Collisionless damping of forced nonlinear plasma oscillations

G. M. Fraĭman
Radiophysics Research Institute
(Submitted September 8, 1972)
Zh. Eksp. Teor. Fiz. **64**, 1589-1595 (May 1973)

The attenuation in a plasma of a nonlinear standing wave, the amplitudes of whose spatial harmonics are coupled owing to nonlinearity, is considered under conditions of independent damping of the individual harmonics. The attenuation of the nonlinear Langmuir waves excited by a homogeneous external electric field is discussed in detail for the case of weak spatial dispersion and weak nonlinearity.

1. In experiments on the interaction of high-frequency radiation with a plasma (see, for example,[1]) that are performable at present, field-intensity values many times exceeding the thresholds for the various three-wave instabilities are comparatively easily attained. Therefore, besides three-wave processes, four-wave self-action effects can also play a significant role. A characteristic feature of these processes lies in the fact that they lead to the excitation of longitudinal virtual waves with the same frequency and large wave numbers. In their turn, these waves efficiently transfer their energy to the resonance particles, so that on the whole the leakage of energy from the large scale can be effective.

In this paper we study forced one-dimensional nonlinear standing Langmuir oscillations of period $\lambda = 2\pi/k \ll r_d$ (r_d is the Debye electron radius) and their collisionless damping. It is assumed that these oscillations are sustained in the plasma by a high-frequency homogeneous electric field. For simplicity, we present only the results obtained in the approximation of weak spatial dispersion and weak nonlinearity[1)]. The important questions connected with the manner in which the periodic, stationary, one-dimensional, density distribution is produced from a distribution which is uniform just before the high-frequency field is switched on are not considered in the paper and, therefore, the rough estimate given in Sec. 4 for the time for such a redistribution should be regarded as a provisional value.

Let us explain the general idea of the paper. We assume that an inhomogeneous periodic density distribution (in the weak-nonlinearity approximation the depth ΔN of the density modulation is small compared to the unperturbed value N) has been produced in the plasma under the action of some nonlinear processes. The self-consistent periodic field (nonlinear wave) corresponding to this distribution can be decomposed into a superposition of partial harmonics with amplitudes E_n ($n = \pm 1, \pm 2, \ldots$ is the number of the harmonic). In the weak-nonlinearity approximation, $\Delta N/N \ll 1$, and the attenuation of the individual spatial harmonics can be considered in the same way as in a homogeneous plasma. In particular, if the number of particles in resonance with the particular plasma wave $E_n e^{i(\omega t - nkx)}$ (ω is the frequency) is sufficiently small, so that the resonance regions in the particle-velocity space do not overlap, then the partial waves attenuate independently of each other (for the formal conditions of such an approximation, see Secs. 2 and 4). If the time for the establishment of the steady-state spatial spectrum of the harmonics is small compared to the damping time, then the amplitudes E_n can be computed with damping neglected. Then, since all the particles participate in the establishment of the steady-state spectrum, while only the resonance particles participate in the damping, the conditions of such an approximation turn out to be comparatively weak (see Sec. 4) even for fairly large total damping constants of the nonlinear wave.

2. Let us first of all derive some general relations for the damping constant of the nonlinear plasma wave. The smallness of the main parameter $\Delta N/N$ in the first approximation, in which the problem under consideration can be solved completely at present reduces, as is well known, to the requirement that the oscillation velocity of the electrons in the wave field be small compared to their thermal velocity:

$$V_\sim = \frac{eE}{m\omega} \ll V_{Te}. \tag{1}$$

Here V_{Te} is the thermal velocity of the electrons.

We shall for simplicity solve the problem in the approximation of independent damping (of the individual harmonics). Since the width of the resonance region with the wave $E_n e^{i(\omega t - nkx)}$ in the velocity space ($\Delta V_n \sim (eE_n/mnk)^{1/2}$) should in this case be sufficiently small:

$$\Delta V_n \ll \omega / nk - \omega / (n+1)k \sim \omega / n^2 k,$$

the required condition can be written in the form

$$eE_n / m\omega \ll \omega / n^3 k. \tag{2}$$

The constraint (2), which is, for $\lambda \gg r_d$, clearly more rigid than (1), can be removed if we use the well-known results of the quasi-linear theory[2] and consider the attenuation of the waves whose resonance regions overlap as the attenuation of a wave packet.

