Nonlinear oscillations of an unstable plasma

E. Abu-Asali, B. A. Al'terkop, and A. A. Rukhadze
P. N. Lebedev Physics Institute, USSR Academy of Sciences
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The nonlinear frequency shift and nonlinear damping of an unstable wave in a plasma due to the generation of its higher stable harmonics are considered as mechanisms for the stabilization of the wave. The time evolution of the amplitude of the unstable wave is studied up to the saturation stage. An example of the stabilization of an instability in the hot beam-cold plasma system is considered.

1. In the study of the behavior of an unstable plasma there naturally arises the problem of finding an effective mechanism for the stabilization of the growing oscillations. Under conditions when a sufficiently broad spectrum of oscillations is excited the solution of this problem is provided by the theory of the weakly turbulent plasma[1,2]. On the other hand, the existence of an excitation threshold and, if the supercriticality of the system is small, the buildup of a narrow spectrum of oscillations are characteristic of many instabilities, especially of a collision plasma, and this allows us to consider such a regime as a single-mode regime. It turns out that in this case the stabilization of the growth of the amplitude of the unstable mode can be achieved owing to the nonlinear frequency shift and nonlinear damping of the mode due to the generation of its higher stable harmonics and the analogous dissipation of energy during the turbulent motion of the fluid[3].

The effect of these factors has been investigated, in particular, in[4,5] in the solution of the problem of the saturation of the drift-dissipative instability, and in[6-8] in the analysis of the ion-sound instability of a nonisothermal, current-carrying plasma. The results obtained in these papers allow us to establish some general relations characterizing the effectiveness of the indicated stabilization mechanism and determining the main characteristics of the unstable wave in the steady state. The analysis of these relations is the object of the present paper.

It should also be noted that the generation of higher harmonics plays an important role in nonlinear optics[9], and that the stabilizing action of the nonlinear frequency shift was recently investigated in the analysis of the parametric interaction of waves in a plasma[10,11] and in the problem of the nonlinear Landau damping[12].

2. In order not to complicate the problem and to expose the effect in its pure form, we shall assume that the nonlinear effects due to the resonant interaction of the particles and waves are suppressed by particle collisions, and thereby restrict ourselves to the consideration of the hydrodynamic nonlinear plasma oscillations. In the single-mode regime all the quantities characterizing the perturbed state of the plasma can be considered as functions of two variables: $\xi = \mathbf{k}\cdot\mathbf{r} - \omega t$ and the time t, the explicit dependence on the time being assumed to be weak. In the expression for ξ the quantity $\mathbf{k}$ corresponds to the wave vector of the mode with the maximum linear increment and ω is an unknown real frequency. If we go over to the Fourier representation in the variable ξ, then the dynamical equation determining the nonlinear evolution of the wave will, in the general case, have the form

$$i\frac{da_n}{dt}=(-\Delta_n+i\Gamma_n)a_n+\sum_m C_{nm}a_m a_{n-m}+\sum_{m,l} C_{nml}a_m a_l a_{n-m-l}, \quad (1)$$

where a_n and Γ_n are the complex amplitude and increment (decrement) of the n-th harmonic of the excitable mode with $\Gamma_1 > 0$ in the instability region; C_{nm} and C_{nml} are the interaction matrix elements for the harmonics; Δ_n is a quantity proportional to Re $\epsilon(n\mathbf{k}, n\omega)$, where $\epsilon(\mathbf{k}, \omega)$ is the permittivity of the plasma. Notice that in the general case, because of the nonlinearity of the dispersion law, $\Delta_n \neq n\Delta_1$ and $\Gamma_n \neq \Gamma_1(n\omega, n\mathbf{k})$.

The system of equations (1) is similar to the dynamical equations that describe the interaction between the natural oscillations of the plasma[1,2], but differs from the latter in that it describes the interaction between the fundamental harmonic of the unstable mode and its overtones, which are forced and, in the general case, not natural oscillations of the system. Notice also that the resonance conditions with respect to the frequencies and wave vectors in (1) clearly are identically fulfilled.

