Singularities of the cosmological model of the Bianchi IX type according to the qualitative theory of differential equations

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An exact definition and calculation of all singular points (in the sense of qualitative theory of differential equations) are presented for the Einstein equations in a homogeneous cosmological model of the Bianchi IX type, as well as their separatrices. This makes possible an exact statement and solution of the problem regarding the initial states of the Universe at early stages of evolution which are "typical" for the sign of time corresponding to expansion (in contrast to contraction, for which the typical states have been found by Belinskiĭ, Lifshitz, and Khalatnikov and the analytically complex structure of the cosmological singularity has been elucidated). The initial typical states for Universe expansion indicated in the paper correspond asymptotically to power-law solutions with three types of time-factor asymptotics: that of the Friedman quasiisotropic type, that of the Taub type and a previously unknown type.

INTRODUCTION

The general anisotropic Bianchi IX model was first investigated by Belinskiĭ, Lifshitz, and Khalatnikov (BLKh) in[1,2], and later by Misner[3,4], by Doroshkevich, Lukash, and I. Novikov[5], and by others, who were initially interested mainly in the asymptotic properties of the components of the spatial metric and the trajectories of light rays when the universe contracts to a point. Another problem, namely the expansion away from the singularity (up to the instant of the maximum expansion) was first considered for the Bianchi IX model only recently (see[6-8]). In each of these problems, it is important to know the "typical" states in which the components of the spatial metric can be situated near the singularity (what is the definition of these typical states, and how does it depend on the sign of the time). For the process of contraction to a point, typical states were already indicated in the language of "Kasner exponents," which resulted from the piecewise approximation of the time dependence of the components of the spatial metric by means of power-law functions, an approximation that turned out to be very successful (see[2]). For the sign of the time on the expansion side, this problem has not been solved. Moreover, there was no exact definition of the concept of the "typical" state during the earlier stages of the evolution near the singularity.

In the present paper we use the method of investigating homogeneous models from the point of view of the qualitative theory of ordinary equations with algebraic right-hand sides, namely, it is necessary in principle to determine and to investigate the singular points of these equations and of their separatrix. It is then necessary to draw the diagram of the transitions along the separatrix and to assess from this diagram the asymptotic behavior and the typical states. However, even the very definition of the singular points for Einstein's equations is far from a trivial matter, since there are no singular points in the region where the metric is positive. Owing to the leeway in the choice of the coordinates and the time, the continuation of the system to the boundary of this region is highly ambiguous and, as a rule, gives singularities that are so degenerate that they cannot be used to assess the asymptotic behavior.

These difficulties can be overcome. We present a correct definition of the singular points and, using certain simple algebraic procedures, supplement physical region with a boundary on which the geometry of the dynamic system becomes perfectly understandable. In particular, the BLKh results[2] in the contraction direction turn out to be a formal consequence of the separatrix diagram of this system (see Sec. 5). By way of another consequence, we present a list of the possible power-law asymptotics near the singularity; for the particular case of the Bianchi IX model with axial symmetry, there are no other asymptotic forms at all (this model admits of exact integration only for zero matter (see Sec. 3)). As a third consequence, we indicate typical initial data (from the point of view of a separatrix diagram on the boundary) on the expansion side (see Sec. 5).

In conclusion, the authors wish to note that the problems and the results of Secs. 1 and 2 belong to S. P. Novikov, whereas the results of Secs. 3 and 4 were obtained mainly by O. I. Bogoyavlenskiĭ. The results of Sec. 5 were obtained by the authors jointly.

1. SINGULARITIES OF THE EINSTEIN EQUATIONS WHERE THE SPATIAL METRIC IS NOT FULLY DEGENERATE

In four-dimensional space-time, there acts on the right a three-parameter group G (with three-dimensional orbits) of one of the nine Bianchi types (Type IX is SU_2 or SO_3); the Einstein metric is assumed to be right-invariant, the orbits of the group are assumed to be spacelike. We choose four right-invariant vector fields X_0, X_1, X_2, and X_3, where the fields X_1, X_2, and X_3 are tangent to the orbits of the group G. Their commutators, by definition, are of the form

$$[X_0, X_\alpha] = 0, \quad [X_\alpha, X_\beta] = C_{\alpha\beta}^{\delta} X_\delta, \tag{1.1}$$

where $\alpha, \beta, \delta = 1, 2, 3$; $C_{\alpha\beta}^{\delta}$ are the structure constants of the group G. Then the scalar products

$$g_{ij} = \langle X_i, X_j \rangle, \quad i, j = 0, 1, 2, 3, \tag{1.2}$$

depend only on the time $g_{ij}(t)$, where the time lines go along the field X_0.

We assume that the trajectories of the field X_0 are geodesics. If $g_{0\alpha} = 0$, then the reference frame is synchronous; if $g_{00} = 0$, then we call the reference frame "light-like." For simplicity we assume that the equation of state takes the form

$$T_k^i = (p+\varepsilon)u_k u^i - p\delta_k^i, \quad u_i = \delta_i^0, \quad p = k\varepsilon. \tag{1.3}$$

The Einstein equations in the synchronous frame have been written out in[2], Starting with the region where the spatial matrix $g_{\alpha\beta}$ is positive, we should like to determine the behavior of the system near the instant of its degeneracy (cosmological singularity).

The singular points or the rest points are determined for a system of ordinary first-order differential equations with smooth time-independent right-hand sides (containing no poles or discontinuities). At these points, all the right-hand sides are equal to zero. If we have a system of equations of order n, then it is necessary to reduce it, by the usual means, to a first-order system of the form $\dot{X}_1 = f_i(x)$; only then can we determine the singular points. If the right-hand sides have poles, then all the right-hand sides must be multiplied, prior to the determination of the singular points, by a common (minimal) factor such that the right-hand sides no longer have poles; only then can we equate all the right-hand sides to zero (this is equivalent to a change of the time in the system). It is then necessary to determine the eigenvalues of the matrix $|\partial f_i/\partial x_j|$ at the singular points and the separatrices corresponding to the nonzero eigenvalues (see[9,10]).

It is easily shown for the Einstein equations in the homogeneous models that there are no singular points at all in the region where the spatial metric $g_{\alpha\beta}$ is positive. All the singular points lie on a surface where the metric $g_{\alpha\beta}$ becomes degenerate. By virtue of the chosen form of the energy-momentum tensor (1.3), we can use a time-independent substitution in the synchronous reference frame to make the metric $g_{\alpha\beta}$ diagonal at all t; the singular points therefore appear on hypersurfaces of the type ($g_{\alpha\alpha} = 0$), for example at $\alpha = 1$. However, the synchronous reference frame itself becomes degenerate on this surface; Einstein's equations lose their physical meaning on this surface.

To determine the singular points we make use of the fact that the light-like reference frame, unlike the synchronous frame, retains a physical meaning on the surface. Let the metric be of the form

$$g_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & g_{11} & 0 & 0 \\ 0 & 0 & g_{nm} & \\ 0 & 0 & & \end{pmatrix}, \quad n, m = 2, 3. \tag{1.4}$$

In Taub's particular case, the transition to a light-like reference frame makes Einstein's metric in empty space analytic at $g_{11} = 0$ and without singularities in the four-dimensional sense[11].

The situation is different in the general Bianchi IX model. As shown in[1], Taub's solutions are unstable in a general model of type IX, in spite of its regularity. The reason lies in the singular points in the sense of the theory of differential equations ([8]). The Ricci tensor in the system (1.4) takes the form

$$R_{00} = \frac{1}{2}(\ln g)^{\cdot\cdot} - \frac{1}{2g} + \frac{1}{4}\varkappa_m{}^n \varkappa_n{}^m,$$

$$R_{10} = \frac{1}{2}\left(\frac{d}{dt} + \frac{\dot g}{2g}\right)\dot g_{11} + \frac{1}{2}(\varkappa_2{}^3 - \varkappa_3{}^2),$$

$$\begin{aligned} R_{11} &= \frac{1}{2}\left(\frac{d}{dt} + \frac{\dot g}{2g}\right)(g_{11}\dot g_{11}) - \frac{\dot g_{11}{}^2}{2} + 1 - \frac{1}{2g}\sum g_{nm}{}^2, \\ R_{22} &= \frac{1}{2}\left(\frac{d}{dt} + \frac{\dot g}{2g}\right)(g_{11}\dot g_{22} - 2g_{23}) - \frac{g_{11}g_{22}}{2g} + 1 \\ &\quad - \frac{g_{11}}{2}\varkappa_2{}^l\varkappa_{l2} - (g_{22}\varkappa_2{}^3 - g_{23}\varkappa_2{}^2), \\ R_{23} &= \frac{1}{2}\left(\frac{d}{dt} + \frac{\dot g}{2g}\right)(g_{11}\dot g_{23} + g_{22} - g_{23}) - \frac{g_{11}g_{23}}{2g} \\ &\quad - \frac{g_{11}}{2}\varkappa_3{}^l\varkappa_{l2} + \frac{1}{2}(g_{33}\varkappa_2{}^2 - g_{23}\varkappa_2{}^3 + g_{23}\varkappa_3{}^2 - g_{22}\varkappa_3{}^3), \\ R_{33} &= \frac{1}{2}\left(\frac{d}{dt} + \frac{\dot g}{2g}\right)(g_{11}\dot g_{33} + 2g_{23}) - \frac{g_{11}g_{33}}{2g} + 1 - \frac{g_{11}}{2}\varkappa_3{}^l\varkappa_{l3} \\ &\quad - (g_{23}\varkappa_3{}^3 - g_{33}\varkappa_3{}^2). \end{aligned} \tag{1.5}$$

