On the nonlinear theory of the beam-plasma instability

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The possibility of radial focusing of a relativistic electron beam in a plasma under contions of beam instability caused by the reaction of the electromagnetic radiation on the beam is demonstrated. The radius of a focused "cold" beam and of a beam with a Maxwellian momentum distribution function is found from the radial-energy equilibrium condition.

1. INTRODUCTION

The idea of the possibility of using the electromagnetic self-fields of charged-particle beams for the realization of radial focusing of high-power relativistic election streams was first put forward by Budker[1]. In the method of the self-stabilized beam discovered by him the radial pinching of a relativistic electron-ion stream occurs under the action of the radiative reaction force that acts on electrons oscillating in the radial potential well of an uncompensated ionic charge, while the energy of the transverse oscillations is continually replenished as a result of the electron-ion scattering processes which change the longitudinal motion of the electrons into transverse motion. Since the energy of the magnetic field arising in the electron-ion beam pinched under conditions of magnetic focusing attains a significant value, the radiative reaction force acting on the electrons leads to a radial compression of the beam into a thin filament. As was shown by Budker[1], an equilibrium state of the beam exists in which the longitudinal momentum acquired by the beam in the external electric field is lost in consequence of the collisions with ions, while the energy gained from the field is carried away by the radial energy flux of the radiation field of the beam.

In order to reduce the beam-focusing time, which is determined by the reciprocal of the electron-ion binary collision rate and which turns out to be quite substantial (of the order of a second and more[1]), Ya. B. Faĭnberg proposed the use of the coherent radiation of electrons for the radial pinching of beams (see[2]). There arises at the same time the possibility of realizing a self-stabilized regime at considerably weaker currents in the beams. Computations carried out in[2-4] confirmed the possibility in principle of radial beam focusing in a plasma under the conditions of a beam-plasma instability. In this case the radial focusing force arises as a result of the reaction on the beam of the inhomogeneous field of the plasma waves excited by the beam, and the radius of the pinched beam is determined by the ratio of the wavelength of the plasma wave to the dimensionless increment of the instability:

$$r_{min} \sim \frac{v_0}{\omega_p}\left(\frac{\omega_p}{\mu}\right) \tag{1}$$

(v_0 is the directed beam velocity and ω_p is the plasma frequency).

However, in the cases considered in[2-4], when the beam excited charge-density waves (potential oscillations) in the plasma, the energy of the field is accumulated in the bulk of the beam (the beam-plasma system is a conservative system), and there develop weakly damped nonlinear radial oscillations (with frequency of the order of the increment of the instability, i.e., $\omega_r \sim \mu$, and with decrement $\delta \sim \mu^2/\omega_p$) in which the beam radius periodically varies within the limits of from r_{min} to the initial value[1)].

The process of radial constriction of a beam under the conditions of a beam-plasma instability can be made irreversible if radiation "removal" from the beam is provided for by creating conditions for the propagation of the "slow" electromagnetic wave out of the beam. The radial field-energy flux leaving the beam then carries away the transverse momentum of the electrons and damps out the transverse oscillations in the beam and in the plasma. The equilibrium beam radius is determined from the condition

$$-\int (\mathbf{j}_b+\mathbf{j}_p)\mathbf{E}\,dV=\frac{c}{4\pi}\oint [\mathbf{EH}]\,d\mathbf{f} \tag{2}^*$$

($\mathbf{j}_b$ is the current density in the beam, $\mathbf{j}_p$ is the plasma-electron current density, and **E** and **H** are the components of the self-consistent field), according to which the energy lost by the beam in the excitation of the plasma waves is carried away by the field-energy flux through the beam boundary. It is clear that such an equilibrium state in the beam can exist only in the presence of an external accelerating field that compensates the energy lost by the beam on radiation.