When the conditions (1)—(2) for the given spectrum of amplitudes E_n are fulfilled, the damping constant γ of the mean energy density of the nonlinear wave is determined by the expression

$$\gamma \sum_{n\neq 0} |E_n|^2 = \sum_{n\neq 0} \gamma_n |E_n|^2. \tag{3}$$

Since γ_n monotonically increases with increasing $|n|$, while $|E_n|^2$ decreases for a sufficiently strong nonlinear coupling (for a more precise statement on this, see Sec. 3) of the spatial harmonics E_n, the damping constant can evidently be substantially larger than γ_1.

Below we carry out the computations for the concrete example of a nonlinear standing wave ($E_{-n} = E_n$). Then, in a number of interesting cases (see Sec. 3)

$$E_n = A \exp(-|n| / n_1) \quad n = \pm 1, \pm 2, \ldots, \tag{4}$$

where n_1 is a characteristic number of coupled harmonics, and the energy density of the plasma wave is given by

$$\frac{1}{8\pi} \sum_{n\neq 0} |E_n|^2 = \frac{A}{4\pi} \left[\exp\left(\frac{2}{n_1}\right) - 1\right]. \tag{5}$$

The damping constant of the n-th partial wave

$$\gamma_n = Q_n(|E_n|)\gamma_n^{\text{л}}, \quad (6)$$

where γ_n^l is the linear Landau-damping constant[3] and Q_n is a factor that takes into account the effect of the quasi-linear relaxation[2]:

$$Q_n \approx 0.63\cdot 10^3 (nkr_d)^{-1/2}\left(\Gamma\frac{E_{cr}}{|E_n|}\right)^{3/2}. \quad (7)$$

Here $\Gamma = e^2N^{1/3}/\chi T_e$ is the plasma constant and $E_{cr} = [4\omega^2 m\chi(T_e + T_i)/e^2]^{1/2}$ is the characteristic plasma field for the strictional effects.

Let us give the upper limit, neglecting the quasilinear-relaxation effects ($Q_n \sim 1$). Let us set

$$n_0 = 1/kr_d. \quad (8)$$

From (3)—(8) it is easy to obtain

$$\gamma = \gamma_0\Phi(n_0; n_1), \quad (9)$$

$$\Phi(n_0;n_1) = 2\sum_{n=1}^{\infty}\left(\frac{n_0}{n_1}\right)^3 \exp\left[-\frac{2n}{n_1}-\frac{n_0^2}{2n^2}\right]\left(\exp\left[\frac{2}{n_1}\right]-1\right), \quad (10)$$

where $\gamma_0 = \sqrt{2}\pi\omega_\infty$ (ω_∞ is the characteristic plasma frequency in the unperturbed plasma).

Let us consider some limiting cases. The maximum value of the exponential function is attained at the point

$$n_m = (n_0^2 n_1/2)^{1/3}. \quad (11)$$

If

$$n_1 < n_0,\ n_m \gg 1,\ n_1^{2/3}n_0^{1/3} \gg 1,$$

then, replacing the summation by integration, we obtain

$$\Phi(n_0;n_1) \approx 4.4\frac{n_0^{4/3}}{n_1^{1/3}}\exp\left[-2.37\left(\frac{n_0}{n_1}\right)^{2/3}\right]\left(\exp\left[\frac{2}{n_1}\right]-1\right). \quad (12)$$

If, however, $n_m \gg 1$ and $n_1^{2/3}n_0^{1/3} \sim 1$, then it is sufficient to consider the dominant term

$$\Phi(n_0;n_1) \approx 4\frac{n_0}{n_1}\exp\left[-2.37\left(\frac{n_0}{n_1}\right)^{2/3}\right]\left(\exp\left[\frac{2}{n_1}\right]-1\right). \quad (13)$$

For a weak coupling between the harmonics ($n_m \lesssim 1$), it is sufficient to limit ourselves to the first terms, i.e., the attenuation occurs only on the large scale.