All the remaining R_{ij} vanish identically; we denote by g the determinant $g = \| g_{nm} \|$,

$$\varkappa_{nm} = \frac{dg_{nm}}{dt}; \quad l, m, n = 2, 3.$$

Einstein's equation (for empty space) takes the form

$$R_{ij} = 0. \tag{1.6}$$

The system (1.6) has two integrals:

$$\begin{aligned} I_1 &= R_{11} - g_{11}R_{10} = 1 - \frac{1}{2g}\sum_{n,m=2,3} g_{nm}{}^2 - \frac{1}{2}g_{11}(\varkappa_2{}^3 - \varkappa_3{}^2), \\ I_2 &= g^{mn}R_{mn} - g_{11}R_{00}. \end{aligned} \tag{1.7}$$

We are interested in this system only at the levels of the integrals $I_1 = 0$ and $I_2 = 0$.

After making the time change

$$d\tau / dt = 1 / g_{11}$$

we obtain, after calculations, a two-dimensional manifold of singular points, which correspond exactly to the limits of Taub's solution in empty space on the surface $g_{11} = 0$:

$$g_{22} = g_{33}, \quad v_{22} = v_{33}, \quad g_{23} = 0, \quad v_{23} = 0, \; v_{11}v_{22} = -2, \quad v_{ij} = dg_{ij}/dt. \tag{1.8}$$

Their nonzero eigenvalues are of the form

$$\lambda_1 = v_{11} < 0, \quad \lambda_2 = \bar\lambda_3 = -v_{11} + 4i, \quad \lambda_4 = -v_{11}, \tag{1.9}$$

and an eigenvalue that is negative on the contraction side corresponds to a Taub solution that enters the singular point in empty space. All the singular points lie at the level $I_1 = 0$, $I_2 = 0$; if $g_{11} = 0$ then $g \neq 0$. They correspond, by virtue of (1.8), to the limits of the so-called "Taub solutions in empty space" ($g_{22} \equiv g_{33}$, $g_{23} = g_{32} \equiv 0$). The surface $g_{11} = 0$ is an invariant manifold or aggregate of the emerging separatrices of this set of singular points.

Simple calculation with the system (1.5) shows, following the substitution

$$d\tau_1 / d\tau = 1 / g,$$

that on the surface $g_{11} = 0$ there are no singular points other than the Taub singularities (1.8), except for the case when all $g_{\alpha\alpha} = 0$. This is the most degenerate singularity, which cannot be conveniently investigated in the light-like reference frame. Since the light-like reference frame has retained a physical meaning on the surface $g_{11} = 0$, the obtained singularities are correctly defined. On the basis of the results we shall subsequently subject the chosen coordinate system to the requirement that it yield on the surface $g_{11} = 0$ the same singularities as the light-like reference frame. It turns out that this can be done even in the synchronous reference frame by using the Hamiltonian formalism and by correctly choosing the phase coordinates.

2. HAMILTONIAN FORMALISM. POWER-LAW ASYMPTOTICS IN THE BIANCHI IX MODEL

The first question we shall consider in the investigation of a strongly degenerate manifold of singular points, where the entire spatial metric is degenerate, is that of the power-law asymptotic expressions in t (in synchronous time t) for an equation of state (1.3). We already know in this case the asymptotic solutions of Friedmann[1)] and Taub

$$q_\alpha^2 = g_{\alpha\alpha} \sim C_\alpha t^{4/3(1+k)}, \quad C_\alpha = \text{const}, \tag{2.1}$$

$$q_\alpha^2 = g_{\alpha\alpha} \sim C_1 t^2, \quad g_{\beta\beta} = g_{\gamma\gamma} = \text{const}. \tag{2.2}$$

As shown by one of us[8], there is one other possible power-law asymptotic form:

$$q_\alpha^2 = g_{\alpha\alpha} \sim C_\alpha t^{(1-k)/(1+k)}, \quad q_\beta^2 = g_{\beta\beta} \sim C_\beta t^{(3+k)/2(1+k)}, \quad q_\gamma^2 = g_{\gamma\gamma} \sim C_\gamma t^{(3+k)/2(1+k)}; \quad \alpha \neq \beta \neq \gamma. \tag{2.3}$$

The asymptotic form (2.2) appears even in empty space, while (2.1) and (2.3) appear only in the presence of matter (we shall show subsequently that there are no other asymptotic forms).

The Hamiltonian formalism in homogeneous models was developed by Misner[3,4] and finally systematized in the most convenient form in[7]. For the equation of state (1.3), the Hamiltonian takes the form

$$H = \frac{1}{2(q_1q_2q_3)^{1-k}}(\overline{U}(P_\alpha) + V(q_\alpha^2)), \quad \frac{dp_\alpha}{d\eta} = -\frac{\partial H}{\partial q_\alpha}, \quad \frac{dq_\alpha}{d\eta} = \frac{\partial H}{\partial p_\alpha}, \tag{2.4}$$

where $P_\alpha = p_\alpha q_\alpha$, α = 1, 2, 3, and p_α and q_α are the momenta and coordinates.

The time η and the polynomial $\overline{U}$ are such that

$$\overline{U}(P_\alpha) = 2P_1P_2 + 2P_2P_3 + 2P_1P_3 - P_1^2 - P_2^2 - P_3^2, \quad \frac{dt}{d\eta} = (q_1q_2q_3)^k, \tag{2.5}$$

and the potential V depends on the group G.

For the Bianchi IX model, the functions $\overline{U}$ and V have the same form. The kinetic energy is indefinite, and the motion is allowed at the levels $2H = A \geq 0$, with the case $A = 0$ corresponding to empty space.

Using the scale group (the homogeneity of this Hamiltonian) we can make the substitution

$$P_\alpha = \lambda^2 b_\alpha, \quad q_\alpha^2 = \lambda^2 \gamma_\alpha^2 \tag{2.6}$$

under the condition $F(\gamma_\alpha) = 1$, where $F(\lambda\gamma_\alpha) = \lambda^m F(\gamma_\alpha)$. We then obtain an equation with energy $\overline{H}$ and friction

$$\frac{dp_n}{ds} = -x_n\frac{\partial \overline{H}}{\partial x_n} \pm 2p_n, \quad \frac{dx_n}{ds} = x_n\frac{\partial \overline{H}}{\partial p_n}, \quad \frac{d\lambda}{ds} = \lambda; \tag{2.7}$$

$$\overline{H} = (4(p_2^2 - p_2p_3 - p_3^2) + 3V(x_2, x_3) + 3Ae^{-\alpha s})^{1/2} = |b_1 + b_2 + b_3|;$$

$n = 2$ and 3, $\alpha = 3k + 1$ (plus on the contraction side, minus on the expansion side), where

$$F(\gamma_\alpha) = \gamma_1\gamma_2\gamma_3 = 1, \ \gamma_2 = x_2, \ \gamma_3 = x_3, \ p_2 = b_2 - b_1, \ p_3 = b_3 - b_1; \ A > 0 \tag{2.8}$$

We have in the system two monotonic functions

$$U^{\text{vac}} = 4(p_2^2 - p_2p_3 + p_3^2) + 3V, \quad U = \overline{H}^2$$

1) $U \to +\infty$ (on the contraction side)

$$dU^{\text{vac}}/ds \geq 0; \tag{2.9}$$

2) $U \to +0$ (on the expansion side)

$$U^{\text{vac}} \to -3Ae^{\alpha s_{max}} < 0.$$

We now derive the asymptotic form (2.3) on the contractor side (in the coordinates (2.6)–(2.8)). We advance the following hypotheses:

$$U^{1/2} \to +\infty, \quad s \to +\infty, \quad p = p_2 + p_3 \gg 1,$$
$$p_2 - p_3 = Bp \ll p_2 + p_3 = p \ \text{or} \ B \ll 1; \ x_2 = \gamma_2 \ll 1, \ x_3 = \gamma_3 \ll 1. \tag{2.10}$$

From (2.7) we readily see that

$$\frac{dx_2}{ds} < 0, \quad \frac{dx_3}{ds} < 0, \quad w = \frac{1}{(x_2x_3)^4} \gg 1,$$
$$\frac{3}{2}\sum_{m=1}^{2} x_m \frac{\partial V}{\partial x_m} = -12 + O\left(\frac{1}{w^{1/2}}\right), \tag{2.11}$$

if $|B| < 1/3$.

