Interaction between solitons in a stable medium

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The nonlinear equation (1) is solved exactly by the inverse scattering problem method. Interaction between solitons is studied within the framework of the equation, the soliton shifts due to their collisions with each other are calculated, and it is shown that only paired collisions occur. The results are applied to the problem of diffraction in a nonlinear defocusing medium.

INTRODUCTION

The equation

$$iu_t + u_{xx} - \varkappa |u|^2 u = 0 \tag{1}$$

is encountered in various physical problems. Gross[1] and Pitaevskiĭ[2], and later on Tsuzuki[3], used this equation to describe the oscillations of a Bose gas at zero temperature. The problem of the evolution of the complex envelope of a monochromatic wave in a nonlinear medium[4,5] also leads to Eq. (1). Finally, interpreting t as a longitudinal coordinate, Eq. (1) can be treated as a two-dimensional variant of the well known "parabolic" equation[6] that describes stationary wave beams in a nonlinear medium.

Equation (1) belongs to a class of equations that can be solved exactly by the method of the inverse scattering problem. This was demonstrated by the present authors[8] in a detailed study of the case $\kappa < 0$ corresponding, when dealing with a Bose gas, to attraction of its particles. In this case the trivial spatially-homogeneous solution of Eq. (1) of the type $u = A\exp(-i\kappa|A|^2 t)$, which we shall call a condensate in analogy with the case of the Bose gas, is unstable. In the nonlinear-optics interpretation of Eq. (1) the condensate has a meaning of a stationary monochromatic wave, and the instability of the condensate at $\kappa < 0$ corresponds to self-modulation instability of a monochromatic wave[7].

In the present paper we study the case $\kappa > 0$ corresponding to repulsion of Bose-gas particles and to stability of a monochromatic wave relative to self-modulation. In the "spatial" interpretation of the parameter t, the case $\kappa > 0$ corresponds to the propagation of a wave beam in a defocusing medium. As will be shown in the present paper, reversal of the sign of κ leads not only to a change in the physical picture of the phenomena described by Eq. (1), but also to a considerable restructuring of the mathematical formalism necessary for its solution.

When the method of the inverse scattering problem is used[9,10], the initial nonlinear equation is associated with a linear differential operator $\hat{L}$, which in our case takes the form (see[7])

$$\hat{L} = i\begin{bmatrix} 1+p & 0 \\ 0 & 1-p \end{bmatrix}\frac{\partial}{\partial x} + \begin{bmatrix} 0 & u^* \\ u & 0 \end{bmatrix}, \quad \varkappa = \frac{2}{p^2-1}. \tag{2}$$

The sought function u is a coefficient of this operator. It can be shown that the scattering matrix of the operator $\hat{L}$ varies in time in accordance with a very simple law, and the solution of (1) reduces to a reconstruction of the "potential" u(x, t) from a given scattering matrix (the solution of the inverse scattering problem). This problem was solved for the operator $\hat{L}$ in[8] under the condition $u(\pm\infty) = 0$. Such a formulation is natural for the case of attraction. Of greater physical interest for the stable case is $|u(x, t)|^2 \to \text{const}$, $u_x \to 0$ as $x \to \pm\infty$, corresponding to the propagation of waves through a condensate of constant density. In such a condensate the soliton

$$\sqrt{\frac{\varkappa}{2}}\,u(x,t) = \frac{(\lambda+i\nu)^2 + \exp\{2\nu(x-x_0-2\lambda t)\}}{1+\exp\{2\nu(x-x_0-2\lambda t)\}}, \quad \nu = \sqrt{1-\lambda^2} \tag{3}$$

can move with constant velocity. In this case

$$\frac{\varkappa}{2}|u(x,t)|^2 = 1 - \frac{\nu^2}{\operatorname{ch}^2\nu(x-x_0-2\lambda t)}.$$

The parameter λ characterizes the amplitude and velocity of the soliton, and x_0 is the position of its center at t = 0.