Below we shall assume that the electromagnetic radiradiation of the beam is ensured by an external decelerating system of effective dielectric constant $\epsilon_e > 1$. The pinching of the beam and the radial equilibrium occur when $\beta_0^2\epsilon_e > 1$, while the equilibrium radius of the beam is determined by the formula

$$a \sim \frac{v_0}{\omega_p}\,\varepsilon_e(\beta_0^2\varepsilon_e-1)^{-1/2}\left(\frac{\omega_p}{\mu}\right), \tag{3}$$

i.e., by the ratio of the wavelength of the radiation to the dimensionless increment. It should be noted that in the case of a relativistic beam ($\beta_0 \approx 1$), a gas with dielectric constant $\epsilon_e \gtrsim 1$ can be used as the decelerating system; the deceleration efficiency of such a system increases if the radiation-wave frequency (the plasma frequency) will be close to one of the natural frequencies of the gas.

2. THE DISPERSION EQUATION

Let the radially constricted relativistic electron beam move along the axis of the plasma cylinder (the plasma and beam radii coincide). The basic system of equations that describes the beam-plasma interaction consists of the kinetic equation for the electron distribution function and the Maxwell equations for **E** and **H**, the components of the self-consistent field. Assuming that the dependence of the wave field on the time t and the z coordinate has the form $\exp(i\omega t - ikz)$ and restricting ourselves to axially symmetric perturbations, we represent the self-consistent system of equations in the form

$$i(\omega - kv_z)f_1 + \frac{v_r}{r}\frac{\partial}{\partial r}(rf_1) + e\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right)\frac{\partial f_0}{\partial \mathbf{p}} = 0,$$
$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\varphi) = i\frac{\omega}{c}\varepsilon E_z + \frac{4\pi e}{c}\int v_z f_1\,d\mathbf{p},$$
$$ikH_\varphi = i\frac{\omega}{c}\varepsilon E_r + \frac{4\pi e}{c}\int v_r f_1\,d\mathbf{p}, \tag{4}$$
$$ikE_r + \frac{\partial E_z}{\partial r} = i\frac{\omega}{c}H_\varphi,$$

where $f_0(\mathbf{r}, \mathbf{p})$ is the equilibrium distribution function, $f_1(\mathbf{r}, \mathbf{p})$ is the amplitude of the oscillating correction to it, $\epsilon = 1 - \omega_p^2/\omega^2$, $\omega_p^2 = 4\pi e^2 n_p/m$ is the plasma frequency, and $\mathbf{v}_0$ and $\mathbf{p}$ are the electron velocity and momentum.

The solution of the kinetic equation can be represented in the form of the following series[2]:

$$rf_1 = e\sum_{s=0}^{\infty} i^{s+1}\frac{v_r^s}{\Delta^{s+1}}\frac{\partial^s}{\partial r^s}\left\{r\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right)\frac{\partial f_0}{\partial \mathbf{p}}\right\}, \quad \Delta = \omega - kv_z, \tag{5}$$

which is an expansion of the function f_1 in powers of the parameter $\eta = v_r/\Delta r$. Assuming $\eta \ll 1$ and retaining only terms with s = 0 and s = 1, we find the components of the beam current:

$$j_z = e^2\int\left\{iE_z\frac{\partial f_0}{\partial p_z} - \frac{v_r}{\Delta r}\frac{\partial}{\partial r}\left[r(E_r - \beta_z H_\varphi)\frac{\partial f_0}{\partial p_r}\right]\right\}v_z\frac{d\mathbf{p}}{\Delta},$$
$$j_r = ie^2\int (E_r - \beta_z H_\varphi)v_r\frac{\partial f_0}{\partial p_r}\frac{d\mathbf{p}}{\Delta}. \tag{6}$$

The system of equations (4) for the field can, with allowance for the currents (6), be reduced to the Bessel equation for the field component E_z:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dE_z}{dr}\right) + \alpha^2 E_z = 0. \tag{7}$$

where

$$\alpha^2 = -k^2\left(\varepsilon + \frac{4\pi e^2}{\omega}\int v_z\frac{\partial f_0}{\partial p_z}\frac{d\mathbf{p}}{\Delta}\right)\left(1 - \beta_\Phi^2\varepsilon - \frac{4\pi e^2}{c^2k^2}\int v_r\frac{\partial f_0}{\partial p_r}d\mathbf{p}\right)$$
$$\times\left[\varepsilon + 4\pi e^2\int v_r(1 - \beta_z^2\varepsilon)\frac{\partial f_0}{\partial p_r}\frac{d\mathbf{p}}{\Delta^2}\right]^{-1},$$
$$\beta_\Phi = \omega/ck, \quad \beta_z = v_z/c.$$