We introduce a new time q such that

$$\frac{d\lambda^{-1}}{dq} = \frac{\lambda^{-1}}{w}U^{1/2}, \quad \frac{dw}{dq} = 8p, \quad \frac{dp}{dq} \sim \frac{2pU^{1/2}}{w} - 12. \tag{2.12}$$

It is easy to show that if the growth of p(q) is faster than linear, then the matter has no influence, for in this case we arrive at the exponential regime p(q), where $|B| \sim 5^{1/2}$, and by the same token the condition (2.11) will sooner or later be violated.

If p(q) grows linearly, then the following hypotheses must be made for the influence of matter (at $k = 0$):

$$\lambda^{-1} \sim \beta p^2, \ w \sim \alpha p^2; \ \alpha > 0, \ \beta > 0. \tag{2.13}$$

In the region where these hypotheses hold, we have

$$\frac{dp}{dq} \sim \frac{2z}{\alpha} - 12; \quad z = (1 + 3B^2 + 3\alpha + 3A\beta)^{1/2};$$
$$\frac{dw}{dq} \sim 2\alpha p\left(\frac{2z}{\alpha} - 12\right), \quad \frac{d\lambda^{-1}}{dq} \sim 2\beta p\left(\frac{2z}{\alpha} - 12\right) \sim \beta p\frac{z}{\alpha}. \tag{2.14}$$

This yields

$$z = 4, \ \alpha = 1/3, \ 16 = 1 + 3B^2 + 3\alpha + 3A\beta; \ dp/dq \sim 12. \tag{2.15}$$

If $p_2 \sim p_3$ (or $B \approx 0$) at all times, then we arrive at the answer

$$p \sim 12q + \text{const}; \ x_\alpha = D_\alpha q^{-1/4}, \ D_\alpha = \text{const}; \ \alpha = 2,3. \tag{2.16}$$

Returning to the synchronous time, we obtain formulas (2.3), where $k = 0$ (dust). For $0 < k < 1$ the situation is analogous, but (2.13) must be replaced by the hypothesis

$$\lambda^{-3k-1} \sim \beta p^2.$$

3. ASYMPTOTIC FORM OF THE TAUB MODEL WITH MATTER NEAR THE SINGULARITY $q_2 \equiv q_3$

We investigate first the simple singular case when $q_2 \equiv q_3$, $P_2 \equiv P_3$, and the Hamiltonian (2.4) takes the form

$$H = \frac{1}{(q_1 q_2^2)^{1-k}}(2P_1\overline{P}_2 - P_1^2 + 4q_1^2q_2^2 - q_1^4), \quad \overline{P}_2 = 2P_2. \tag{3.1}$$

We introduce formally new variables (without discussing their meaning):

$$u = \frac{P_1}{2P_2}, \quad w = \frac{q_1^2}{2P_2}, \quad v = \frac{q_1q_2}{2P_2} \tag{3.2}$$

and the time τ, where

$$\frac{d\tau}{dt} = -\frac{w}{q_1v^2}. \tag{3.3}$$

In terms of the new variables, we obtain the equations

$$\frac{dv}{d\tau} = v(-k - (1-k)(u-1)^2 - (1-k)w^2 - 4kv^2), \quad \frac{dq_1}{d\tau} = q_1(u-1),$$
$$\frac{du}{d\tau} = -2w^2 + (2u-1)H_0 + 4v^2 - 4uv, \quad \frac{dw}{d\tau} = w(2(u-1) + 2H_0 - 4v^2),$$

$$H_0 = {}^1/_2(1-k)(1-(u-1)^2 - w^2 + 4v^2),$$
$$H_0 \geqslant 0, \quad \frac{1}{v}\frac{dv}{d\tau} \leqslant 0, \quad w \leqslant 0, \quad \frac{d\tau}{dt} > 0 \tag{3.4}$$
(H = 0 for empty space).

When the singularity is approached, we find that the asymptotic forms are all determined by the system (3.4), which is correctly defined in the semicircle

$$v = 0, \quad H_0 \geqslant 0, \quad w \leqslant 0 \tag{3.5}$$

(the plane v = 0 did not appear in the physical region). The singular points and their eigenvalues take the form (in terms of the variables (u, w)):

$$1)\ u=0, \quad w=0 \quad (H_0=0), \quad \lambda_1 = -(1-k), \quad \lambda_2 = -2 \quad \text{(node)};$$

$$2)\ u={}^1/_2,\ w=0,\ H_0 = {}^3/_8(1-k),\ \lambda_1 = {}^3/_4(1-k);\ \lambda_2 = -{}^1/_4(1+3k) \quad \text{(saddle)};$$

$$3)\ u=\frac{3+k}{5-k}, \quad w=-\frac{1}{5-k}((1-k)(1+3k))^{1/2}, \quad H_0=\frac{2(1-k)}{5-k}$$
$$\lambda_{1,2} = \frac{2}{5-k}\left(1-k \pm i\left(\left(\frac{1-k}{2}\right)(3+16k-3k^2)\right)^{1/2}\right) \quad \text{(focus)}; \tag{3.6}$$

$$4)\ u=2, \quad w=0, \quad H_0=0, \quad \lambda_1 = -3(1-k), \quad \lambda_2 = 2 \quad \text{(saddle)}.$$

The phase diagram takes the form shown in Fig. 1.

It is easy to show that the "whiskers" (or separatrices) of the singular point 1), which come from the region v < 0, are of the Taub type (2.1), those of singular point 2) are of the Friedmann type (2.2), and those of singular point 3) are of the type (2.3). The whiskers of the singular point 4) come from the unphysical region $q_1 \equiv 0$.

This investigation provides rigorous proof that no other power-law asymptotic forms are possible in this model. At $0 \leq k \leq {}^1/_3 + \alpha$, where $\alpha > 0$, it can be shown additionally that the system (3.4) has no limit cycles in the region (3.5). It follows directly from this that in Taub's model with matter this interval in k contains no asymptotics at all other than (2.1), (2.2), (2.3). (There are probably no limit cycles at all k < 1 either.)

Let us prove that the system (3.4) has no limit cycles for $0 \leq k \leq {}^1/_3$: we consider a function R such that

$$R = (u-1)^2 + w^2, \quad \frac{dR}{d\tau} = (1-k)(1-R)[2R + (u-1)]. \tag{3.7}$$

Obviously $dR/d\tau$ reverses sign inside the unit circle only on the circle $dR/d\tau = 0$ on which $R \leq {}^1/_4$. Therefore the cycle, if it does exist, lies in the region $R \leq {}^1/_4$, which contains the circle $dR/d\tau = 0$. The divergences of the right-hand sides of the system (3.4) are given by

$$\operatorname{div} f = (1-k)\left[2 - 4R + \frac{1+k}{1-k}(u-1)\right].$$

For the interval $0 \leq k \leq {}^1/_3$, the region lying inside the circle div f = 0 contains an internal subregion $R \leq {}^1/_4$. Therefore in the region $R \leq {}^1/_4$, according to the Bendixson criterion, there are no limit cycles.

The phase diagram indicated in Fig. 1 at v = 0 leads to the following conclusions: on going towards the singularity in Taub's model with matter, we "almost always" obtain (on the contraction side) an asymptotic form (2.1) of the Taub type in empty space; to the contrary, on moving away from the singularity on the expansion side during the earlier stages of development, the "typical" state is near the singular point of type 3)

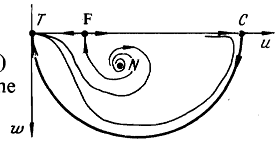

FIG. 1. Trajectories of the system (3.4) in the semicircle $H_0 \geqslant 0$, $w \leqslant 0$ on the plane v = 0 at $0 \leqslant k \leqslant 1/3$.

(see (3.6)), since time reverses sign and the singular point of type 3) becomes a tightening point with respect to the variables (u, w) so long as the entire action takes place close to the singularity.