Solitons play the same role in Eq. (1) as in the Korteveg—de Vries (KdV) equation[9,11,12]. They are directly connected with the discrete spectrum of the operator $\hat{L}$, namely, each soliton corresponds to a time-independent eigenvalue of the operator. Just as in the KdV case, the solitons determine the asymptotic behavior of an arbitrary initial condition as $t \to \infty$ and behave in simple fashion when scattered by each other.

In Secs. 1 and 2 of the present paper we consider the direct and inverse scattering problems for the operator $\hat{L}$ at $\kappa > 0$. The condition $|u(\pm\infty)|^2 = \text{const}$ calls for the study of the analyticity of the scattering matrix on a two-sheet Riemann surface, and this makes the theory of the inverse problem much more complicated in comparison with the case $u(\pm\infty) = 0$. The results of Secs. 1 and 2 are used in Sec. 3 to study soliton collisions. The scattering of solitons is considered and the shifts of the centers of the solitons after their collisions are calculated. Unlike Tsuzuki[3], we calculate not only the relative values of the shifts, which follow from the conservation law for the mass center, but also their absolute magnitudes. We prove also the additivity of the shifts in collisions of a large number of solitons. In Sec. 4 we consider the problem of reflection of a soliton from a wall on which the wave function of the condensate vanishes.

In Sec. 5 we apply the results to the problem of the diffraction of a plane wave by an opaque band in a nonlinear defocusing medium.

1. THE DIRECT SCATTERING PROBLEM

We consider the problem of the eigenvalues of the operator $\hat{L}$:

$$\hat{L}\chi = E\chi, \quad \chi = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}.$$

We make the change of variables

$$\chi_1=(p-1)^{1/2}\exp\left\{i\frac{E}{p^2-1}x\right\}v_1,\quad \chi_2=(p+1)^{1/2}\exp\left\{i\frac{E}{p^2-1}x\right\}v_2. \tag{4}$$

The system $\hat{L}\chi = E\chi$ is reduced by it to the form

$$i\frac{\partial v_1}{\partial x}+q^*v_2=\lambda v_1,\quad \lambda=\frac{pE}{p^2-1},$$
$$-i\frac{\partial v_2}{\partial x}+qv_1=\lambda v_2,\quad q=\frac{u}{(p^2-1)^{1/2}}. \tag{5}$$

We assume that $|q| \to 1$ as $x \to \pm\infty$. Generally speaking, in this case $q \to e^{i\alpha_\pm}$ as $x \to \pm\infty$. However, the change of variables $q \to qe^{-i\alpha_+}$, $v_1 \to v_1$, $v_2 \to v_2 e^{-i\alpha_+}$ makes it possible to put $q \to 1$ as $x \to +\infty$. Then $q \to e^{i\alpha}$ as $x \to -\infty$. The quantity $\alpha = \alpha_- - \alpha_+$, as will be shown below, does not depend on the time. The asymptotic solution of the system (5) as $x \to \infty$ is given by

$$v \to c_1^+X_1^+ + c_2^+X_2^+,$$
$$X_1^+=e^{-i\zeta x}\begin{bmatrix}1\\ \zeta-\lambda\end{bmatrix},\quad X_2^+=e^{i\zeta x}\begin{bmatrix}\zeta-\lambda\\ 1\end{bmatrix}. \tag{6}$$

Here $\zeta(\lambda) = (\lambda^2-1)^{1/2}$ is a double-valued function of λ. Analogously, as $x \to -\infty$ we have

$$v \to c_1^-X_1^- + c_2^-X_2^-,$$
$$X_1^-=e^{-i\zeta x}\begin{bmatrix}1\\ e^{i\alpha}(\zeta-\lambda)\end{bmatrix},\quad X_2^-=e^{i\zeta x}\begin{bmatrix}e^{-i\alpha}(\zeta-\lambda)\\ 1\end{bmatrix}. \tag{7}$$

We determine, for real ζ, the Jost functions ψ_1 and ψ_2 as solutions of the system (5) with asymptotic form

$$\psi_1 \to X_1^+,\quad \psi_2 \to X_2^+ \quad \text{as} \quad x \to +\infty.$$

Analogously

$$\varphi_1 \to X_1^-,\quad \varphi_2 \to X_2^- \quad \text{as} \quad x \to -\infty.$$