The solution of Eq. (7) in the region $r \le a$ (a is the beam radius) which is finite at the point r = 0 has the form

$$E_z^i = AJ_0(\alpha r), \tag{8}$$

while the E_r^i and H_φ^i components are given in terms of E_z by the formulas

$$E_r^i = -i\frac{\alpha}{k}A\frac{\mathscr{L}_1}{\mathscr{L}_2}J_1(\alpha r),$$
$$H_\varphi^i = -i\frac{\alpha}{k}A\frac{\mathscr{L}_3}{\mathscr{L}_2}J_0(\alpha r);$$
$$\mathscr{L}_1 = 1 + \frac{4\pi e^2}{ck}\int \beta_z v_r\frac{\partial f_0}{\partial p_r}\frac{d\mathbf{p}}{\Delta}, \tag{9}$$
$$\mathscr{L}_2 = 1 - \beta_\Phi^2\varepsilon - \frac{4\pi e^2}{c^2k^2}\int v_r\frac{\partial f_0}{\partial p_r}d\mathbf{p},$$
$$\mathscr{L}_3 = \beta_\Phi\varepsilon + \frac{4\pi e^2}{ck}\int v_r\frac{\partial f_0}{\partial p_r}\frac{d\mathbf{p}}{\Delta}.$$

Correspondingly, the fields in the region outside the beam and plasma, i.e., in the region $r > a$, filled with a dielectric of dielectric constant ϵ_e, are given by the formulas (7)—(9) in which we should set $f_0 = 0$ and $\epsilon = \epsilon_e$:

$$E_z^e = BK_0(\sigma r), \quad E_r^e = -iB\frac{k}{\sigma}K_1(\sigma r), \tag{10}$$
$$H_\varphi^e = -iB\varepsilon_e\beta_\Phi\frac{k}{\sigma}K_1(\sigma r),$$

where $\sigma^2 = k^2(1 - \beta_\varphi^2\epsilon_e)$.

Substituting the fields from the formulas (8)—(10) into the boundary conditions

$$E_z^i(a) = E_z^e(a),$$
$$H_\varphi^i(a) + \frac{\mathscr{L}_4}{\mathscr{L}_1}E_r^i(a) = H_\varphi^e(a),$$
$$\mathscr{L}_4 = 4\pi e^2\int (1 - \varepsilon\beta_z\beta_\Phi)\beta_z v_r\frac{\partial f_0}{\partial p_r}\frac{d\mathbf{p}}{\Delta^2}, \tag{11}$$

we obtain the dispersion equation

$$\alpha\frac{J_0(\alpha a)}{J_1(\alpha a)} = -\frac{\sigma}{\varepsilon_e}\left(\varepsilon + \frac{4\pi e^2}{\omega}\int v_z\frac{\partial f_0}{\partial p_z}\frac{d\mathbf{p}}{\Delta}\right)\frac{K_0(\sigma a)}{K_1(\sigma a)}, \tag{12}$$

describing the dependence of the frequency ω on the wave vector k and on the beam and plasma parameters.

Under the conditions of a plasma resonance, $|\epsilon| \ll 1$ and the solution of Eq. (12) can be represented in the form of an expansion in powers of this parameter. Taking note of this, we set

$$\alpha a = \lambda_p + \lambda_p^{(1)}, \quad J_0(\lambda_p) = 0, \quad \lambda_p^{(1)} \ll \lambda_p. \tag{13}$$

Substituting (13) into (12), we find

$$\lambda_p^{(1)} = -\frac{\sigma a}{\lambda_p\varepsilon_e}\left(\varepsilon + \frac{4\pi e^2}{\omega}\int v_z\frac{\partial f_0}{\partial p_z}\frac{d\mathbf{p}}{\Delta}\right)\frac{K_0(\sigma a)}{K_1(\sigma a)}. \tag{14}$$