It is important that the coordinate change (3.2) is regular outside the singularity, where the entire spatial metric is equal to zero. The substitution (3.2) replaces the singular point in phase space, in terms of the new coordinates, by an entire two-dimensional manifold v = 0 that is "glued in" in the phase space with the dynamic system (3.4) on it. This system, under the conditions (3.5), produces a "boundary" that encloses the physical region completely.

Note. When the Taub model with matter is imbedded in the general Bianchi IX model, the mapping on the boundary becomes multiple-valued; the point F blows up into a segment on the triangle Δ, the point T goes over into a point F_α with a half-line T_α^u together with the ends T_α^0 and T_α^∞, and the point C goes over into C_α and the middle of the segment $\overline{AE}_\alpha$ (see Fig. 3 and the start of Sec. 5).

4. GENERAL BIANCHI MODEL IX

Starting from the Hamiltonian H (2.4) with time η (see (2.5)), let us determine and investigate the singular points, especially where the entire spatial metric vanishes. We are interested in the region of nonnegative matter and spatial metric. We make the time substitution

$$\frac{d\tau_0}{d\eta} = \frac{2}{(q_1q_2q_3)^{1-k}} = \frac{2}{\lambda^{3(1-k)}}. \tag{4.1}$$

We consider the phase coordinates

$$P_\alpha, \quad Q_\alpha = q_\alpha^2. \tag{4.2}$$

We then change over from them, using gauge invariance, to new coordinates in accordance with (2.6):

$$b_\alpha = \frac{P_\alpha}{Q_1}, \quad F(\gamma_\alpha) = \gamma_1 = 1, \quad \alpha = 1,2,3;$$
$$y_2 = \gamma_2^2 = Q_2/Q_1, \quad y_3 = \gamma_3^2 = Q_3/Q_1; \tag{4.3}$$
$$u_1 = b_1, \quad v_1 = b_2 + b_3, \quad v_2 = b_2 - b_3; \quad d\tau_1/d\tau_0 = Q_1.$$

In terms of the new coordinates and the time τ_1 we obtain the system

$$\frac{dQ_1}{d\tau_1} = -Q_1(u_1 - v_1), \quad \frac{dy_2}{d\tau_1} = y_2(2u_1 - v_1 - v_2), \quad \frac{dy_3}{d\tau_1} = y_3(2u_1 - v_1 + v_2),$$
$$\frac{du_1}{d\tau_1} = u_1(u_1 - v_1) - (y_2 + y_3 - 1) + H_1, \tag{4.4}$$
$$\frac{dv_1}{d\tau_1} = v_1(u_1 - v_1) - y_2(1 + y_3 - y_2) - y_3(1 + y_2 - y_3) + 2H_1,$$
$$\frac{dv_2}{d\tau_1} = v_2(u_1 - v_1) - y_2(1 - y_2) + y_3(1 - y_3);$$

$$H_1 = {}^1/_4(1-k)(2u_1v_1 - u_1^2 - v_2^2 + 2y_2 + 2y_3 + 2y_2y_3 - 1 - y_2^2 - y_3^2). \tag{4.5}$$

We note that in terms of these coordinates, the monotonic function H (see (2.7)) takes the form

$$-\overline{H} = \frac{u_1 + v_1}{(y_2y_3)^{1/4}}, \quad -\overline{H} \to -\infty \tag{4.6}$$

(on the contraction side).

An essential fact is that v_1 is always less than zero. If $-\bar{H} < -B^2$, then

$$v_1 \leqslant -f^2(B, \bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3) < 0, \quad \sum_{\alpha=1}^{3} \bar{\gamma}_\alpha^2 = 1, \quad q_\alpha = \mu \bar{\gamma}_\alpha, \quad \bar{\gamma}_\alpha \geqslant 0, \tag{4.7}$$

with the exception of points of the type $(0, 2^{-1/2}, 2^{-1/2})$. The constant B in the inequality (4.7) is of the order of 1. In addition

$$f(B, \bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3) \to 0, \tag{4.8}$$

if $\bar{\gamma}_\alpha \to 0$, $\bar{\gamma}_\beta \to 2^{-1/2}$, and $\bar{\gamma}_\delta \to 2^{-1/2}$. This result will be important when it comes to proving that we have obtained all the singularities (see the end of Sec. 4), and also to maintain monotonicity under the time change (4.12).

Getting ahead of ourselves, we note that (4.7) and (4.8) can be derived from the inequalities (4.6) in terms of the coordinates (4.12), by using the fact that matter is positive, $H_2 \geq 0$ (4.13). (Naturally, we shall use the coordinates (4.3) and (4.12) only in the region where $\bar{\gamma}_1 > 0$, and in other regions, such as $\bar{\gamma}_2 > 0$ or $\gamma_3 > 0$, we shall introduce analogous coordinates by making the corresponding permutations.)

We now obtain the singular points of the system in terms of the coordinates (4.3). First, the system has three invariant manifolds:

$$1)\ y_2 = y_3,\ v_2 = 0;\ 2)\ y_2 = 1,\ v_1 + v_2 = 2u_1;\ 3)\ y_3 = 1,\ v_1 - v_2 = 2u_1, \tag{4.9}$$

corresponding to the Taub model $q_\alpha \equiv q_\beta$. The simplest singular points of the system in terms of the coordinates (4.3) in a finite region are given by

$$\begin{gathered} 1)\ y_2 = 1,\ y_3 = 0,\ u_1 = v_1 = v_2;\ 1)'\ y_3 = 1,\ y_2 = 0,\ u_1 = v_1 = -v_2; \\ 2)\ y_2 = 0,\ y_3 = 0,\ u_1(u_1 - v_1) + 1 + H_1 = 0;\ v_1(u_1 - v_1) + 2H_1 = 0. \end{gathered} \tag{4.10}$$

Obviously, singular points of types 1) and 1)′ belong to the intersection of the Taub manifolds (4.9) with edge $y_i = 0$ and are equivalent under the permutation of q_1, q_2, and q_3. Singular points of type 2) appear only in filled space, and the whiskers produced in them upon contraction yield the power-law asymptotic form (2.3). Calculating the characteristic polynomial in the singular points 1) and 1)′ from (4.10), we obtain the eigenvalues

$$\lambda_1 = 2u_1, \quad \lambda_{2,3} = \pm 2i, \quad \lambda_4 = u_1(1-k), \quad \lambda_5 = 0, \tag{4.11}$$

where the parameter u_1 defines a singular point of type 1) or 1)′ in accordance with formulas (4.10). Singular points of these types form a single-parameter family, where all the remaining eigenvalues are different from zero. These singular points coincide with those defined in Sec. 1 in the light-like reference frame; our coordinate system is such that there are no "extra" singular points in the region where not all the q_i vanish (see the requirement at the end of Sec. 1).

These eigenvalues contain a new one, not considered in (1.9), namely $\lambda_4 = u_1(1-k)$, which is due to the matter. The values $\lambda_{2,3} = \pm 2i$ correspond to the complex-conjugate pair in (1.9). We see that $\operatorname{Re} \lambda_{2,3} = 0$ in formulas (4.11), unlike in the light-like reference frame; this is no more than a manifestation of the poor properties of the synchronous reference frame on the surface $g_{11} = 0$.[2)] The extra positive eigenvalue contained in (1.9) is due to the increase in the dimensionality of the phase space.

It will be convenient to investigate the singular points of type 2) in (4.10) later on.