For a small supercriticality of a system in the single-mode regime, we can, as usual, assume that the main contribution to the profile of a finite-amplitude wave is made by a few harmonics. Limiting ourselves to the first two harmonics, we obtain from (1)

$$ida_1/dt=(-\Delta_1+i\Gamma_1)a_1+V_1a_1^*a_2+V_2|a_1|^2a_1,$$
$$ida_2/dt=(-\Delta_2+i\Gamma_2)a_2+V_3a_1^2,$$
$$V_1=C_{12}+C_{1-1},\quad V_3=C_{111}+C_{11-1}+C_{1-11}+C_{1-1-1},\quad V_2=C_{21}, \quad (2)$$

where the star denotes complex conjugation. Under conditions, when the dissipative processes are weak and determine only the linear growth rate of the instability, we can neglect the contribution of the dissipative terms to the matrix elements and assume them to be real, as obtains in conservative systems. The last circumstance allows us to establish, in particular, some properties of the matrix elements V_1 and V_3.

To show this, let us neglect in (2) dissipation and cubic nonlinearities:

$$ida_1/dt=-\Delta_1a_1+V_1a_1^*a_2,\quad ida_2/dt=-\Delta_2a_2+V_3a_1^2. \quad (2')$$

These equations have the integral

$$|a_1|^2/V_1+|a_2|^2/V_3=\text{const}, \quad (3)$$

It follows from (3), when energy conservation in the conservative system is taken into account, that V_1 and V_3 have the same sign, i.e.,

$$V_1V_3>0. \quad (4)$$

The same property can be obtained by transforming Eqs. (2′) to the Hamiltonian form. In fact, let us set

$$a_j=\alpha_jb_j\exp(i\Delta_jt), \quad (5)$$

where α_j can, without loss of generality, be assumed to be real quantities, Substituting (5) into (2′), we can easily establish that only when the condition

$$ {}^1\!/_2\alpha_2 V_1 = \alpha_1^2 V_3 / \alpha_2 \equiv W \tag{6} $$

is fulfilled will Eqs. (2′) assume the Hamiltonian form (in the interaction representation):

$$ idb_j/dt = \partial H/\partial b_j^*, $$
$$ H = W\exp[i(\Delta_2 - 2\Delta_1)t] b_2 b_1^{*2} + \text{c.c.} \tag{7} $$

From (6) follows

$$ V_1/V_3 = 2\alpha_1^2/\alpha_2^2 > 0, $$

which agrees with (4).

If the overtones are natural oscillations of the system (linear dispersion law), then the time-averaged intensity of the wave can be expressed in the form of a sum of the squares of the moduli of the amplitudes of the individual harmonics[3]. Then, it follows from (3) that for a linear dispersion law we must have

$$ V_1 = V_3. \tag{8} $$

In particular, this relation is valid for long-wave drift and ion-sound plasma oscillations[4-7], as well as for sound vibrations in a solid (see, for example,[9], p. 147). Let us also note that the sign rule (4) and the equality (8) correspond to symmetry conditions for the matrix elements of the three-wave interaction of the natural oscillations of a plasma[2].

Let us now return to the system (2). Assuming that the condition $|-\Delta_2 + i\Gamma_2| \gg \Gamma_1$ (Γ_1 is proportional to the supercriticality, which we assume to be small) is fulfilled, we easily derive from (2) the equations describing the time evolution of the amplitude and of the frequency shift of the unstable mode:

$$ \frac{1}{2}\frac{dA_1}{d\tau} = A_1 + \frac{V_1V_3}{\Delta_2^2+\Gamma_2^2}\frac{\Gamma_2}{\Gamma_1}A_1^2, \tag{9} $$

$$ \Delta_1(\tau) = \left(\Delta_2\frac{V_1V_3}{\Delta_2^2+\Gamma_2^2} + V_2 + \Gamma_1\frac{d}{d\tau}\arg a_1\right)A_1. \tag{10} $$

Here $\tau = \Gamma_1 t$, $A_1(\tau) = |a_1|^2$, $\Delta_1 \equiv \Delta = \omega - \Omega(\mathbf{k})$, where $\Omega(\mathbf{k})$ is the system's natural frequency corresponding to the unstable harmonic.