Further, from the formulas (13) and (7) follows

$$\left(k^2 + \frac{\lambda_p^2}{a^2}\right)\varepsilon + 4\pi e^2\int\left(\frac{k^2}{\omega}v_z\frac{\partial f_0}{\partial p_z} + \frac{\lambda_p^2}{a^2\Delta}v_r\frac{\partial f_0}{\partial p_r}\right)\frac{d\mathbf{p}}{\Delta} = 0. \tag{15}$$

This relation is an approximate relation, valid up to terms $\sim \epsilon^2$.

Because of the anisotropy in the longitudinal and transverse masses, the last term in the dispersion equation (15) exceeds the second by a factor of $\gamma_0^2 \gg 1$ (γ_0 is the relativistic factor) and, consequently, transverse oscillations are excited in the beam[6]. Since, moreover, the increments of the harmonics with $\lambda_p \gg ka$ turn out to be the largest, then the relations (14) and (15) can be represented in the form

$$\varepsilon + 4\pi e^2\int v_r\frac{\partial f_0}{\partial p_r}\frac{d\mathbf{p}}{\Delta^2} = 0, \quad \lambda_p^{(1)} = -\frac{\sigma a}{\lambda_p\varepsilon_e}\frac{K_0(\sigma a)}{K_1(\sigma a)}\varepsilon. \tag{16}$$

In the same approximation, the E_z and H_φ fields, as well as the j_r and j_z currents in the region $r < a$ are equal to

$$E_z = \sum_{k,p} E_{kp}J_0(\alpha_p r)\exp(i\Phi_k), \qquad \Phi_k = \omega t - kz,$$
$$H_\varphi = -i\sum_{k,p}\beta_\Phi\frac{\lambda_p}{ka}\varepsilon E_{kp}J_1(\alpha_p r)\exp(i\Phi_k),$$
$$E_r = -i\sum_{k,p}\frac{\lambda_p}{ka}E_{kp}J_1(\alpha_p r)\exp(i\Phi_k), \tag{17}$$
$$j_z = ie^2\sum_{k,p}\frac{\lambda_p^2}{a^2k}E_{kp}\int\frac{v_r v_z}{\Delta^2}\frac{\partial f_0}{\partial p_r}d\mathbf{p}\,J_0(\alpha_p r)\exp(i\Phi_k),$$
$$j_r = e^2\sum_{k,p}\frac{\lambda_p}{ka}E_{kp}\int v_r\frac{\partial f_0}{\partial p_r}\frac{d\mathbf{p}}{\Delta}J_1(\alpha_p r)\exp(i\Phi_k).$$

The formula (17) will be used below in the computation of the beam-energy losses and the radial field-energy flux entering into (2).

3. RADIAL EQUILIBRIUM IN THE BEAM

As was noted in the Introduction, with the development of the instability is established an equilibrium state in which the energy lost by the beam is carried away by the field-energy flux through the beam boundary. The equilibrium radius of the beam is then determined by the formula (2). In the case of the infinitely long plasma being considered the radiation occurs through the side surface and the field-energy flux turns out to be equal to

$$S_r = \frac{c}{8\pi} \operatorname{Re} E_z H_\varphi^*. \tag{18}$$

We are interested in the value of the quantity S_r at the surface of the beam. Substituting the fields E_r and H_φ from the formula (17) into the expression (18) and setting $r = a$, we obtain[2)]

$$S_r(a) = -\frac{c}{8\pi} \operatorname{Re} i \sum_k \frac{\sigma|\varepsilon|^2}{\varepsilon_e k} \frac{K_0(\sigma a)}{K_1(\sigma a)} \Sigma_k,$$

$$\Sigma_k = \sum_{n,p} \frac{\lambda_n}{\lambda_p} J_1(\lambda_n) J_1(\lambda_p) E_{kp} E_{kn}^*. \tag{19}$$

In deriving the formula (19) we set $J_0(\alpha_p a) = -\lambda_p^{(1)} J_1(\lambda_p)$ in the formula for E_z and neglected the quantity $\lambda_p^{(1)}$ in comparison with λ_p in the expression for H_φ.