We assume that the Jost function ψ_1 can be expressed in the triangular representation

$$\psi_1(x,\lambda)=X_1^+(x,\lambda)-\int_x^\infty \hat{\Psi}(x,s)X_1^+(s,\lambda)\,ds. \tag{8}$$

Substituting (8) in (5) we verify, after transformations, that the matrix $\hat{\Psi}(x, y)$ should satisfy the following system of partial differential equations

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\begin{bmatrix}\Psi_{11}(x,y)\\ \Psi_{22}(x,y)\end{bmatrix}=i\begin{bmatrix}1 & -q^*(x)\\ q(x) & -1\end{bmatrix}\begin{bmatrix}\Psi_{12}(x,y)\\ \Psi_{21}(x\ y)\end{bmatrix},$$
$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)\begin{bmatrix}\Psi_{12}(x,y)\\ \Psi_{21}(x,y)\end{bmatrix}=i\begin{bmatrix}1 & -q^*(x)\\ q(x) & -1\end{bmatrix}\begin{bmatrix}\Psi_{11}(x,y)\\ \Psi_{22}(x\ y)\end{bmatrix}, \tag{9}$$

with boundary conditions

$$2\Psi^*_{12}(x,\ x)=2\Psi_{21}(x,\ x)=i(q(x)-1);$$
$$\Psi_{jk}(x,\ y)\to 0 \quad \text{as} \quad y\to+\infty. \tag{9a}$$

It can be shown (cf.[13]) that the system (9) with these boundary conditions can be uniquely solved, from which follows the existence of the representation (8). The system (9) has a symmetry, by virtue of which

$$\Psi_{21}=\Psi^*_{12},\quad \Psi_{22}=\Psi^*_{11}. \tag{10}$$

The system (5) is invariant with respect to the involution

$$u=\begin{pmatrix}u_1\\ u_2\end{pmatrix}\to\tilde{u}=\begin{pmatrix}u_2^*\\ u_1^*\end{pmatrix}, \tag{11}$$

in particular, $X_2 = \tilde{X}_1$ and $\psi_2 = \tilde{\psi}_1$. This circumstance, as well as the relations (10), enable us to find a triangular representation for ψ_2. It is effected with the same matrix $\hat{\Psi}(x, y)$ as used for ψ_1.

The scattering problem for the system (5) consists of constructing a matrix S that transforms the asymptotic form as $x \to -\infty$ into the asymptotic form as $x \to +\infty$. In the notation of (6) and (7) we have

$$\begin{bmatrix}c_1^+\\ c_2^+\end{bmatrix}=\hat{S}\begin{bmatrix}c_1^-\\ c_2^-\end{bmatrix}. \tag{12}$$

From the invariance to involution, it follows that

$$S_{11}=S^*_{22}=a,\quad S_{21}=S^*_{12}=b.$$

The asymptotic relation (12) can be rewritten in terms of relations between the Jost functions

$$\varphi_1=a\psi_1+b\psi_2, \tag{13}$$
$$\varphi_2=a^*\psi_2+b^*\psi_1. \tag{13a}$$

We denote the Wronskian of two functions by $\{u, v\} = u_1v_2 - u_2v_1$. From (13) we have

$$a(\lambda,\zeta)=\frac{\{\varphi_1,\psi_2\}}{\{\psi_1,\psi_2\}}=-\frac{\{\varphi_1,\psi_2\}}{2\zeta(\zeta-\lambda)}. \tag{14}$$