To roughly take the quasi-linear effects into account, it is sufficient to multiply the corresponding expression (11)—(13) by Q_n for $n = n_m$ (11).

Notice that we did not take into consideration above the influence of the effects of the periodic modulation of the plasma concentration on the expression for the damping constants. It can be shown that for a weak nonlinearity the corrections connected with this influence are small

3. Let us consider stationary, nonlinear, periodic, plasma waves excited in a plasma by a homogeneous electric field of induction D and frequency ω close to the unperturbed plasma frequency. Specifically, we can consider a plasma layer located between the plates of a parallel-plate capacitor, etc. The equation describing the forced nonlinear plasma oscillations in the weak nonlinearity and weak spatial dispersion approximation, but with allowance for linear damping, can, according to Gurevich and Pitaevskiĭ[4], be written in the form

$$3r_d^2\frac{d^2E}{dx^2}+\left(\varepsilon+\frac{|E|^2}{E_{cr}^2}\right)E = D + i\theta(x), \quad (14)$$

where $\epsilon = 1 - \omega_\infty^2/\omega^2$, ω_∞ being the plasma frequency of the unperturbed plasma[2)].

The energy dissipated by the plasma wave is given by the function

$$\theta(x) = -\frac{1}{\omega_\infty}\int_{-\infty}^{+\infty} G\left(\frac{x'}{r_d}\right)E(x-x')dx',$$

$$G(u) = \frac{\sqrt{2\pi}\,\omega_\infty}{r_d}\int_0^\infty \frac{\cos pu}{p^3}\exp\left(-\frac{1}{p^2}\right)dp. \quad (15)$$

As long as the damping is weak (for greater detail see Sec. 4), it can be determined according to (2) by substituting the amplitudes of the individual harmonics found without allowance for losses.

The solution of Eq. (14) for a real field ($\theta \equiv 0$) is determined by the two parameters ϵ and D. We shall discuss only two important cases: $\epsilon < 0$, $D = D_{cr} = (4/27)^{1/2}|\epsilon|^{3/2}E_{cr}$, when there are two equilibrium states and the phase plane (E, dE/dx) is divided into two connected regions by a separatrix and the periodic solutions are characterized by strong coupling and, secondly, $\epsilon > -(27/4)^{1/3}(D/E_{cr})^{2/3}$, when there is accordingly only one equilibrium state of the "center" type.

In the first case the solution E_c corresponding to the separatrix has the form

$$E_c(x) = \frac{|\varepsilon|^{1/2}E_{cr}}{\sqrt{3}}\left(\frac{4}{1+{}^2/_9|\varepsilon|x^2/r_d^2}-1\right). \quad (16)$$

From the exact solution near the separatrix (it can be written in terms of the elliptic function) we can determine the wave number

$$k = \frac{2\pi}{\lambda} = \sqrt{\frac{2}{3}}\frac{|\varepsilon|^{1/2}}{r_d}\left(\frac{\sqrt{3}W}{\varepsilon^2E_{cr}^2}\right)^{1/4}, \quad (17)$$

$$W = 3r_d^2\left(\frac{dE}{dx}\right)^2+\varepsilon E^2+\frac{1}{2}\frac{E^4}{E_{cr}^2}-2DE+\text{const},$$

where W is the integral of Eq. (14) with $\theta \equiv 0$ (W = 0 on the separatrix),

$$n_0 = \left(\frac{2|\varepsilon|}{3}\right)^{-1/2}\left(\frac{\sqrt{3}W}{\varepsilon^2E_{cr}^2}\right)^{-1/4},\quad n_1 = \frac{1}{kL_c} = \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}W}{\varepsilon^2E_{cr}^2}\right)^{-1/4}, \quad (18)$$

where $L_c = 3r_d/(2|\epsilon|)^{1/2}$ is the characteristic scale of the separatrix.