It is easy to see that in the absence of an external decelerating system (or for $\beta_\varphi^2 \epsilon_e < 1$), when $\sigma > 0$ and the wave attenuates exponentially in the region $r > a$ outside the beam, the energy flux $S_r(a)$ is equal to zero. Therefore, let us set $\sigma = ik_\perp$. Then, using the relation

$$K_n(ik_\perp a) = -i\frac{\pi}{2} e^{-in\pi/2} H_n^{(2)}(k_\perp a),$$

we find

$$S_r(a) = \frac{c}{8\pi\varepsilon_e} \sum_k \frac{k_\perp}{k} \frac{2}{\pi k_\perp a} \frac{|\varepsilon|^2}{J_1^2(k_\perp a) + N_1^2(k_\perp a)} \Sigma_k, \tag{20}$$

where J_n and N_n are the Bessel and Neumann functions, $H_n^{(2)} = J_n - iN_n$ is the Hankel function, and K_n is the modified Hankel function.

Estimates show that the equilibrium beam radius a considerably exceeds the wavelength, i.e., $k_\perp a \gg 1$, and we can use the asymptotic forms of the Bessel functions

$$J_1^2(k_\perp a) + N_1^2(k_\perp a) \approx 2/\pi k_\perp a.$$

Substituting (20) into (19), we obtain the following expression for the field-energy flux through the beam boundary:

$$S_r(a) = \frac{c}{8\pi\varepsilon_e} \sum_k \frac{k_\perp}{k} |\varepsilon|^2 \Sigma_k. \tag{21}$$

The beam- and plasma-energy losses (per unit length) are determined by the expression

$$W_{b,p} = \pi \operatorname{Re} \int \mathbf{j}_{b,p} \mathbf{E}^* r\, dr. \tag{22}$$

Substituting into the formula (22) the expressions for the fields and currents from the formula (17) and carrying out the integration over the variable r, we obtain the following expression determining the beam-energy losses:

$$W_b = \frac{\pi e^2 c}{2} \Sigma,$$

$$\Sigma = \sum_{k,p} \beta_\Phi \frac{\lambda_p^2}{k} |E_{kp}|^2 J_1^2(\lambda_p) \operatorname{Re} i \int v_r \frac{\partial f_0}{\partial p_r} \frac{d\mathbf{p}}{\Delta^2}. \tag{23}$$

The current generated by the plasma electrons is equal to

$$j_{rp} = -\frac{e^2 n_p}{m} \sum_{k,p} \frac{\lambda_p}{\omega k a} E_{kp} J_1\left(\lambda_p \frac{r}{a}\right) \exp(i\Phi_k).$$

In computing the quantity W_p from the formula (22), we set $\omega = \omega_p + i\mu_{kp}$ ($|\mu_{kp}| \ll \omega_p$) and determine μ_{kp} from the dispersion equation (15):

$$i\mu_{kp} = -2\pi e^2 \omega_p \int v_r \frac{\partial f_0}{\partial p_r} \frac{d\mathbf{p}}{\Delta^2}.$$

The kinetic energy imparted to the plasma electrons is proportional to μ_{kp} and is equal to $W_p = -\frac{1}{2} W_b$.

Equating, according to the formula (2), the quantities $2\pi a S_r(a)$ and $-(W_b + W_p)$, we find the equilibrium beam radius:

$$a = -\pi e^2 \varepsilon_e \Sigma \Big/ \sum_k \frac{k_\perp}{k} |\varepsilon|^2 \Sigma_k, \tag{24}$$

where Σ and Σ_k are respectively defined in (23) and (19).