If u and v are the solutions of the system (5), then their Wronskian does not depend on x. From (13) we obtain

$$\{\varphi_1,\ \varphi_2\}=\det\hat{S}\{\psi_1,\ \psi_2\}.$$

Since $\{\varphi_1, \varphi_2\} = \{\psi_1, \psi_2\} = -2\zeta(\zeta-\lambda)$, it follows that

$$\det\hat{S}=|a|^2-|b|^2=1. \tag{15}$$

So far we defined the Jost functions for real λ and ζ. The function $\zeta(\lambda)$ is defined on a two-sheet Riemann surface with cuts $(-\infty, -1)$ and $(1, \infty)$. On the upper sheet of the surface we have $\mathrm{Im}\,\zeta > 0$ and on the lower $\mathrm{Im}\,\zeta < 0$. From the triangular representation for the function ψ_2

$$\psi_2(x,\lambda)=X_2^+(x,\lambda)-\int_x^\infty\hat{\Psi}(x,s)X_2^+(s,\lambda)\,ds, \tag{16}$$

it follows that this function can be analytically continued to the upper sheet of the Riemann surface, and has as $|\lambda| \to \infty$ the asymptotic form

$$\psi_2(x,\ \lambda)\to X_2^+(x,\ \lambda)[1+O(1/|\lambda|)]. \tag{17}$$

Analogous properties are possessed by the Jost function $\varphi_1(x, \lambda)$. Its asymptotic form is

$$\varphi_1(x,\ \lambda)\to X_1^-(x,\ \lambda)[1+O(1/|\lambda|)]. \tag{18}$$

The functions ψ_1 and ψ_2 are analytic on the lower sheet of the Riemann surface.

It follows from (14) that $a(\lambda, \zeta)$ is analytic on the upper sheet of the Riemann surface. The zeros of a correspond to the eigenvalues of the system (5). From the self-adjoint character of the system and from relation (15) it follows that they lie on the segment $-1 < \lambda < 1$ of the real axis and are simple. From (14), (17), and (18) it follows that as $|\lambda| \to \infty$ the coefficient $a(\lambda, \zeta)$ behaves asymptotically like

$$a(\lambda,\zeta)\approx[(\zeta-\lambda)^2e^{i\alpha}-1]/2\zeta(\zeta-\lambda). \tag{19}$$

Of particular interest is the case $b(\lambda) \equiv 0$. Then the coefficient $a(\lambda, \zeta)$ is determined completely by its zeros λ_k ($k = 1, \ldots, N$) and is a meromorphic function on the complete Riemann surface. Indeed, in this case we have $|a|^2 = 1$ on the cuts and $a(\lambda)$ can be continued to the lower sheet in accordance with the symmetry principle

$$a(\lambda^*,\ \zeta^*)=1/a^*(\lambda,\ \zeta). \tag{20}$$

In the lower sheet, at the points λ_j, the coefficient a has simple poles. This circumstance and the condition (19) at infinity enable us to determine the form of $a(\lambda, \zeta)$:

$$a(\lambda,\zeta)=\prod_{j=1}^N\frac{\zeta-\zeta_j+\lambda-\lambda_j}{\zeta-\zeta_j^*+\lambda-\lambda_j},\quad \zeta_j=i[1-\lambda_j^2]^{1/2}=i\nu_j. \tag{21}$$

From a comparison of (19) and (21) it follows that the total change of the phase of the condensate q(x) is determined by the distribution of the eigenvalues

$$e^{i\alpha}=\prod_{j=1}^{N}\frac{\lambda_j+i\nu_j}{\lambda_j-i\nu_j}. \qquad (22)$$

The use of the operator $\hat{L}$ for the solution of Eq. (1) is based on the identity of this equation with the operator relation

$$\partial\hat{L}/\partial t=i[\hat{L},\hat{A}], \qquad (23)$$

where (see[8])

$$\hat{A}=-p\begin{bmatrix}1&0\\0&1\end{bmatrix}\frac{\partial^2}{\partial x^2}+\begin{bmatrix}\frac{|u|^2}{1+p} & iu_x^* \\ -iu_x & \frac{|u|^2}{p-1}\end{bmatrix}.$$

From (23) we find that the eigenfunctions χ of the operator $\hat{L}$ satisfy the equation

$$i\partial\chi/\partial t=\hat{A}\chi, \qquad (24)$$

which goes over as $|x|\to\infty$ into an equation with constant coefficients. In (24) we obtain for v_1 at large $|x|$

$$i\frac{\partial v_1}{\partial t}=\left(\frac{\lambda^2}{p}+\frac{1}{1+p}\right)v_1-2i\lambda\frac{\partial v}{\partial x_1}-p\frac{\partial^2 v_1}{\partial x^2},$$

from which it follows that

$$a(\lambda,t)=a(\lambda,0)=\text{const},\quad b(\lambda,t)=b(\lambda,0)e^{-4i\lambda\zeta t}. \qquad (25)$$

The eigenvalues λ_i, as the zeros of $a(\lambda,\zeta)$, do not depend on the time.