Notice that, knowing k (15), we can easily find the relations (18) by representing the solution as a sequence of the bell-shaped pulses (14). In this case

$$\bar{E} = E_0 \approx -\frac{|\varepsilon|^{1/2}}{\sqrt{3}}E_{cr},\quad A = \frac{4|\varepsilon|^{1/2}}{\sqrt{3}}\left(\frac{\sqrt{3}W}{\varepsilon^2E_{cr}^2}\right)^{1/4}E_{cr} \quad (19)$$

and, according to (11),

$$n_m = \left(\frac{\sqrt{3}}{4|\varepsilon|}\right)^{1/3}\left(\frac{\sqrt{3}W}{\varepsilon^2E_{cr}^2}\right)^{-1/4}. \quad (20)$$

For small W, according to (12),

$$\Phi(n_0;n_1) = 5.8\exp(-3.95|\varepsilon|^{-1/3})\left(\frac{\sqrt{3}W}{\varepsilon^2E_{cr}^2}\right)^{-1/4}\left\{\exp\left[3.5\left(\frac{\sqrt{3}W}{\varepsilon^2E_{cr}^2}\right)^{1/4}\right]-1\right\} \quad (21)$$

and in the limit as $W \to 0$

$$\gamma_m \approx 50|\varepsilon|^{-2/3}\exp[-3.95|\varepsilon|^{-1/3}]\cdot\omega_\infty \quad (22)$$

The relations (21) and (22) allow us to estimate the value of the dissipated energy for distributions whose phase trajectories are located near the separatrix. In using them we must remember that they have been derived under the assumptions of weak spatial dispersion ($n_0 \gg 1$) and sufficiently strong coupling ($n_1 > 1$). Therefore, they can be used when $|\epsilon|^{1/2} \ll 1$ (for greater detail, see Sec. 4). Since the quasi-linear effects were not taken into account in the derivation of (22), this relation, (22), is valid only for distributions whose trajectories are slightly removed from the separatrix, such that the independent damping condition (2), (32), is fulfilled. Also, since the dependence on the quantity W in (21) is weak, the relation (22) correctly estimates the value of the maximum damping constant.

In the other limiting case, in the vicinity of equilibrium states of the "center" type E_0, the equation $u(x) = E(x) - E_0$ for the deviation from the equilibrium state assumes the form

$$3r_d^2\frac{d^2u}{dx^2}+\left(\varepsilon+3\frac{E_0^2}{E_{cr}^2}\right)u+3\frac{E_0}{E_{cr}^2}u^2+u^3=0, \tag{23}$$

where E_0 is found from the equation

$$(\varepsilon+E_0^2/E_{cr}^2)E_0=D. \tag{24}$$

Let us estimate the quantities n_1 and n_0 in terms of the amplitude E_1 of the first harmonic. It is not difficult to show that for small E_1 and E_0 the general solution of (23) has the form

$$u(x)\approx E_1\mathrm{cn}(kx,q)\{[1+\beta\mathrm{cn}(kx,q)]+O(\beta^2,q^2)\}+u_0, \tag{25}$$

where $\mathrm{cn}(kx, q)$ is the elliptic cosine with modulus $q \ll 1$,

$$\beta=\frac{E_0E_1}{4E_{cr}^2(\varepsilon+3E_0^2/E_{cr}^2)},\quad \beta_2\ll 1;\quad k=k_0\left(1+\frac{q^2}{4}\right),$$

$$k_0^2=\frac{\varepsilon+3E_0^2/E_{cr}^2}{3r_d^2},\quad u_0\sim q^2E_0$$

Here k_0 is the wave number with allowance for the nonlinear correction.