The solution of Eq. (9) is well known:

$$ A_1(\tau) = A_\infty A_0 e^{2\tau} / [A_\infty + A_0(e^{2\tau} - 1)], \tag{11} $$

where $A_0 = A_1(\tau = 0)$, while the steady-state value of the square of the amplitude of the unstable mode will be

$$ A_\infty = -\frac{\Gamma_1\Gamma_2}{V_1V_3}\left[1 + \left(\frac{\Delta_2}{\Gamma_2}\right)^2\right]. \tag{12} $$

Taking (4) and the instability condition $\Gamma_1 > 0$ into account, we obtain from (12) the necessary condition for the stabilization of the unstable mode:

$$ \Gamma_2 < 0, \tag{13} $$

which amounts to the requirement that the second harmonic be damped, and is physically perfectly natural. The condition (13), like (4), is a consequence of the energy conservation law.

Notice that when (13) is fulfilled the conditions we are considering correspond to a regime of self-oscillations: energy pumping occurs in the oscillations with a wave vector satisfying the linear buildup condition, while on a smaller scale energy is dissipated. The solution (11) also shows that the condition (13) is sufficient, since the steady-state value (12) is attained for any (nonvanishing) initial value of the amplitude[1)]. It also follows from (11) that the characteristic time of emergence to the steady state $\sim\Gamma_1^{-1}$, the steady-state value of the amplitude being proportional to the square root of the increment, which, in turn, is proportional to the supercriticality of the system. It is also easy to derive from the Eqs. (2) that

$$ |a_2|^2 \sim A_1^2 \ll A_1. $$

It can be shown that the square of the amplitude of the n-th harmonic $A_n \sim \epsilon^n$ ($\epsilon \ll 1$ is the supercriticality of the system), which justifies the neglect of the contribution of the higher harmonics.

From Eq. (10) we obtain for the stationary-wave frequency shift

$$ \Delta = -\frac{\Gamma_1}{\Gamma_2}\left[\Delta_2 + \frac{V_2}{V_1V_3}(\Delta_2^2 + \Gamma_2^2)\right]. \tag{14} $$

It can be seen that the frequency shift is determined by the dispersion of the natural plasma oscillation and the cubic nonlinearity of the dynamical equations. In the particular case of the linear dispersion law, when $\Omega(n\mathbf{k}) = n\Omega(\mathbf{k})$ and the harmonics are natural oscillations of the system, we have

$$ \Delta_2 = 2\Delta, \tag{15} $$

and from (14) and (12)

$$ \Delta = \frac{V_2A_\infty}{1 + 2\Gamma_1/\Gamma_2}. \tag{16} $$

Hence it follows, in particular, that in the approximation being considered there is no frequency shift for waves with a linear dispersion law under conditions when the cubic nonlinearity can be neglected. In this case the stabilization of the instability is completely due to the transfer of energy from the unstable mode to its damped higher harmonics[4-7].

It also follows from (12) and (14) that for $|\Delta_2| \gg |\Gamma_2|$ the steady-state value of the wave amplitude is essentially determined by the frequency shift. Such a situation is realized, for example, in the instability of a current-carrying plasma executing ion Langmuir oscillations[8].

3. Let us illustrate the effectiveness of the obtained relations on a concrete example.

Let us consider the instability of a beam-plasma system in the case of a tenuous and hot (with a larger velocity spread) beam (b) and a cold plasma (p):

$$ N_{0b}/N_{0p} \ll (v_{Tb}/u)^2, \quad \nu_b \gg v_{Tb}\omega_p/u, \quad \omega_p > \nu_p. \tag{17} $$

Here u is the directed velocity of the beam electrons, v_{Tb} and ν_b are their thermal velocity and collision rate, ω_p is the Langmuir frequency of the plasma electrons, and v_{Tp} and ν_p are their thermal velocity and collision rate. The potential oscillations of such a system can be described on the basis of the hydrodynamics equations

$$ \partial\mathbf{v}_p/\partial t + (\mathbf{v}_p\nabla)\mathbf{v}_p = -\frac{e}{m}\mathbf{E} - \nu_p\mathbf{v}_p, $$
$$ e\mathbf{E} + T_b\nabla N_b/N_b + m\nu_b\mathbf{v}_b = 0, $$
$$ \partial N_\alpha/\partial t + \operatorname{div} N_\alpha\mathbf{v}_\alpha = 0, \qquad \operatorname{div}\mathbf{E} = 4\pi\sum_\alpha e_\alpha N_\alpha. \tag{18} $$