It is seen from (14) that $\lambda=\lambda_i$ the functions φ_1 and ψ_2 are proportional to each other

$$\varphi_1(x,\lambda_i)=b_i\psi_2(x,\lambda_i).$$

In those cases when $b(\lambda)$ can be analytically continued in the vicinity of the point λ_i, we have $b_i=b(\lambda_i)$. This enables us to find the dependence of b_i on the time:

$$b_i(t)=b_i(0)\exp\{4\lambda_i\nu_i t\}. \qquad (26)$$

The independence of the coefficient a of the time shows that Eq. (1) has a continual set of conservation laws. It is possible to distinguish among them a denumerable set of conservation laws that are expressed in explicit form in terms of q(x).

We consider the asymptotic form of the quantity

$$\ln a(\lambda,\zeta)\approx\sum_{k=1}^{\infty}\frac{I_k}{(2i\lambda)^k}.$$

as $|\lambda|\to\infty$ and at $\operatorname{Re}\lambda>0$. The coefficients I_k can be found directly from the system (5). To this end we make the change of variables

$$e^{\Phi}=\varphi_{1,1}(x,\lambda)e^{i\lambda x}.$$

From the system (5) we get an equation for Φ:

$$q^*\frac{d}{dx}\frac{1}{q^*}\Phi'+\Phi'^2-|q|^2=2i\lambda\Phi'. \qquad (27)$$

From (27) we easily obtain an asymptotic series for Φ':

$$\Phi'(x,\lambda)=\sum_{k=1}^{\infty}\frac{f_k(x)}{(2i\lambda)^k},$$

where the quantities $f_k(x)$ are determined by the recurrence formula

$$f_n=q^*\frac{d}{dx}\frac{1}{q^*}f_{n-1}+\sum_{j+k=n-1}f_jf_k,\quad f_1=-|q|^2.$$

It is obvious that

$$\ln a(\lambda,\zeta)=\int_{-\infty}^{\infty}[\Phi'(x,\lambda)-\Phi'(-\infty,\lambda)]dx,$$

from which we get

$$I_n=\int_{-\infty}^{+\infty}[f_n(x,\lambda)-f_n(-\infty,\lambda)]dx.$$

All the quantities I_n are conservation laws of (1).

We present in explicit form the first field I_n:

$$I_1=-\int_{-\infty}^{+\infty}(|q|^2-1)dx,$$

$$I_2=\frac{1}{2}\int_{-\infty}^{+\infty}\left\{q\frac{dq^*}{dx}-q^*\frac{dq}{dx}\right\}dx,$$

$$I_3=\int_{-\infty}^{+\infty}\left\{|q|^4+\left|\frac{dq}{dx}\right|^2-1\right\}dx,$$

$$I_4=\int_{-\infty}^{+\infty}\left\{q^*\frac{d}{dx}\left(|q|^2q-\frac{d^2}{dx^2}q\right)+2|q|^2q^*\frac{dq}{dx}\right\}dx,$$

$$I_5=-\int_{-\infty}^{+\infty}\left\{2|q|^6+6|q|^2\left|\frac{dq}{dx}\right|^2+\left(\frac{d}{dx}|q|^2\right)^2+\left|\frac{d^2q}{dx^2}\right|^2-2\right\}dx.$$

The first three quantities I_1, I_2, and I_3 have, apart from the numerical coefficients, the meaning of the conservation laws for the particle number, for the momentum, and for the energy of the Bose gas. The remaining conservation laws have no simple physical meaning.