In order to get rid of the sums over the transverse wave numbers, let us assume that the amplitude of the initial perturbation does not depend on the variable r. Then the coefficients E_{kn}, which are the terms of a series expansion of the function E_k in terms of the functions $J_0(\lambda_n r/a)$, are equal to

$$E_{kn} = \frac{2}{\lambda_n J_1(\lambda_n)} E_k. \tag{25}$$

Substituting into the formula (24) the quantities E_{kn} from the formula (25) and performing the summation over n with the aid of the relation[7]:

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = \frac{1}{4},$$

we arrive at the formula

$$a = -4\pi e^2 \varepsilon_e A_1 / A_2,$$

$$A_1 = \sum_k \frac{\beta_\Phi}{k} |E_k|^2 \operatorname{Re} i \int v_r \frac{\partial f_0}{\partial p_r} \frac{d\mathbf{p}}{\Delta^2},$$

$$A_2 = \sum_k \frac{k_\perp}{k} |\varepsilon|^2 |E_k|^2. \tag{26}$$

The expression obtained for the beam radius in the form (26) can be used to compute the radius of the "cold" beam (see Sec. 4). However, in the case of a beam with a finite temperature it turns out to be more convenient to express the dielectric constant ϵ in terms of the beam parameters from the dispersion equation (16). The formula (26) then assumes the form:

$$a = -\frac{\varepsilon_e}{4\pi e^2} \frac{A_3}{A_4},$$

$$A_3 = \sum_{k,p} \frac{\beta_\Phi}{k} |E_k|^2 \operatorname{Re} i q_k,$$

$$A_4 = \sum_{k,p} \frac{k_\perp}{k} |E_k|^2 |q_k|^2,$$

$$q_k \equiv \int v_r \frac{\partial f_0}{\partial p_r} \frac{d\mathbf{p}}{\Delta^2}. \tag{27}$$

4. THE COLD BEAM

As follows from the formulas (26) and (27), the equilibrium beam radius is determined by the specific form of the electron distribution function. In the case when the longitudinal velocity of the electrons is sufficiently low

$$v_T \ll \frac{\mu}{\omega_p} v_0, \tag{28}$$

the oscillation spectrum turns out to be fairly narrow, so that practically only the regular wave is excited: $\Delta k \sim \mu/v_0 \ll k$. Substituting into the formulas (16) and (26) the distribution function in the form

$$f_0 = n_b \delta(\mathbf{p} - \mathbf{p}_0) \tag{29}$$

and performing the integration over the momenta with the aid of the δ-function, we obtain

$$\varepsilon - \frac{\omega_b^2}{\gamma_0(\omega - kv_0)^2} = 0, \qquad \omega_b^2 = \frac{4\pi e^2 n_b}{m}, \tag{30}$$

$$a = \frac{\omega_b^2 \varepsilon_e}{k_\perp |\varepsilon|^2} \mathrm{Re} \frac{i}{(\omega - kv_0)^2}. \tag{30'}$$

Solving the dispersion equation (30), we find the root corresponding to the solutions that increase in time:

$$\omega - kv_0 = -\frac{1}{2^{4/3}} \left(\frac{n_b}{n_p \gamma_0}\right)^{1/3} (1 + i\sqrt{3}) \omega_p, \qquad \omega_p = kv_0. \tag{31}$$

The dielectric constant ϵ of the plasma at the frequency (31) of the instability turns out to be equal to $|\epsilon|^2 = 4(n_b/2n_p\gamma_0)^{2/3}$, while from the formula (30′) follows the expression for the beam radius:

$$a = \frac{\sqrt{3}}{4} \frac{\varepsilon_e}{k_\perp} \left(2 \frac{n_p \gamma_0}{n_b}\right)^{1/3}. \tag{32}$$

According to the estimates of [4], during a time interval of the order of the inverse increment $\mu^{-1} \sim \omega_p^{-1}(n_p\gamma_0/n_b)^{1/3}$ of the instability the energy density of the plasma oscillations attains the value $(8\pi)^{-1}E^2 \sim (n_b/n_p\gamma_0)^{1/3} n_b mc^2 \gamma_0$. Therefore to sustain the equilibrium in the beam, we must compensate with the aid of an external field the energy losses due to radiation.