The "nonlinear" period of the wave is easily found up to quantities of the order of E_1^2/E_{cr}^2 (see, for example,[5]), in which case

$$q^2\approx\frac{E_1^2/E_{cr}^2}{\varepsilon+3E_0^2/E_{cr}^2}. \tag{26}$$

Using further the Fourier-series expansion of the elliptic functions, we find

$$n_1\approx\ln^{-1}\frac{16}{q^2},\quad n_0\approx\frac{\sqrt{3}}{(\varepsilon+3E_0^2/E_{cr}^2)^{1/2}};\quad \sum_{n\neq0}|E_n|^2\approx\frac{1}{2}E_1^2. \tag{27}$$

In particular, for $\epsilon \ll 3E_0^2/E_{cr}^2 \ll 1$, we obtain from (24)–(27)

$$E_0^2\approx(D^2E_{cr}^4)^{1/3},\quad n_1\approx\ln^{-1}\frac{48E_0^2}{E_{cr}^2};\quad n_0\approx\frac{E_{cr}}{E_0}. \tag{28}$$

The cited relations show that in the case of weak coupling ($n_1 \ll 1$) the growth of the damping constant of the nonlinear wave may turn out to be substantial.

4. Let us discuss the conditions of applicability of the above-obtained relations.

First, these limitations are connected with the fact that we did not consider the establishment processes. We can distinguish two characteristic time scales: the time τ_k required for the establishment of the equilibrium density distribution (i.e., the time for the establishment of the nonlinearity as a whole) and the time τ_c required for the establishment of the steady-state spatial spectrum of the harmonics (i.e., the time for energy transfer across the spectrum). In other words, τ_c is the steady-state distribution establishment time for small perturbations, while τ_k is the corresponding time for large perturbations. The specific value of τ_k is determined by the nonlinearity-relaxation mechanism. Thus, in the considered model for the striction nonlinearity in an isothermal plasma, τ_k can, in order of magnitude, be estimated as the time an ion takes to travel the spatial period of the wave:

$$\tau_k\approx\frac{1}{\omega_\infty}\left(\frac{M}{m}\right)^{1/2}\frac{1}{kr_d}. \tag{29}$$

Whereas τ_k is inversely proportional to the ion-sound velocity, τ_c is evidently determined by the group velocity v_g of the plasma wave. According to the linear theory, for a weak coupling between the spatial harmonics,

$$\tau_c\simeq\frac{\lambda}{v_g}=\frac{2\pi}{3\omega_0}\frac{1}{(kr_d)^2}. \tag{30}$$

As the coupling increases, τ_c increases. In the considered case of the nonlinear standing plasma wave, the quantity

$$v_g\equiv\delta\omega/\delta k|_{|E|}$$

is easily found from the nonlinear dispersion relation. Let us multiply (14) by $E(x)$ and averaging the resulting expression over the period of the standing wave. Then, performing a variational calculation at a fixed value of the nonlinear-wave amplitude, we easily obtain that $v_g \approx 3\omega_\infty r_d/L_c$ and, consequently

$$\tau_c=\frac{2\pi}{3\omega_\infty}\frac{1}{kr_d}\frac{L_c}{r_d} \tag{31}$$

tends to infinity as we approach the separatrix.

Thus, the ion-density redistribution takes place within the time interval $\Delta t \gtrsim \tau_k$, and our analysis is henceforth valid under the condition that the energy-dissipation characteristic time $1/\gamma$ is long compared to the equilibrium-spectrum establishment time τ_c. Since, according to (30) and (31), $1/\tau_c$ is a comparatively large quantity, this condition is easily met.

Another limitation consists in the conditions under which the approximation for the given field can be used. The formal criterion is not difficult to derive, if we consider the damping in (14) as a perturbation. Seeking the solution in the form $E_1 + iE_2$, we obtain for the imaginary part of the amplitude of the harmonic with the number n_m in which the main damping occurs the estimate

$$E_{2nm}\approx\frac{\gamma_{nm}}{\omega}\frac{E_{1nm}}{3(n_m^2-1)(kr_d)^2}. \tag{32}$$

Consequently, according to (19), in the vicinity of the separatrix, even under conditions of strong coupling, it is sufficient for

$$\frac{E_{2nm}}{E_{1nm}}\approx 0.6|\varepsilon|^{-1/2}\frac{\gamma_{nm}}{\omega}\ll 1, \tag{33}$$

which can easily be fulfilled when $n_1 \ll n_0$.

