Scattering of an electron in an electromagnetic wave and in a homogeneous magnetic field

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An exact formula for the probability of quantum transitions, incuded by an external electromagnetic wave of arbitrary intensity, of an electron in a homogeneous magnetic field is derived and investigated. The case of a monochromatic electromagnetic wave and the case of a "pulse" of electromagnetic radiation of finite duration are both considered. The criterion for the validity of an expansion of the transition matrix element in powers of the perturbation causing the transition is obtained.

A characteristic feature of the behavior of an electron in an external electromagnetic wave and in a homogeneous magnetic field is the appearance of cyclotron resonance at $\omega = \omega_H$ (ω and ω_H denote, respectively, the frequency of the electromagnetic wave and the electron cyclotron frequency); the electron is therefore strongly affected by the external field and may gain considerable energy.[1-3] The interaction of the electron with an electromagnetic wave may turn out to be so strong under the conditions for cyclotron resonance that it becomes necessary to go outside the framework of perturbation theory to investigate physical phenomena in this region.[4,5] In this connection an exact calculation of the interaction with the external field, based on the exact solution of the Dirac equation in the presence of an electromagnetic wave and a homogeneous magnetic field,[6,7] is of special interest. The existence of this solution has permitted us to clarify the mechanism for the interaction of an electron with an electromagnetic wave and to predict a number of new physical effects, which appear under the conditions of cyclotron resonance.[4,8]

In the present article we consider the quantum transition of an electron in a homogeneous magnetic field from one Landau level to another, the transition being caused by a plane electromagnetic wave propagating along the direction of the magnetic field. If the electron energies are small in the initial and final states, then the probability for this process is determined by the parameter

$$\zeta = (\omega_H / m)^{1/2} |e\tilde{A}(\omega_H)|$$

(e and m denote the electron charge and mass, and $\tilde{A}(\omega)$ is the Fourier transform of the vector potential of the electromagnetic field). For $\zeta \ll 1$ it is the transition from a given level to the nearest neighboring Landau level which occurs with the maximum probability; however, in the opposite limiting case ($\zeta \gg 1$) the probability for the transition $n \to n'$ (n and n′ label the Landau levels) has its maximum value for $|n - n'| \sim \zeta^2/2 \gg 1$.

The characteristic feature of the scattering process under consideration is the fact that its probability is determined only by the component of the electromagnetic wave corresponding to the cyclotron frequency ω_H, and according to the exact formula which we have derived, in the monochromatic limit ($\tilde{A}(\omega) \to \delta(\omega - \omega_0)$, where ω_0 is the frequency of the wave) the probability of the transition $n \to n'$ vanishes for any finite value of $n - n'$. This result cannot be derived within the framework of perturbation theory, since the criterion for the applicability of the latter to the present case is the inequality $\zeta \ll 1$, whereas in the monochromatic limit $\zeta \to \infty$.

Let us consider the quantum transition of an electron in a static external field $\mathbf{A}_1(\mathbf{r})$ from one energy level to another, the transition being due to an external electromagnetic field $\mathbf{A}_2(\mathbf{r}, t)$. As usual we assume that at the initial moment of time ($t \to -\infty$) and at the moment of observation of the scattering process ($t \to +\infty$) the transition-inducing field $\mathbf{A}_2(\mathbf{r}, t)$ is absent and the system is described by a zero-order Hamiltonian which includes the interaction with the field $\mathbf{A}_1(\mathbf{r})$. As the static external field let us consider a homogeneous magnetic field of intensity H directed along the z axis:

$$\mathbf{A}_1(\mathbf{r}) = (-yH, 0, 0), \tag{1}$$

and as the field causing the transition we assume a plane electromagnetic wave

$$\mathbf{A}_2(\tau) = (0, A(\tau), 0) \quad (\tau = t - z). \tag{2}$$

Here $A(\tau)$ is an arbitrary function satisfying the condition $A(\tau) = 0$ for $\tau \to \pm\infty$.

