-\delta x/2}, \quad \alpha^2 = 4\tilde{V}_0/\delta^2, \quad \nu^2 = 4\tilde{\varkappa}^2\delta^{-2},$$

$J_\nu(\xi)$ is a Bessel function with index ν. The boundary condition $\varphi(0) = 0$ leads to the equation $J_\nu(\alpha) = 0$, which is satisfied only for definite positive values of ν. Having the set ν_0, ν_1, ..., and expressing with $\varkappa_{nl}$ in terms of ν_n and l the aid of (A.4), we arrive at the following definition of the spectrum:

$$\varkappa_{nl}^2 = \frac{1}{4}\nu_n^2\delta^2 - l(l+1), \tag{A.5}$$

which is analogous to expression (4) of the main text. The quantity $l_{max}^{(n)}$ for each value of n is determined by the requirement $\varkappa_{nl}^2 > 0$.

It should be noted that the behavior of the wave function as a function of r is determined only by the constants ν_n, α, and δ, and does not depend on $\varkappa_{nl}$ or l.

separately. This means, in particular, that all the wave functions with given ν_n have the same damping depth, which does not depend on l. Using this remark, we can easily state a criterion for the applicability of approximation (A.3). This approximation is valid for all ν_n that ensure a damping length $\lambda_n \ll R_0$ for the wave functions.

In conclusion, let us estimate the probability of the existence of levels with n > 0 for a problem with parameters $R \approx 20$ Å, $m_s \approx 10^{-23}$ g, and $\epsilon_0 \approx 2-3°$ K. The damping length is $\lambda_0 = \hbar(2m_s\epsilon_0)^{-1/2} \approx 2-3$ Å, i.e., the inequality $\lambda_0/R \ll 1$ is satisfied. The parameter ν_0 is connected with ϵ_0 by the relation

$$\epsilon_0 = \hbar^2\nu_0^2/8m_s\lambda_0^2. \tag{A.6}$$

From this we get $\nu_0 \approx 3-4$.

Turning now to the dependence of $J_\nu(\alpha)$ on the index ν, we choose from among the curves the first to vanish at $\nu_0 \approx 3-4$ on moving from the direction of large ν. This curve coincides with the curve J_ν (7) shown in Fig. 3 of[12]. We see that in the region of smaller ν, there can be on the given line one more point ν_1 satisfying the condition $J_{\nu_1}(\alpha) = 0$. But a small change of the parameters R, ϵ_0, and m_s, whereby ν_0 is shifted closer to 3 than to 4, leads to the vanishing of the intersection ν_1. For this reason, it is difficult to insist that there is a level with n = 1, although its existence is in principle possible.

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1)It should be noted that the real number of impurities on an individual anion cannot be much smaller than unity. It either has an impurity or has not. Formula (7a) and all the other relations of this type should be regarded as averaged over a large number of anions, each having or not having an impurity.

2)The height U_0 of the principal barrier in the problem of the scattering of impurities by an anion consists of two parts: $U_0 = U_0^{(1)} + U_0^{(2)}$, where $U_0^{(1)} \approx 2.7°$ K is the usual barrier for the impurity and the liquid-vapor boundary and $U_0^{(2)} \approx p_{el}v_3$ is a barrier of electronic origin ($p_{el} = 3\pi\hbar^2/8m_eR_0^5$ is the effective electron pressure on the surface of the bubble and v_3 is the volume of the v-impurity). Using the constants that enter in $U_0^{(2)}$ and p_{el} (m_e is the electron mass), we obtain for $U_0^{(2)}$ the estimate $U_0^{(2)} \approx 2 - 3°$ K. The resultant sum is $U_0 = U_0^{(1)} + U_0^{(2)} \approx 5 - 6°$ K.

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