To investigate the singularity in v_1 it is convenient, on the basis of (4.6) and (4.7), to introduce new coordinates and time:

$$\bar{u} = \frac{u_1}{v_1}, \quad \bar{v}_2 = \frac{v_2}{v_1}, \quad \bar{w} = \frac{1}{v_1}, \quad \frac{d\tau_2}{d\tau_1} = -v_1 > 0. \tag{4.12}$$

We then obtain the system

$$\begin{gathered} \frac{dQ_1}{d\tau_2} = Q_1(\bar{u}-1), \quad \frac{dy_2}{d\tau_2} = y_2(1+\bar{v}_2-2\bar{u}), \quad \frac{dy_3}{d\tau_2} = y_3(1-\bar{v}_2-2\bar{u}), \\ \frac{d\bar{u}}{d\tau_2} = (y_2+y_3-1)\bar{w}^2 - \bar{u}\bar{w}^2(y_2(1+y_3-y_2)+y_3(1+y_2-y_3)) + (2\bar{u}-1)H_2, \\ \frac{d\bar{v}_2}{d\tau_2} = \bar{w}^2(y_2(1-y_2)-y_3(1-y_3)) - \bar{v}_2\bar{w}^2(y_2(1+y_3-y_2) + y_3(1+y_2-y_3)) + 2\bar{v}_2H_2, \\ \frac{d\bar{w}}{d\tau_2} = \bar{w}(\bar{u}-1-\bar{w}^2(y_2(1+y_3-y_2)+y_3(1+y_2-y_3)) + 2H_2), \\ H_2 = {}^1/_4(1-k)(1-(\bar{u}-1)^2-\bar{v}_2^2+\bar{w}^2(2y_2+2y_3 + 2y_2y_3 - 1 - y_2^2 - y_3^2)). \end{gathered} \tag{4.13}$$

The condition that matter be positive delineates a circle on the infinitely remote plane $\bar{w} = 0$:

$$(\bar{u}-1)^2 + \bar{v}_2^2 \leqslant 1 \quad (H_2 \geqslant 0). \tag{4.14}$$

The region where y_2 and y_3 vanish takes, under the condition $H_2 \geq 0$, the form

$$(\bar{u}-1)^2 + \bar{v}_2^2 + \bar{w}^2 \leqslant 1, \quad \bar{w} \leqslant 0 \tag{4.15}$$

(the sign of $\bar{w}$ is given here for the contraction side; in the case of expansion $\bar{w} \geq 0$).

It is easily seen that in the region (4.15) the equations take the form

$$\begin{gathered} \frac{d\bar{u}}{d\tau_2} = -\bar{w}^2 + (2\bar{u}-1)H_3, \quad \frac{d\bar{w}}{d\tau_2} = \bar{w}(\bar{u}-1+2H_3), \quad \frac{d\bar{v}_2}{d\tau_2} = 2\bar{v}_2H_3, \\ H_3 = {}^1/_4(1-k)(1-(\bar{u}-1)^2-\bar{v}_2^2-\bar{w}^2). \end{gathered} \tag{4.16}$$

It is seen from (4.16) that the plane $\bar{v}_2 = 0$ is an invariant submanifold. The singular points of the system (4.16) (at $\bar{w} \neq 0$) are singular points of type 2) of (4.10):

$$\bar{u} = \frac{3+k}{5-k}, \ \bar{w} = -\frac{1}{5-k}((1-k)(1+3k))^{1/2}, \ \bar{v}_2 = 0, \ H_3 = \frac{1-k}{5-k} \tag{4.17}$$

On the plane $\bar{v}_2 = 0$ this singular point is a repelling focus with eigenvalues

$$\lambda_{1,2} = \frac{1-k}{5-k} \pm \frac{i}{5-k}\left(\left(\frac{1-k}{2}\right)(3+16k-3k^2)\right)^{1/2}, \quad 0 \leqslant k < 1 \tag{4.18}$$

(see also Sec. 3).

The remaining eigenvalues are given by

$$\begin{gathered} \lambda_3 = \frac{2(1-k)}{5-k} > 0 \quad (\text{variable } \bar{v}_2), \\ \lambda_4 = \lambda_5 = -\frac{1+3k}{5-k} \quad (\text{variable } y_2 y_3), \\ \lambda_6 = -\frac{2(1-k)}{5-k} \quad (\text{variables } Q_1). \end{gathered} \tag{4.19}$$

This singular point is nondegenerate. From the form of the eigenvalues it follows that a three-dimensional manifold of solutions enters in this singular point (on the contraction side). Calculation shows that these solutions have power-law asymptotic forms of the type (2.3), whereas in Taub's model with matter only a two-dimensional manifold of solutions entered in these singular points (see Sec. 3).

The singular points at infinity (at $\bar{w} = 0$) should be sought for the system (4.13), since matter is positive, only in the circle (4.14) under the condition that the

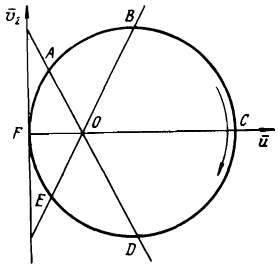

FIG. 2. Singular points (ψ, α) $(\bar{w} = 0)$

spatial metric is nonnegative. Direct calculation in the circle (4.14) yields for $\bar{w} = 0$ the following (see Fig. 2):

1) y_2 and y_3 arbitrary, $\bar{v}_2 = 0$, $\bar{u} = 1/2$ (point O at fixed y_2 and y_3);

$$
\begin{aligned}
&2)\ \ y_2 \neq 0 \text{ arbitrary}, \ y_3 = 0,\ \bar{u} = 1,\ \ \bar{v}_2 = 1 \ \text{(point } B);\\
&2)'\ y_2 \neq 0 \text{ arbitrary} \ \ y_3 = 0,\ \bar{u} = 1/5,\ \ \bar{v}_2 = 3/5 \ \text{(point } A);\\
&3)\ \ y_3 \neq 0 \text{ arbitrary}, \ y_2 = 0,\ \bar{u} = 1,\ \ \bar{v}_2 = -1 \ \text{(point } D);\\
&3)'\ y_3 \neq 0 \text{ arbitrary}, \ y_2 = 0\ \ \bar{u} = 1/5,\ \ \bar{v}_2 = -3/5 \ \text{(point } E);\\
&4)\ \ y_2 = 0,\ y_3 = 0, \qquad (\bar{u} - 1)^2 + \bar{v}_2^2 = 1 \ \text{(boundary of circle)}.
\end{aligned}
\tag{4.20}
$$

We consider the singular points of type 1) in (4.20). This is a two-dimensional manifold, since y_2 and y_3 are arbitrary. Their eigenvalues are given by

$$
\begin{gathered}
\lambda_1 = \lambda_2 = 0 \ \text{(variables } y_2, y_3),\\
\lambda_3 = 3/8(1 - k) \ \text{(variable } \bar{u}),\\
\lambda_4 = 3/8(1 - k) \ \text{(variable } \bar{v}_2),\\
\lambda_5 = -1/2 + 3/8(1 - k) \ \ \text{(variable } \bar{w}),\\
\lambda_6 = -1/2 \ \text{(variable } Q_1).
\end{gathered}
\tag{4.21}
$$

This manifold of singular points includes (in contraction) a four-dimensional family of solutions of power-law form, with all the exponents equal (see (2.2)). This is a generalization of Friedmann's solutions.

The singular points of types 2), 2)′, 3), and 3)′ are the most degenerate; it is easy to show that "whiskers" of minimal dimensionality enter in these points (on either side of the time). Singular points of type 4) of (4.20) have eigenvalues whose signs depend on their location in the circle. These signs are given by the matrix

	$\widehat{EA}$	$\widehat{AB}$	$\widehat{BD}$	$\widehat{DE}$	A	B	D	E
$\lambda_{\bar{w}}$	−	−	+	−	−	0	0	−
λ_{y_2}	+	+	−	−	+	0	−	0
λ_{y_3}	+	−	−	+	0	−	0	+
λ_{Q_1}	−	−	+	−	−	0	0	−

(4.22)

The missing eigenvalues correspond to variables in the $(\bar{u}, \bar{v}_2)$ plane: one of them is equal to zero, and the other is negative. The singular points A, B, D, and E are degenerate.

Before we proceed to integrate the separatrices of the singular points obtained by us, and then to draw conclusions, it is natural to raise the following question: did we find all the singular points? To answer this question we shall show that the "boundary" which we attached during the course of change of coordinates to the physical region S, where $H \geq 0$ and $q_\alpha \geq 0$, makes this physical region S together with the boundary Γ (after taking gauge invariance into account) a compact manifold with an edge (apart from three exclusive points); our system is correctly defined and continuous, including the boundary, and the boundary is an invariant manifold of the system. The monotonic functions contained in the system press the phase point towards the boundary, and by the same token the asymptotic form in contraction is determined by the behavior of this system on the boundary (analogously, the behavior of the system during the earlier stages of expansion).

Let us describe this manifold S. By virtue of the gauge invariance of Eqs. (2.4), we can normalize the components of the metric by the condition

$$
q_\alpha = \mu \bar{\gamma}_\alpha, \ \sum_{\alpha=1}^{3} \bar{\gamma}_\alpha^2 = 1, \ \bar{\gamma}_\alpha \geq 0,
\tag{4.23}
$$

and this yields the spherical triangle Δ. By virtue of (4.7), we can use the variables $(\bar{u}, \bar{v}_2, \bar{w})$. At $\bar{w} = 0$ we have the circle (4.14), and at all $\bar{w} < 0$ satisfying (4.7) we obtain a compact region in $(\bar{u}, \bar{v}_2, \bar{w})$ (with the exception of the exclusive points $\bar{\gamma}_\alpha = \bar{\gamma}_\beta$, $\bar{\gamma}_\delta = 0$, $\alpha \neq \beta \neq \delta$), as follows from the form of the function H_2 and from the inequalities (4.7) and (4.8). We use the coordinates (4.3) in that part of Δ where $\bar{\gamma}_1 \neq 0$. In the remaining regions we use analogous coordinates, replacing $\bar{\gamma}_1$ by $\bar{\gamma}_\alpha$. Thus, the constructed manifold with the boundary is compact after closure by the Taub limits at the three exclusive points on the triangle Δ.