The amplitude for the electron's transition $p_x, p_z, n, \sigma \to p_x', p_z', n', \sigma'$ (in what follows we shall denote the set of variables p_x, p_z, n, σ briefly by P) under the influence of the electromagnetic field (2) is given by the formula

$$M(p_x, p_z, n, \sigma \to p_x', p_z', n', \sigma') \equiv M(P \to P') = \int d\mathbf{r} [\chi_{P'}^{(+)}(\mathbf{r}, t)]^+ \Psi_P^{(+)}(\mathbf{r}, t) \quad (t \to +\infty), \tag{3}$$

where $\Psi_P^{(+)}(\mathbf{r}, t)$ ($\chi_{P'}^{(+)}(\mathbf{r}, t)$) denotes the solution of the Dirac equation in the presence of the field $\mathbf{A}_1 + \mathbf{A}_2$, which goes over into the wave function for a stationary state of the electron in the magnetic field (1) at the initial moment of time (at the moment of observation). The wave functions $\Psi_P^{(\pm)}(\mathbf{r}, t)$ and $\chi_P^{(\pm)}(\mathbf{r}, t)$ have the following form:

$$\Psi_P^{(\pm)}(\mathbf{r}, t) = \psi_P^{(\pm)}(\mathbf{r}, t | \tau_0)|_{\tau_0 \to -\infty},$$
$$\chi_P^{(\pm)}(\mathbf{r}, t) = \psi_P^{(\pm)}(\mathbf{r}, t | \tau_0)|_{\tau_0 \to +\infty}, \tag{4}$$

Here the wave functions given in[4] (see formula (3)) are denoted by $\psi_P^{(\pm)}(\mathbf{r}, t | \tau_0)$, in which the functions $N_p(\tau)$ and $g_p(\tau)$ are defined by the expressions

$$N_p(\tau) = \frac{\beta}{\bar{n}p} \int_{\tau_0}^{\tau} d\tau' \, eA(\tau') \sin \frac{\beta}{\bar{n}p}(\tau - \tau'),$$
$$g_p(\tau) = \frac{\beta}{\bar{n}p} \int_{\tau_0}^{\tau} d\tau' \, eA(\tau') \cos \frac{\beta}{\bar{n}p}(\tau - \tau'). \tag{5}$$

and $\bar{n}p = p_0 - p_z$.

By using the wave functions (4) we obtain (j, k = ±)

$$\int d\mathbf{r} [\chi_{P'}^{(k)}(\mathbf{r}, t)]^+ \Psi_P^{(j)}(\mathbf{r}, t) = 2\delta_{\sigma\sigma'} C_{p'n'\sigma} C_{pn\sigma}.$$

$$\times \exp\{i(p_0'-p_0)t\}\int d\mathbf{r}\exp\{-i(p_x'-p_x)x-i(p_z'-p_z)z\}$$
$$\cdot K^*_{p'n'}(\tau,y|\tau_0')\left[2i\bar{n}p\frac{d}{d\tau}+p_0^2-p_z^2+(\bar{n}p)(\bar{n}p')\right]$$
$$\times K_{pn}(\tau,y|\tau_0)\quad(\tau_0\to-\infty,\ \tau_0'\to+\infty);$$
$$K_{pn}(\tau,y|\tau_0)=\exp\left\{iyN_p(\tau)-\frac{i}{2\bar{n}p}\int_{\tau_0}^{\tau}dt\,R_p(t)-\frac{\xi_p^2}{2}\right\}H_n(\xi_p),$$
$$p_0=\varepsilon_{pn\sigma}^{(j)},\quad p_0'=\varepsilon_{p'n'\sigma'}^{(k)},\quad \varepsilon_{pn\sigma}^{(\pm)}=\pm[p_z^2+m^2+\beta(2n+1-\sigma)]^{1/2}\quad(6)$$

(the functions $R_p(t)$ and ξ_p are defined in[4], and $H_n(\xi_p)$ is the Hermite polynomial).