The most rigid condition of applicability of the relations in the strong-coupling case turns out to be the requirement (2) of independent damping of the harmonics. For the fundamental (in the damping process) harmonic with the number $n = n_m$, (11), this condition has the form

$$E_1^2/E_{cr}^2\ll\frac{\exp(2n_0/n_1)^{1/2}}{n_0^6n_1^2}. \tag{34}$$

Under conditions of weak coupling ($n_1 \ll 1$), we have $n_0, n_m \gg 1$, and the condition (34) can easily be fulfilled because of the presence of the exponential function. Under conditions of strong coupling, we can, according to (17)–(20), write

$$n_1\ll 2|\varepsilon|^{-1/2}\exp(0.3|\varepsilon|^{-1/2}) \tag{35}$$

The last inequality determines the maximum value of n_1, (18), at which the independent-damping requirement is still fulfilled. For example, for $\epsilon = -0.1$, $n_1 \lesssim 12$.

5. The comparatively large value of the damping constant allows us to expect a significant manifestation of the considered mechanism, even at a low plasma-oscillation density, in experiments on the interaction between an opaque plasma and microwave electromagnetic-field pulses of duration longer than τ_k (29). In contrast to the decay and induced-scattering processes, the enrichment of the spatial spectrum in this case occurs in the direction of increasing plasma-oscillation wave numbers, owing to the excitation of higher spatial harmonics. For such a pumping direction, the problem of large-scale energy absorption is solved without the inclusion of the additional processes, owing to the presence of an effective, collisionless, damping mechanism.

The main restrictions on the application of the relations obtained are determined by the requirements of: 1) weak spatial dispersion ($|\epsilon|^{-1/2} \gg 1$), since in the opposite case damping on the large scale is already so strong that the effects of the nonlinear coupling are insignificant; 2) small quasi-linear effects of the distribution-function relaxation in the region of considerable damping.

The author is grateful to V. B. Gel'denburg and A. G. Litvak for interest to the work and numerous discussions.

[1)]We note that by weak nonlinearity we mean here only that $\Delta N/N \ll 1$ (N is the concentration of the unperturbed plasma and ΔN is its perturbation by the field of the nonlinear wave). In particular, our analysis includes the case $\Delta N/N \sim (kr_d)^2$.

[2)]We note that such an expression for ϵ is, as is well known, valid only in the case when the particles are expelled from the strong-field region in the process of its establishment. It is clear how everything must be changed in the case when such a redistribution is not realized. In the weak-nonlinearity approximation, it is sufficient to replace ϵ everywhere by $\epsilon_\infty - \overline{|E|^2}/E_{cr}^2$, where ϵ_∞ is the unperturbed value and $\overline{|E|^2}$ is the square of the field averaged over the period of the wave. In particular, the magnitude of the field averaged over the spatial period is, in contrast to (24), found from the equation

$$(\varepsilon_\infty + \overline{(E-\bar{E})^2} / E^2_{cr})\bar{E} = D.$$

[1]Yu. Ya. Brodskiĭ, B. G. Eremin, A. G. Litvak, and Yu. A. Sakhonchik, ZhETF Pis. Red. **13**, 136 (1971) [JETP Lett. **13**, 95 (1971)]; G. M. Batanov and V. A. Silin, ZhETF Pis. Red. **14**, 445 (1971) [JETP Lett. **14**, 303 (1971)].

[2]A. A. Vedenov, in: Voprosy teorii plazmy (Problems of Plasma Theory), Vol. 3, Gosatomizdat, 1963; V. E. Zakharov and V. I. Karpman, Zh. Eksp. Teor. Fiz. **43**, 490 (1962) [Sov. Phys.-JETP **16**, 351 (1963)].

[3]V. L. Ginzburg, Rasprostranenie voln v plazme (Propagation of Electromagnetic Waves in a Plasma), Fizmatgiz, 1960, p. 88 (Eng. Transl., Gordon and Breach, New York, 1962).

[4]A. V. Gurevich and L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **45**, 1243 (1963) [Sov. Phys.-JETP **18**, 855 (1964)].

[5]L. D. Landau and E. M. Lifshitz, Mekhanika (Mechanics), Nauka, 1965, p. 112 (Eng. Transl., Pergamon Press, New York, 1960).

Translated by A. K. Agyei
173