The boundary Γ of this manifold has corners Y_1, Y_2, and Y_3. At the corner Y_1, the coordinates $\bar{\gamma}_1$, $\bar{\gamma}_2$, and $\bar{\gamma}_3$ run along the sides of Δ, while $\bar{u}$ and $\bar{v}_2$ belong to the circle (4.14) and $\bar{w} = 0$; at the corner at Y_2 the coordinates $\bar{\gamma}_1$, $\bar{\gamma}_2$, and $\bar{\gamma}_3$ are arbitrary, $\bar{u}$ and $\bar{v}_2$ belong to the boundary of the circle (4.14), and $\bar{w} = 0$; at the corner Y_3 we have $\bar{\gamma}_\alpha = 1$, $\bar{\gamma}_\delta = \bar{\gamma}_\beta = 0$ (vertex of Δ), while $\bar{u}$, $\bar{v}_2$, and $\bar{w}$ belong to the sphere (4.15). These corners and their intersections Y_{12}, Y_{23}, and Y_{13} are invariant manifolds of the system.

5. SEPARATRIX DIAGRAM OF THE SINGULARITIES AND ITS APPLICATIONS

We use the following notation for the singular points: $\Phi^{\gamma}_{\mathrm{LKh}}$, where $\bar{\gamma}$ is an internal point of the spherical triangle Δ (type 1) from (4.20), where $\bar{\gamma}_\alpha > 0$). $\Phi^{\gamma}_{\mathrm{LKh}}$ lies on the boundary Γ away from the corners;

$\Phi^{0\gamma}_\alpha$, where $\bar{\gamma}$ is on the boundary of Δ, $\bar{\gamma}_\alpha = 0$, $\bar{\gamma}_\beta > 0$, $\bar{\gamma}_\delta > 0$ (type 1) from (4.20)). $\Phi^{0\gamma}_\alpha$ lies in the corner Y_1;

Φ^{00}_α—angles of the triangle (type 1) from (4.20)), lies in the intersection of the corners Y_{31};

T^0_α—type 1) from (4.10), where $u_1 = 0$, $\bar{\gamma}_\alpha = 0$ (exclusive point, arbitrarily assumed to belong to S);

$T^{u_1}_\alpha$—type 1) from (4.10), lies on Γ, where $\bar{\gamma}_\alpha = 0$, $-\infty < u_1 < 0$;

T^{∞}_α—type 1) of (4.10), lies in the corner Y_{12}, $\bar{\gamma}_\alpha = 0$, $u_1 = -\infty$ $(\bar{w} = 0, \bar{u} = 1)$.

N_α—type 2) of (4.10) is located in the corner Y_3, $\bar{\gamma}_\alpha = 1$; (ψ, α)—type 4) of (4.20), where $\bar{u} - 1 = \cos\psi$, $\bar{v}_2 = \sin\psi$, $0 \leq \psi \leq 2\pi$, lies in the intersection $Y_{23} = Y_{123}$.

$\overline{BD}_\alpha$—point of segment BD over the side of Δ: $\bar{\gamma}_\alpha = 0$ (type 2)—3) of (4.20)); $\overline{BD}_\alpha$ lies in the corner Y_{12};

$\overline{AE}_\alpha$—points of segment AE over the side of Δ: $\bar{\gamma}_\alpha = 0$ (type 2)′—3)′ of (4.20)); $\overline{AE}_\alpha$ lies in the corner Y_{12}. The singular points are shown in Fig. 3.

The power-law asymptotics yield whiskers that emerge from the physical region S, where $q_\alpha > 0$, to the boundary (see (2.1)—(2.3)), into singular points of the type $\Phi^{\gamma}_{\mathrm{LKh}}$, N_α, and $T^{u_1}_\alpha$:

$$
T^{u_1}_\alpha \leftarrow S \begin{matrix} \nearrow \Phi^{\gamma}_{\mathrm{LKh}} \\ \searrow N_\alpha \end{matrix}
\tag{5.1}
$$

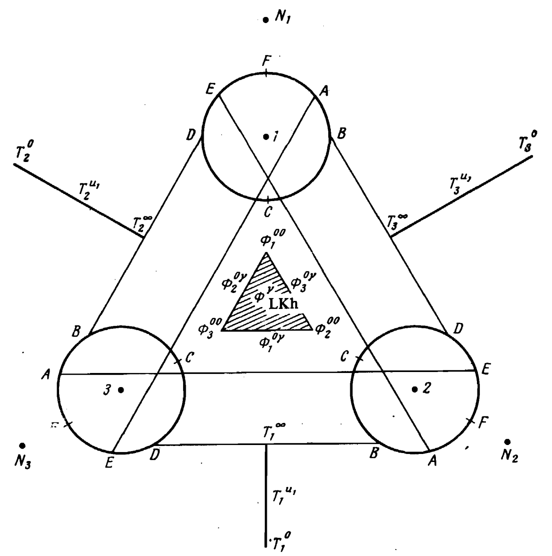

FIG. 3. General arrangement of the singular points of the dynamical system of the Bianchi IX model after taking into account scale invariance and adding a boundary to the physical region.

	S	T_α^0	Φ_{LKh}^γ	$T_\alpha^{u_1}$	N_α	$\Phi_\alpha^{0\gamma}$	Φ_α^{00}	$\widehat{AB}_\alpha$	$\widehat{BC}_\alpha$	C_α	$\widehat{CD}_\alpha$	$\widehat{DE}_\alpha$	$\widehat{EF}_\alpha$	F_α	$\widehat{FA}_\alpha$	$\overline{AE}_\alpha$	$\overline{BD}_\alpha$
Φ_{LKh}^γ	3																
$T_\beta^{u_1}$	3 $\beta=1,2,3$	2 $\beta=\alpha$															
N_β	2 $\beta=1,2,3$	1 ? $\beta=\alpha\pm1$															
$\Phi_\beta^{0\gamma}$		2 ? $\beta=\alpha$															
Φ_β^{00}					1 $\beta=\alpha$												
$\widehat{AB}_\beta$		4 ? $\beta=\alpha-1$		3 ? $\beta=\alpha-1$	3 $\beta=\alpha$	3 $\beta=\alpha-1$	2 $\beta=\alpha$		2 θ $\beta=\alpha$				2 T $\beta=\alpha+1$			? $\beta=\alpha-1$	? $\beta=\alpha-1$
$\widehat{BC}_\beta$			4 $\beta=1,2,3$			3 $\beta=\alpha\pm1$	2 $\beta=\alpha$					2 T^2 $\beta=\alpha-1$			3 T $\beta=\alpha+1$		
C_β			3 $\beta=1,2,3$			2 $\beta=\alpha\pm1$	1 $\beta=\alpha$									2 $\beta=\alpha$	
$\widehat{CD}_\beta$			4 $\beta=1,2,3$			3 $\beta=\alpha\pm1$	2 $\beta=\alpha$	2 T $\beta=\alpha+1$					3 T^2 $\beta=\alpha-1$				
$\widehat{DE}_\beta$		4 ? $\beta=\alpha+1$		3 ? $\beta=\alpha+1$	3 $\beta=\alpha$	3 $\beta=\alpha+1$	2 $\beta=\alpha$				2 θ $\beta=\alpha$				2 T^2 $\beta=\alpha-1$	? $\beta=\alpha+1$	? $\beta=\alpha+1$
$\widehat{EF}_\beta$					3 $\beta=\alpha$		2 $\beta=\alpha$				2 θ $\beta=\alpha$						
F_β					2 $\beta=\alpha$		1 $\beta=\alpha$			1 θ $\beta=\alpha$							
$\widehat{FA}_\beta$					3 $\beta=\alpha$		2 $\beta=\alpha$		2 θ $\beta=\alpha$								
$\overline{AE}_\beta$		3 ? $\beta=\alpha$		2 ? $\beta=\alpha$		2 $\beta=\alpha$											
$\overline{BD}_\beta$			3 $\beta=1,2,3$			2 $\beta=\alpha$								2 $\beta=\alpha$			

FIG. 4. Separatrix diagram on the contraction side.