In order to evaluate the integral appearing on the right hand side of expression (6), it is convenient to use the relation

$$\int d\mathbf{r}\exp\{-i(p_x'-p_x)x\}\frac{2i(\bar{n}p)(\bar{n}p')}{\bar{n}p-\bar{n}p'}$$
$$\times\frac{d}{dz}[\exp\{-i(p_z'-p_z)z\}K^*_{p'n'}(\tau,y|\tau_0')K_{pn}(\tau,y|\tau_0)]$$
$$=\int d\mathbf{r}\exp\{-i(p_x'-p_x)x-i(p_z'-p_z)z\}K^*_{p'n'}(\tau,y|\tau_0')$$
$$\cdot\left[2i\bar{n}p\frac{d}{d\tau}+p_0^2-p_z^2+(\bar{n}p)(\bar{n}p')\right]K_{pn}(\tau,y|\tau_0),\quad(7)$$

which one can easily prove by taking the Hermitian nature of the operator $p_y=-i\partial/\partial y$ and the equation

$$\left[2i\bar{n}p\frac{d}{d\tau}+\beta(2n+1)-(p_x+\beta y)^2-\left(-i\frac{\partial}{\partial y}-eA(\tau)\right)^2\right]K_{pn}(\tau,y|\tau_0)=0.$$

into consideration.

On the basis of relations (6) and (7) we find (L is a quantity of the order of the linear size of the system)

$$\int d\mathbf{r}[\chi_{P'}^{(k)}(\mathbf{r},t)]^+\Psi_P^{(j)}(\mathbf{r},t)=8\pi i\left(\frac{\pi}{\beta}\right)^{1/2}C_{p'n'\sigma}C_{pn\sigma}$$
$$\times\delta_{\sigma'\sigma}\delta(p_x'-p_x)\exp\{i(p_0'-p_0)t\}\frac{(\bar{n}p)(\bar{n}p')}{\bar{n}p-\bar{n}p'}$$
$$\times[\exp\{-i(p_z'-p_z)z\}F_{p'n',\,pn}(\tau)]_{z=-L}^{z=+L},$$
$$F_{p'n',\,pn}(\tau)\equiv\left(\frac{\beta}{\pi}\right)^{1/2}\int dy\,K^*_{p'n'}(\tau,y|\tau_0')K_{pn}(\tau,y|\tau_0).\quad(8)$$

It is not difficult to verify that, for $\bar{n}p-\bar{n}p'\neq 0$, the substitution that enters in (8) is effectively equal to zero for $L\to+\infty$: it reduces to a sum of terms of the type $e^{i\alpha L}$ ($\alpha\neq 0$), and every integral, whose integrand contains such a function multiplied by smooth functions, vanishes as $L\to+\infty$. Since $\bar{n}p>0$ for the electron states, and $\bar{n}p<0$ for the positron states, hence follows the equation

$$\int d\mathbf{r}[\chi_{P'}^{(-)}(\mathbf{r},t)]^+\Psi_P^{(+)}(\mathbf{r},t)=0,$$

which means that the field $\mathbf{A}_1+\mathbf{A}_2$ does not create electron-positron pairs—a conclusion which was reached earlier in[4] on the basis of other considerations.