2. THE INVERSE SCATTERING PROBLEM

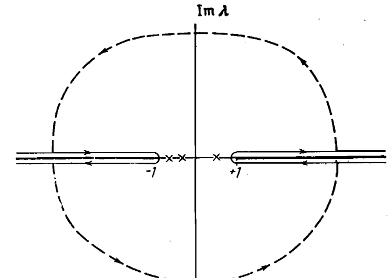
Formulas (25) and (26) enable us to calculate the scattering matrix at an arbitrary instant of time from the initial data. This yields the quantity u(x, t) at an arbitrary instant of time, provided that the inverse scattering problem is solved for the system (5). In the present section we construct integral equations (Marchenko equations) which make it possible to determine the potential q(x) from the scattering data.

We start from relation (13), which we rewrite in the form

$$\frac{1}{2\pi\zeta}\left(\frac{1}{a}\varphi_1-X_1^+\right)e^{i\zeta y}=\left(\psi_1-X_1^++\frac{b}{a}\psi_2\right)e^{i\zeta y}\frac{1}{2\pi\zeta},\quad y>x. \qquad (28)$$

The left-hand side of (28) is analytic on the upper sheet of the Riemann surface, with the exception of the points λ_n, at which it has simple poles. As $|\lambda|\to\infty$, the left-hand side of (28) behaves like $\exp\{-\operatorname{Im}\zeta(y-x)\}O(1/|\lambda|)$.

We integrate relation (28) along the contour indicated in the figure. In the integration of the left-hand side, the contour can be closed through infinity, so that the integral of the left-hand side is equal to the sum of the residues in λ_n:



$$\frac{1}{2\pi}\int\frac{1}{\zeta}\left(\frac{1}{a}\varphi_1 - X_1^+\right)e^{i\zeta y}\,d\lambda = \sum\frac{\varphi_1(x,\lambda_n)}{\nu_n a'(\lambda_n, i\nu_n)} = \sum b_n\frac{\psi_2(x,\lambda_n)}{\nu_n a'(\lambda_n, i\nu_n)}e^{-\nu_n y}. \tag{29}$$

The integral of the right-hand side can be transformed into

$$\frac{1}{2\pi}\int\frac{e^{i\zeta y}}{\zeta}\left(\psi_1 - X_1^+ + \frac{b}{a}\psi_2\right)d\lambda$$
$$= \hat{\Psi}(x,y)\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}-F_1^{(1)}(x+y) + iF_2^{(1)'}(x+y)\\ F_2(x+y)\end{bmatrix} \tag{30}$$
$$-\int_x^\infty\hat{\Psi}(x,s)\begin{bmatrix}-F_1^{(1)}(s+y) + iF_2^{(1)'}(s+y)\\ F_2(s+y)\end{bmatrix}ds.$$

Here

$$F_1^{(1)}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}c_1(\xi)e^{-i\xi x}d\xi, \quad F_2^{(1)}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}c_2(\xi)e^{-i\xi x}d\xi,$$

$$c_1(\zeta) = \frac{c(\lambda,\zeta) + c(-\lambda,\zeta)}{2}, \quad c_2(\zeta) = \frac{c(\lambda,\zeta) - c(-\lambda,\zeta)}{2\lambda},$$

$$c = b/a.$$

From the system (5) (see (13a)) we get directly

$$\varphi_1(\lambda,-\zeta) = \frac{e^{i\alpha}}{\zeta - \lambda}\varphi_2(\lambda,\zeta), \quad \psi_1(\lambda,-\zeta) = \frac{1}{\zeta-\lambda}\psi_2(\lambda,\zeta).$$

Substituting these formulas in (13) and (13a), we readily obtain

$$a(\lambda,-\zeta) = a^*(\lambda,\zeta)e^{i\alpha}, \quad b(\lambda,-\zeta) = b^*(\lambda,\zeta)e^{i\alpha},$$

whence

$$c(\lambda,-\zeta) = c^*(\lambda,\zeta), \quad c_{1,2}(-\zeta) = c_{1,2}^*(\zeta). \tag{31}$$

It follows from (31) that the functions $F_{1,2}^{(1)}(z)$ are real.