The remaining singular points have no separatrices that emerge from S—all their separatrices lie on the boundary Γ (we recall that the important singular points (ψ, α) lie on the corner of the edge Y_{123} of our manifold).

The separatrices lead from each singular point to another. Their integration is quite laborious but straightforward; omitting the calculations, we obtain the separatrix diagram shown in Fig. 4. The following notation is used in the diagram:

a) A filled square denotes a separatrix that goes on the contraction side from one set of singular points in the upper row to another set in the column; the number in it denotes the dimensionality of this separatrix, and an empty square means absence of a separatrix.

b) The question mark denotes that the corresponding separatrix has not been fully integrated or that the corresponding set of singularities has zero eigenvalues whose number exceeds its dimensionality.

c) On the singular points of type (ψ, α) from the three circles Y_{123}, the transitions in the diagram to other singular points of the same type are given by the mappings indicated in the squares:

$$T(\bar{u}, \bar{v}_2) = (\bar{u}^0, \bar{v}_2^0); \quad \bar{u}^0 = \frac{1+\bar{v}_2}{1-\bar{v}_2+2\bar{u}}, \quad \bar{v}_2^0 = \frac{1-\bar{v}_2-2\bar{u}}{1-\bar{v}_2+2\bar{u}}; \tag{5.2}$$

$$\theta(\bar{u}, \bar{v}_2) = (2-\bar{u}, \bar{v}_2); \quad T^3 = 1, \quad \theta^2 = 1. \tag{5.3}$$

Using the monotonic function (4.6) contained in the system, we can justify the separatrix approximation on moving towards the singularity. We shall express the motion near the singularity by means of the sequence of singular points and separatrices near which this trajectory passes.

Properties of the separatrix diagram on the contraction side. Comparison with the BLKh model. From the diagram (Fig. 4) we see that when moving on the contraction side, sufficiently close to the cosmological singularity, we can confine ourselves to consideration of only singular points of the type (ψ, α), since they and their separatrices form (on this side of the time) a closed system (together with the segments $\overline{BD}_\alpha$). Thus, we have trajectories of the type

$$(\psi_0, \alpha_0) \to (\psi_1, \alpha_1) \to (\psi_2, \alpha_2) \to \ldots, \tag{5.4}$$

where $\alpha = 1, 2, 3$.

According to the diagram of Fig. 4, $(\psi_{s+1}, \alpha_{s+1})$ is a single-valued function of (ψ_s, α_s), if ψ_s is on the arc $\widehat{ACE}$; if ψ_s lies on the arc $\widehat{EFA}$, then the transition to $(\psi_{s+1}, \alpha_{s+1})$ is ambiguous:

$$\begin{aligned} &(\widehat{DE}, \alpha_s+2) \xleftarrow{\text{II}} (\widehat{FA}, \alpha_s) \xrightarrow{\text{I}} (\widehat{BC}, \alpha_s+1), \\ &(\widehat{AB}, \alpha_s+1) \xleftarrow{\text{II}} (EF, \alpha_s) \xrightarrow{\text{I}} (CD, \alpha_s+2). \end{aligned} \tag{5.5}$$

It follows from the diagram that in the next step, starting from the results of path II, we arrive at the same result we obtained in one step on path I.

We now compare the results of this separatrix diagram with the combinatorial "model of Kasner exponents" of Belinskiĭ, Lifshitz, and Khalatnikov[2], which describes the regular regime in their sense and which has resulted from the idea of piecewise approximation of the components of the metric of the Bianchi IX model by the power-law functions

$$t^{p_1}, \ t^{p_2}, \ t^{p_3},$$

which result from the Bianchi model I.

We introduce the parameter κ, where $1 \le \kappa < \infty$:

$$p_1(\varkappa) = \frac{-\varkappa}{1+\varkappa+\varkappa^2}, \quad p_2(\varkappa) = \frac{1+\varkappa}{1+\varkappa+\varkappa^2}, \quad p_3(\varkappa) = \frac{\varkappa(1+\varkappa)}{1+\varkappa+\varkappa^2}. \tag{5.6}$$

We consider the "exponent alternation" transformation

$$\begin{aligned} &\varkappa \to \varkappa - 1, \quad \text{if} \quad 2 \leqslant \varkappa < \infty; \\ &\varkappa \to \frac{1}{\varkappa - 1}, \quad \text{if} \quad 1 \leqslant \varkappa \leqslant 2. \end{aligned} \tag{5.7}$$

The state is described by the pair

$$(\varkappa, \sigma); \ \sigma = \begin{pmatrix} 1\,2\,3 \\ i\,j\,k \end{pmatrix}, \ \sigma_{12} = \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix}, \ \sigma_{23} = \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix}, \tag{5.8}$$

where σ is a permutation.

The transformation ("oscillation" or "alternation of Kasner epochs") takes the following form in the BLKh model:

$$(\varkappa, \sigma) \xrightarrow{K} (\varkappa - 1, \sigma\sigma_{12}) \quad (2 \leqslant \varkappa < \infty);$$

$$(\varkappa, \sigma) \xrightarrow{\mathrm{K}} \left(\frac{1}{\varkappa - 1}, \sigma\sigma_{12}\sigma_{23}\right) \quad (1 \leqslant \varkappa \leqslant 2). \tag{5.9}$$

The sequence of states

$$(\varkappa_0, \sigma_0) \xrightarrow{\mathrm{K}} (\varkappa_1, \sigma_1) \xrightarrow{\mathrm{K}} (\varkappa_2, \sigma_2) \to \ldots \tag{5.10}$$

codes the asymptotic of the typical trajectory in accordance with BLKh[2].

When comparing this model with (5.5), we choose only path I in (5.5); we note that after one step path II gives the same result). By choosing path I, we obtain an unambiguous model (the next state is determined by the preceding one):

$$(\psi_s, \alpha_s) \xrightarrow{\overline{\mathrm{K}}} (\psi_{s+1}, \alpha_{s+1}),$$

$$\overline{\mathrm{K}}(\psi_s, \alpha_s) = (\psi_{s+1}, \alpha_{s+1}) = \begin{cases} (T\psi_s, \alpha_s + 1) & (\psi_s \text{ in arc } \widehat{FB}) \\ (T^2\psi_s, \alpha_s + 2) & (\psi_s \text{ in arc } \widehat{DF}), \\ (\theta\psi_s, \alpha_s) & (\psi_s \text{ in arc } \widehat{BCD}) \end{cases} \tag{5.11}$$

where $T^3 = 1$ and $\theta(\psi) = \pi - \psi$.

We define the function

$$\varkappa(\psi) = \frac{1 + \bar{u} + 2\bar{v}}{1 + \bar{v} - 2\bar{u}} \quad (\text{in arc } \widehat{FB}),$$
$$\varkappa(\psi) = \frac{1 + \bar{u} - 2\bar{v}}{1 - \bar{v} - 2\bar{u}} \quad (\text{in arc } \widehat{DF}), \tag{5.12}$$
$$\varkappa(\psi) = \varkappa(\pi - \psi) \quad (\psi \text{ in arc } \widehat{BCD}),$$

where $\bar{v} = \bar{v}_2 = \sin\psi$ and $\bar{u} - 1 = \cos\psi$. We can consider next only the arcs $\widehat{DFB}$, carrying out the transition in the model (5.11) in two steps at a time (along path I):

$$\overline{\overline{\mathrm{K}}} = \overline{\mathrm{K}}^2(\psi_s, \alpha_s) = \begin{cases} (\theta T\psi_s, \alpha_s + 1) \, (\text{in arc } \widehat{FB}) \\ (\theta T^2\psi_s, \alpha_s + 2) \, (\text{in arc } \widehat{DF}) \end{cases} \tag{5.13}$$

Identifying the points B, F, and D as a single point, we obtain a smooth transformation K. Formulas (5.12) show the isomorphism of the model of (5.13) with the BLKh model (5.9).

The presence of a path II such that

$$\varkappa \to \frac{1}{\varkappa - 1} + 1, \quad 1 \leqslant \varkappa \leqslant 2 \tag{5.14}$$

is obtained in one step complicates the model somewhat in those places where, for example, the "long era" in the sense of BLKh terminates[2], i.e., where $1 \leq \kappa \leq 2$.