By using the foregoing relations we obtain the following formula for the matrix element (3):

$$M(P\to P')=(2\pi)^2\delta_{\sigma\sigma'}\delta(p_x'-p_x)\delta\left(p_z'-p_z-\frac{\beta}{\bar{n}p}(n'-n)\right)M_{nn'}(\bar{n}p),$$
$$M_{nn'}(\bar{n}p)=\frac{1}{2}(p_0+p_0')(p_0p_0')^{-1/2}\left(\frac{2^{n'}n!}{2^n n'!}\right)^{1/2}$$
$$\times L_n^{n'-n}\left(\frac{\beta}{2(\bar{n}p)^2}\left|e\tilde{A}\left(\frac{\beta}{\bar{n}p}\right)\right|^2\right)\left[-\frac{\beta^{1/2}}{2\bar{n}p}e\tilde{A}\left(\frac{\beta}{\bar{n}p}\right)\right]^{n'-n}$$
$$\times\exp\left\{i\Phi(\bar{n}p)-\frac{\beta}{4(\bar{n}p)^2}\left|e\tilde{A}\left(\frac{\beta}{\bar{n}p}\right)\right|^2\right\},$$
$$p_0=\varepsilon_{pn\sigma}^{(+)},\quad p_0'=\varepsilon_{p'n'\sigma'}^{(+)},\quad(9)$$

where $L_n^{n'-n}(x)$ is the generalized Laguerre polynomial,

$$e\tilde{A}(\omega)=\int_{-\infty}^{+\infty}d\tau\,e^{i\omega\tau}eA(\tau),$$
$$\Phi(\bar{n}p)=-\frac{1}{4\pi}\frac{1}{\bar{n}p}\mathrm{P}\int d\omega|e\tilde{A}(\omega)|^2\frac{\omega^2}{\omega^2-(\beta/\bar{n}p)^2}$$

(the symbol P denotes the principal value of the integral). The equation ($p_0=\epsilon_{pn\sigma}^{(\pm)}$, $p_0'=\epsilon_{p'n'\sigma}^{(\pm)}$)

$$p_z'-p_z-\frac{2\beta(n'-n)}{\bar{n}p+\bar{n}p'}=\frac{\bar{n}p-\bar{n}p'}{\bar{n}p+\bar{n}p'}(p_0+p_0').$$

was taken into consideration in deriving formula (9).

The probability of the quantum transition $n,\sigma\to n',\sigma'$, summed over the final state momenta of the electron, is determined by the formula

$$W_{n\sigma\to n'\sigma'}(\bar{n}p)\equiv\frac{(2\pi)^2}{L_xL_z}\int dp_x'\,dp_z'\left|\frac{M(P\to P')}{(2\pi)^2}\right|^2=\delta_{\sigma\sigma'}|M_{nn'}(\bar{n}p)|^2.\quad(10)$$

According to expression (9), for $n'\neq n$ an electron transition occurs only if there is a component with frequency $\omega=\beta/\bar{n}p$ in the Fourier expansion of the potential $A(\tau)$, and it does not depend on the components with other frequencies (the phase factor $\exp\{i\Phi(\bar{n}p)\}$ is not essential in connection with the investigation of transitions between stationary states). In other words, only the component of the electromagnetic field which is in resonance with the electron cyclotron frequency gives any contribution to the probability for the scattering process. The law of energy conservation,

$$\varepsilon_{p'n'\sigma}^{(+)}-\varepsilon_{pn\sigma}^{(+)}=\omega(n'-n)\quad(\omega=\beta/\bar{n}p),$$

is satisfied in the scattering process, from which we conclude that the transition from the n-th Landau level to the n-th level is the result of an $(n'-n)$-quantum process with the quantum of energy equal to $\beta/\bar{n}p$.

We note that if we set

$$A(\tau)=e^{-\varepsilon|\tau|}a\cos\omega_0\tau\quad(a=\text{const},\ \omega_0=\text{const},\ \varepsilon>0),$$

then in the monochromatic limit $\epsilon\to+0$ the matrix element (9) vanishes for any finite value of $n'-n$. This fact deserves a more detailed discussion. Let us present the first term of the expansion of formula (9) in powers of $e\tilde{A}(\omega)$:

$$M^{(1)}(P\to P')=\frac{(2\pi)^2\beta^{1/2}(2^{n'}n'!)^{1/2}}{(p_0p_0'2^n n!)^{1/2}}e\tilde{A}(p_0'-p_0)$$
$$\times\delta(p_x'-p_x)\delta(\bar{n}p'-\bar{n}p)\left(-\frac{1}{2}\delta_{n',n+1}+n\delta_{n',n-1}\right)\delta_{\sigma'\sigma}.\quad(11)$$