5. BEAM WITH A FINITE TEMPERATURE

To conclude the paper, let us find the equilibrium radius of a beam with the Maxwellian momentum distribution function:

$$f_0(\mathbf{p}) = \frac{n_b}{\pi p_T p_\perp} \exp\left[-\frac{p_r^2}{p_\perp^2} + \frac{(p_z - p_0)^2}{p_T^2}\right], \tag{33}$$

using for the computations the formula (27).

It is convenient to represent the momentum integral figuring in the formula (27) in the form

$$q_k = -\frac{d}{dk} \int \frac{1}{v_z} \frac{\partial v_r}{\partial p_r} \frac{f_0 \, d\mathbf{p}}{\omega - kv_z}. \tag{34}$$

Further, assuming that $p_0 \gg p_T$, we expand the integrand in a series at the point $p_z = p_0$:

$$\frac{\partial v_r}{\partial p_r} \approx \frac{1}{m\gamma_0}, \qquad v_z = v_0 + \frac{dv_z}{dp_0}(p_z - p_0) \tag{35}$$

and introduce the integration variable $x = p_z - p_0$. After integrating with respect to the variable p_r the formula assumes the form

$$q_k = -\frac{n_b}{\sqrt{\pi}\, p_T} \frac{1}{m\gamma_0 v_0} \int_{-\infty}^{\infty} \frac{\exp(-x^2/p_T^2)}{\omega^* - \alpha x} dx, \qquad \omega^* = \omega - kv_0, \qquad \alpha = k\frac{dv_z}{dp_0} \tag{36}$$

To evaluate the integral (36), let us use the relation

$$(\omega^* - \alpha x)^{-1} = i \int_0^\infty \exp[i(\omega^* - \alpha x)\xi] d\xi, \qquad \mathrm{Im}\, \omega^* < 0.$$

Changing the order of integration and evaluating the integral over the variable x, we reduce the expression (36) to the following form:

$$q_k = -\frac{n_b}{m\gamma_0 v_0} \frac{d}{dk} \left[\frac{1}{kv_T} \exp\left\{-\left(\frac{\omega^*}{kv_T}\right)^2\right\} \left(i\sqrt{\pi} + 2\int_0^{\omega^*/kv_T} e^{z^2} dz\right)\right], \tag{37}$$

where $v_T \equiv p_T dv_z/dp_0$.

The formula (27) determines together with (37) the equilibrium beam radius. Since the dominant contribution to the sums (27) is made by the resonant harmonics $\omega^*/kv_T \ll 1$, then, expanding the first part of the formula in powers of this parameter and retaining the leading terms of the expansion, we find

$$q_k = \frac{n_b}{m\gamma_0 k^2 v_0 v_T} \left(i\sqrt{\pi} + 2\frac{v_0}{v_T}\right). \tag{38}$$

Since we assume that the inequality $1 - \beta_0^2 \ll 1$ is fulfilled, the k-spectrum of the oscillations is sufficiently narrow and, to estimate the sums in the formula (26), it is sufficient to set $k \approx \omega_p/v_0$. Then, dividing the numerator and denominator of the formula by the number of oscillations, we obtain (under the assumption that $v_0/v_T \gg 1$)

$$a = \frac{\sqrt{\pi}}{4} \frac{\varepsilon_e}{k_\perp} \left(\frac{v_T}{v_0}\right)^3 \frac{n_p \gamma_0}{n_b}. \tag{39}$$

To compensate the energy losses of a kinetic (quasilinear) beam, an external electric field of energy density comparable with the energy density of the beam is necessary.

It should be noted that for the "skew" Langmuir oscillations ($ka \ll \lambda_p$) under consideration, the condition for the expansion of the kinetic equation in powers of the parameter

$$\eta = \frac{\lambda_p}{ka} \frac{v_\perp}{v_T} \ll 1$$

is valid only when the longitudinal and transverse temperatures of the beam are strongly anisotropic: $v_T \gg v_\perp$.

*$[EH] \equiv E \times H$.

[1]As has been shown by Ivanov and Rudakov [5], the magnetic-focusing effect in a plasma weakens to a considerable degree because of the appearance of a reverse current.

[2]The averaging is carried out over the spatial period in the case of the regular wave, and over the phases if there is a wide spectrum of oscillations.

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