We note some properties of the isomorphic models (5.9) and (5.13).

a) All the trajectories are repelled from one another.

b) There is a denumerable everywhere-dense set of periodic points.

c) $$\left|\frac{dT(\psi)}{d\psi}\right| > 1 \quad \text{on arc } \widehat{FB},$$

$$\left|\frac{dT^2(\psi)}{d\psi}\right| > 1 \quad \text{on arc } \widehat{DF}.$$

Under these conditions, as is well known, the transformation $\overline{\overline{\mathrm{K}}}$ will conserve a smooth measure and have the "coarseness" property in the sense of Andronov (the Anosov criterion), namely, each small perturbation of the transformation reduces to the transformation itself by a continuous change of variables (it is a requirement that the derivatives in the perturbation be small; it is assumed also that the degree of proximity of the perturbed transformation to the initial $\overline{\overline{\mathrm{K}}}$ improves sufficiently rapidly on approach to the points D, F, and B

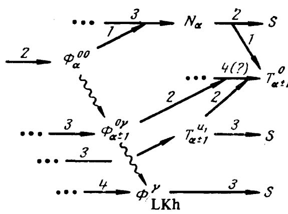

FIG. 5. Paths on the separatrix diagram on the expansion side prior to the departure from the singularity.

on the semicircle). This property justifies to a certain degree the approximate application of combinatorial models (5.9) and (5.13), including their statistical properties, to the description of the behavior of true trajectories.

<u>Conclusions of separatrix diagram on the expansion side</u>. On going to the problem of typical initial data near a cosmological singularity on the side of expansion of the Universe, it is necessary to reflect the diagram of Fig. 4 about the principal diagonal. We call attention to the fact that during the concluding stage, prior to the departure from the singularity, the only paths that are possible in the separatrix diagrams are those shown in Fig. 5, where the arrow denotes the separatrix going from one set of singular points to another set, and the figure over the arrow denotes its dimensionality. The ellipsis stands for all possible singular points (ψ, α), AE_α, or $\overline{\mathrm{BD}}_\alpha$, and the wavy line shows a transition along the continuity as a result of the inclusion

Thus, in the separatrix approximation, typical initial data are obtained in the vicinity of the singular points of the types Φ_{LKh}, $\mathrm{T}_\alpha^{u_1}$, and N_α, with singular points of the type $\Phi^\gamma_{\mathrm{LKh}}$ running through a two-dimensional set (spherical triangle Δ) and the singularities $\mathrm{T}_\alpha^{u_1}$ through a one-dimensional set, while the three points N_α are isolated.

It is appropriate to stop and discuss here the question of how "typical" initial data are generally defined and why they depend on the sign of the time. On the contraction side, this question does not arise by virtue of the BLKh results, namely, on approaching the cosmological singularity along practically any trajectory, the regular regime described by their combinatorial model will sooner or later become established. On the expansion side, as already indicated, we regard as the typical initial state a sojourn near the last singularity along a given trajectory of the separatrix. This separatrix leads from the boundary Γ over to the physical region S. Of course, it is assumed here that the observer himself is far from the cosmological singularity, during the later stage of development. This definition of typical initial conditions is based on the following: The physical region of possible values of the components of the metric and their first derivatives, after taking the scale invariance into account, reduces to a five-dimensional manifold S with boundary Γ on which the spatial metric degenerates, with a dynamic Einstein system on S. One specifies initially the small parameter ρ, namely the distance from the random initial condition in the region S to the boundary Γ. The time is then started in the expansion direction. The trajectory moves for some time still in the vicinity of Γ, until it passes near a singular point of the type $\Phi^\gamma_{\mathrm{LKh}}$, $\mathrm{T}_\alpha^{u_1}$, or N_α, and then starts to go off into the region S away from the boundary Γ along the separatrix of one of these singular points.

By the same token, even close to the boundary Γ of the physical region S, random initial data accumulate in the vicinity of regimes that have power-law behavior in t ((2.1), (2.2), (2.3)), and correspond to separatrices of these singular points emerging from Γ into S.[3)]

The three exclusive points T_α^0, which should be assigned to S, form a special case. Several separatrices go into these points, which are limiting for Γ (see Fig. 4). The special character of this point lies in the fact that a trajectory passing in their vicinity (on the expansion side) remains in the vicinity of these points up to the instant of maximum expansion.

Let us compare these results with the previously developed[7] friction formalism for the expansion process up to the instant of maximum expansion. According to (2.6)—(2.9), we have

$$\lambda^3 = e^{3\varphi} = q_1q_2q_3 = (-g)^{1/2},\ \gamma_1 = (\gamma_2\gamma_3)^{-1},$$

$$U = \bar{H}^2 = K + 3V + 3A\exp(-\alpha s),\ \alpha = 3k+1,\ U^{\rm LKh} = K + 3V,$$

$$K = 4(p_2^2 - p_2p_3 + p_3^2),\quad V = -\sum_{\alpha=1}^{3}\gamma_\alpha^4 + 2\sum_{\alpha\neq\beta}\gamma_\alpha^2\gamma_\beta^2,\quad \frac{dU^{\rm LKh}}{ds} = -4K \leqslant 0,$$

where K is the kinetic energy and 3V is the potential energy; the phase region $U^{\rm vac} < 0$ is a trap in the terminology of[7].

After recalculating the asymptotic expressions (2.1)—(2.3) in terms of the coordinates (2.6)—(2.9), we obtain the following:

1. For quasiisotropic regimes $\Phi^\gamma_{\rm LKh}$

$$\gamma_\alpha = q_\alpha\lambda^{-1} \sim C_\alpha(C_1C_2C_3)^{-1/3} = B_\alpha,\quad K\sim 0,$$

where the constants C_α are arbitrary and positive; the kinetic energy is small here and the potential energy arbitrary.

For the regime N_1 we have

$$K\sim V,\quad p_2\sim p_3,\quad \lambda\sim DK^{-1/(3k+1)},\quad \gamma_2\sim C\gamma_3,$$

where γ_1 is arbitrary but sufficiently small and the constants C and D are arbitrary and positive. The kinetic energy is here equal to one-third of the potential energy, and both are large.

2. For the singular regime T_1^0, both the kinetic energy K and the potential energy 3V are small in absolute value, with

$$3V<0,\quad \gamma_1 \ll \gamma_2 \sim \gamma_3.$$

We see that in the power-law regimes the kinetic energy is not large compared with the potential energy, and in the states $\Phi^\gamma_{\rm LKh}$ it is very small. To the contrary, for the contraction process, by virtue of the BLKh results, in typical states the kinetic energy becomes periodically infinitely larger than the potential one.

The statistical properties of the Bianchi IX model can be determined in this case by resorting to information on the distribution of the probabilities among the typical initial data obtained in the present paper. We can then find the probable instant of falling into the trap $U^{\rm vac} < 0$, and the distribution in this trap during the later development stage preceding the instant of maximum expansion.

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Note added in proof (March 1973). In a simpler homogeneous model (axially symmetrical, with group of motions of type II after Bianchi), an asymptotic form of the type (2.3) was obtained by Ellis and McCollum (Comm. Math. Physics 12, 108 (1969) as an exact solution. Collins (ibid. 23, 137 (1971) was the first to use the two dimensional qualitative Poincare-Bendixson theory for the study of axially-symmetrical models of Bianchi types II - VII. Not even the axially symetrical model of type IX reduces to the two-dimensional problem (see Sec. 3). Only on the boundary v = 0, which we have "glued in," is the problem two-dimensional (see Fig. 1).

1)These asymptotic forms, which generalize the Friedmann solution, were first obtained by Lifshitz and Khalatnikov [12] and called "quasiisotropic."

2)The eigenvalues (4.11) correspond to the picture obtained by Belinskiĭ and Khalatnikov [1] in a different language in a study of small perturbations of the Taub model in a synchronous reference frame in empty space.

3)The time interval between the stay on one of the seperatrices of the BLKh model (the singular points (ψ, α)) and the establishment of a power-law asymptotic form of one of the three types (all this occurs near the boundary Γ) is divided into three parts: 1) the passage from the separatrix of the BLKh model to a separatrix going from the singular points (ψ, α) into one of the singular points of the types $\Phi^\gamma_{\rm LKh}$, T_α^{u}, or N_α; 2) the time of motion along this seperatrix to a fixed vicinity of the final points; 3) the passage to a separatrix going from Γ into S. It can be shown that the time 2) tends to zero with ρ, and that time 3) has a finite limit as $\rho\to 0$. As to the time 1), which can naturally be called the time of "turning on the matter," it can have many limits (from zero to ∞) as $\rho \to 0$, depending on the method of approaching the limit. This pertains also to the volume $(-g)^{1/2} = q_1q_2q_3$, with one important exception, namely, the transition from a separatrix of the BLKh model to a separatrix of singular points of the type $\Phi^\gamma_{\rm LKh}$ yields a small finite volume $(-g)^{1/2}$ as $\rho\to 0$.

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