Using the triangular representation for ψ_2 in (29), we observe that (29) reduces to the form (30) if $F_{1,2}^{(1)}(z)$ is replaced by $F_{1,2}^{(2)}(z)$, where

$$F_1^{(2)}(z) = -\sum\mu_n\lambda_n\exp(-\nu_n z), \quad F_2^{(2)}(z) = -\sum\mu_n\exp(-\nu_n z),$$
$$\mu_n = b_n/\nu_n a'(\lambda_n, i\nu_n). \tag{32}$$

Combining both parts of (28), we obtain a system of two integral equations (Marchenko equations) for the quantities Ψ_{11} and Ψ_{12}, which determine the triangular representation of the Jost functions ψ_1 and ψ_2:

$$\Psi_{12}(x,y) - \int_x^\infty\{\Psi_{12}(x,s)F_2(s+y) + \Psi_{11}(x,s)[-F_1(s+y) + iF_2'(s+y)]\}ds$$
$$= F_1(x+y) - iF_2'(x+y),$$
$$\Psi_{11}(x,y) = -\int_x^\infty\{\Psi_{11}(x,s)F_2(s+y) \tag{33}$$
$$+\Psi_{12}(x,s)[-F_1(s+y) - iF_2'(s+y)]\}ds = -F_2(x+y).$$

Here

$$F_1(z) = F_1^{(1)}(z) + F_1^{(2)}(z), \quad F_2(z) = F_2^{(1)}(z) + F_2^{(2)}(z).$$

From (25) and (26) we can obtain the time dependences of the functions $F_{1,2}$. It is easy to verify that these functions satisfy a system of linear requations

$$\frac{\partial F_1}{\partial t} + 4\frac{\partial F_2}{\partial z} = 4\frac{\partial^3 F_2}{\partial z^3}, \quad \frac{\partial F_2}{\partial t} + 4\frac{\partial F_1}{\partial z} = 0. \tag{34}$$

The system (33) contains the solution of the inverse scattering problem for the operator L.

In the particular case when $b(\lambda) \equiv 0$, $F_{1,2}^{(1)} \equiv 0$, the integral equations are degenerate, and the system (33) reduces to a system with a finite number of linear algebraic equations (2N if N is the number of eigenvalues). In the simplest case of a single eigenvalue λ, we have

$$F_1(z) = -\mu\lambda e^{-\nu z}, \quad F_2(z) = -\mu e^{-\nu z}, \quad \nu = (1-\lambda^2)^{1/2}.$$

Solving the system (33), we get

$$\Psi_{12}(x,y) = -\frac{\nu(\lambda + i\nu)}{1 + (\nu/\mu)e^{2\nu x}}e^{\nu(x-y)},$$
$$q(x) = 1 - 2i\Psi_{12}^*(x,x) = \frac{(\lambda + i\nu)^2 + (\nu/\mu)e^{2\nu x}}{1 + (\nu/\mu)e^{2\nu x}}. \tag{35}$$

We note that in accordance with (26) and (32) we have $d\ln\mu/dt = 4\lambda\nu$. Introducing

$$\ln\mu = 2\nu(x_0 + 2\lambda t) \tag{36}$$

and comparing (35) with (2), we verify that the eigenvalue λ corresponds to a soliton moving with velocity 2λ. Unlike the KdV case, the velocity can be of arbitrary sign.

We denote the single-soliton solution (35) by $q_0(x - 2\lambda t, \lambda, x_0)$. In the more general case when $b(\lambda) \equiv 0$ and there are N eigenvalues, the system (33) determines a 2N-parameter particular solution of Eq. (1), which can be expressed in explicit form. Just as in[8,11], we shall call this an N-soliton solution.

3. INTERACTION OF SOLITONS

Let us consider the problem of the interaction of two solitons with velocities $2\lambda_1$ and $2\lambda_2$. To this end it is necessary to study the corresponding two-soliton solution. Owing to the complexity of the explicit formula for this solution, we confine ourselves to an investigation of its asymptotic behavior as $t \to \pm\infty$.