The last expression coincides with the corresponding term of the expansion of the transition amplitude (3) according to perturbation theory. In the monochromatic limit $e\tilde{A}(p_0'-p_0)\sim\delta(p_0'-p_0\pm\omega_0)$. If we now calculate the transition probability $W^{(1)}=|M^{(1)}|^2$ by the standard method, then it turns out to be proportional to the time (the duration of the scattering process). In actual fact, however, according to formula (9) expression (11) in the monochromatic limit would not be correct in any approximation (with the exception of the trivial case $a=0$). According to (9), the criterion whereby we can confine our attention to the first few terms of the expansion is given by the inequality

$$\zeta(\bar{n}p)\equiv\left|\frac{\beta^{1/2}}{\bar{n}p}e\tilde{A}\left(\frac{\beta}{\bar{n}p}\right)\right|\ll 1,\quad(12)$$

which is clearly not satisfied in the case of a monochromatic electromagnetic wave. Formally the crux of the matter is that in the monochromatic limit the perturbation-theory series has a zero radius of convergence. It should be emphasized that the above-indicated property of the expansion of the matrix element (3) is not in any way unexpected: it is a consequence of a radical realignment of the electron's energy spectrum in the field of the monochromatic electromagnetic wave and in a homogeneous magnetic field.[4,7]

For simplicity we confine the investigation to transitions of the electron from the Landau level n = 0. According to (9), the probability that the electron remains in its initial state is close to unity for $\zeta \equiv \zeta(\overline{n}p) \ll 1$. The probability for a transition to the level n′ is proportional to $\zeta^{2n'}$ and, therefore, falls off rapidly with increasing values of n′. The situation changes for $\zeta \gg 1$: the transition to the level $n' \approx \zeta^2/2$ occurs with maximum probability (in the case of a weak dependence of $\overline{n}p$ on n).

As an example let us consider the quantum transitions of an electron under the influence of a "pulse" of electromagnetic radiation of the following form:

$$A(\tau) = a \exp\{-\tau^2/\tau_0^2\} \cos \omega_0\tau \qquad (a = \text{const},\ \omega_0 = \text{const},\ \tau_0 = \text{const}). \tag{13}$$

Then

$$\zeta = \frac{\sqrt{\pi}}{2}\frac{|ea|}{\overline{n}p}\frac{(\omega_H m)^{1/2}}{\omega_0}\omega_0\tau_0\left[\exp\left\{-\frac{\tau_0^2}{4}(\omega+\omega_0)^2\right\}+\exp\left\{-\frac{\tau_0^2}{4}(\omega-\omega_0)^2\right\}\right]$$

($\omega_H = |e|H/m$, $\omega = \beta/\overline{n}p$). It is natural to assume that the condition $\omega_0\tau_0 \gg 1$ is satisfied (the duration of the "pulse" is large in comparison with the period of the monochromatic wave a cos $\omega_0\tau$). In this case the parameter ζ is exponentially small for $\omega - \omega_0 \sim \omega_0$. The maximum value of this parameter is reached for $|\tau_0(\omega_0 - \omega)| \lesssim 1$, that is, in the cyclotron resonance region. In the frequency region of electromagnetic waves satisfying the condition $\omega_0 \gg \omega_H$, the cyclotron resonance and consequently the maximum value of the parameter ζ as well are reached only for relativistic electrons having momenta $p_z \approx m\omega_0/2\omega_H \gg m$. In conclusion, we note that according to formulas (9), (10), and (13) the p probability of the process depends on the duration of the pulse τ_0 in a very complicated manner.

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