Here N_α and $\mathbf{v}_\alpha$ are the density and hydrodynamic velocity (α = p, b) and $\mathbf{E}$ is the oscillation field. In the linear approximation, we obtain from (18) for perturbations of the form $\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})$ the following oscillation spectrum ($\omega \to \omega + i\gamma$):

$$\omega^2 \approx \omega_p^2, \qquad \gamma = \nu_b \frac{\omega_p^2 \omega_b^2}{2k^4 v_{Tb}^4}\left(\frac{\mathbf{ku}}{\omega_p} - 1\right) - \frac{\nu_p}{2}. \qquad (19)$$

It is not difficult to determine from (19) the excitation threshold for a beam of Langmuir plasma oscillations:

$$u_* = \frac{4}{3} v_{Tb}\left(3\frac{\nu_p}{\nu_b}\frac{N_{0p}}{N_{0b}}\right)^{1/4}, \qquad k_* = \frac{4}{3}\frac{\omega_p}{u_*}, \qquad (20)$$

the maximum increment corresponding to the perturbations that propagate in the direction of motion of the beam. The increment can be expressed with the aid of (20) in terms of the supercriticality ϵ:

$$\gamma = 2\nu_p \varepsilon, \qquad \varepsilon = u / u_* - 1. \qquad (21)$$

Let us now consider the nonlinear phase of the development of the instability. For a low-density beam it is sufficient to take into account in (18) only the nonlinearities due to the perturbations in the plasma. Moreover, near the threshold ($\epsilon \ll 1$) we can restrict ourselves to the analysis of the mode with the maximum increment and go over to the one-dimensional case. The system of equations (18) then reduces to a single nonlinear equation for the perturbation of the hydrodynamic velocity of the plasma electrons:

$$\frac{\partial^4}{\partial x^4}\left\{\frac{\partial^2 v}{\partial t^2} + \omega_p^2 v + \nu_p \frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v^2}{\partial t\,\partial x} + v\frac{\partial^2 v}{\partial t\,\partial x} + \frac{\mathrm{v}}{2}\frac{\partial^2 v^2}{\partial x^2}\right\}$$
$$-\frac{\nu_b \omega_b^2}{v_{Tb}^4}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\frac{\partial^2 v}{\partial t^2} = 0. \qquad (22)$$

Let us seek near the instability threshold, where $0 < \epsilon \ll 1$, the solution of Eq. (22) in the form

$$v = v_1(t)\exp[i(k_* x - \omega t)] + v_2(t)\exp[2i(k_* x - \omega t)] + \text{к.с.},$$

where $v_{1,2}(t)$ is a slowly varying amplitude and ω is a real frequency, for the determination of which we derive a system of equations of the type (2) with the coefficients

$$\Gamma_1 = 2\nu_p\varepsilon, \quad \Gamma_2 = -{}^3/_8\nu_p, \quad \Delta_2 = {}^3/_4\omega_p,$$
$$V_1 = 3k_*, \quad V_2 = -k_*^2/\omega_p, \quad V_3 = {}^3/_4 k_*. \qquad (23)$$

It can be seen from (23) that $\Gamma_2 < 0$, i.e., the condition (13) for the existence of a stationary wave is fulfilled and, moreover, $|\Delta_2| \gg |\Gamma_2|$, i.e., the value of the steady-state amplitude is determined by the nonlinear frequency shift. Using the definitions (12) and (14), we find the characteristics of the stationary wave:

$$|v_1|^2 = {}^3/_4 u_*^2 \varepsilon, \qquad \omega = \omega_p(1 + {}^8/_3\varepsilon). \qquad (24)$$

The time dependence of the amplitude of the unstable wave is described by the expression (11).

In conclusion, let us note that there arises in the wave field an averaged motion of the plasma with velocity $v_0 = -{}^8/_3\epsilon\omega_p/k_*$. The solution found above has been written in the moving coordinate system and the obtained frequency shift is due to the Doppler effect; in the stationary coordinate system the frequency is equal to $\omega' = \omega + k_* v_0 = \omega_p$[14].

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1) The condition (13) corresponds to a soft excitation regime. If it is not fulfilled (the soft regime), then an additional analysis of the nonlinearities of the dynamical equations is needed [13].

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