Proceeding as in[11], we verify that as $t \to \pm\infty$ the two-soliton solution breaks up into individual solitons:

$$q(x,t) \to q_0(x - 2\lambda_1 t, \lambda_1, x_1^+) + q_0(x - 2\lambda_2 t, \lambda_2, x_2^+), \quad t \to +\infty,$$
$$q(x,t) \to q_0(x - 2\lambda_1 t, \lambda_1, x_1^-) + q_0(x - 2\lambda_2 t, \lambda_2, x_2^-), \quad t \to -\infty.$$

The scattering of the solitons by each other gives rise to shifts of their centers

$$\delta x_1 = x_1^+ - x_1^-, \quad \delta x_2 = x_2^+ - x_2^-.$$

Let us calculate these shifts. We note first that, as seen from (35), each soliton with velocity $2\lambda_i$ represents a discontinuity in the phase of the wave function of the condensate

$$\arg q(-\infty) - \arg q(+\infty) = 2\,\mathrm{arctg}(\nu_i/\lambda_i) = \alpha_i.$$

The total phase discontinuity in the presence of N eigenvalues is

$$\alpha = \sum_{i=1}^N\alpha_i.$$

It is interesting to note that it does not depend on the relative positions of the solitons.

Returning to the case of two solitons, we consider the Jost function $\varphi_1(x,\lambda)$. As $x \to -\infty$, it takes the asymptotic form

$$\varphi_1(x,\lambda) \to e^{\nu x}\begin{bmatrix}1\\ \exp\{i(\alpha_1+\alpha_2)\}(\nu - \lambda)\end{bmatrix}, \quad -1 < \lambda < 1.$$

The asymptotic form of $\varphi_1(x,\lambda)$ as $x \to +\infty$ is

$$\varphi_1(x,\lambda) \to a(\lambda, i\nu)e^{\nu x}\begin{bmatrix}1\\ i\nu - \lambda\end{bmatrix}, \quad \lambda \neq \lambda_1, \lambda_2, \tag{37}$$

$$\varphi_1(x,\lambda) \to b_{1,2}(t)e^{-\nu x}\begin{bmatrix}i\nu_{1,2} - \lambda_{1,2}\\ 1\end{bmatrix}, \quad \lambda = \lambda_1, \lambda_2.$$

We represent the coefficient a in the form $a = a_1a_2$ (see (21)), where

$$a_{1,2}(\lambda,\nu) = \frac{i\nu + \lambda - i\nu_{1,2} - \lambda_{1,2}}{i\nu + \lambda + i\nu_{1,2} + \lambda_{1,2}}$$

is the component of the "single-soliton" scattering matrix.

Let $\lambda_1 > \lambda_2$. As $t \to -\infty$, the solitons move apart to large distances and $|q| \to 1$ in the region between solitons. As $t \to -\infty$, soliton 2 (with velocity $2\lambda_2$) is located on the right, so that $q \to \exp(i\alpha_2)$ in the region between the solitons. As $t \to \infty$, soliton 1 is located on the right, and $q \to \exp(i\alpha_1)$ in the region between the solitons.

We consider in this region the asymptotic form of the Jost functions $\varphi_1(x, \lambda)$ as $t \to -\infty$. We have

$$\varphi_1 \approx a_1(\lambda, i\nu) e^{\nu x} \begin{bmatrix} 1 \\ e^{i\alpha_2}(i\nu - \lambda) \end{bmatrix}, \quad \lambda \neq \lambda_1,$$

$$\varphi_1 \approx b_1^-(t) e^{-\nu_1 x} \begin{bmatrix} e^{i\alpha_2}(i\nu_1 - \lambda_1) \\ 1 \end{bmatrix} \quad \lambda = \lambda_1.$$

Here $b_1^-(t)$ is a function as yet unknown. To calculate it we note that by virtue of (20) we have $a(\lambda, -i\nu) = 1/a^*(\lambda, i\nu)$. "Passing" the function φ_1 through soliton 2 in accordance with this rule, and comparing with (37), we obtain

$$b_1^-(t) = a_2^*(\lambda_1, i\nu_1) b_1(t)$$

In the same manner we determine the quantity $b_2^-(t) = b_2(t)/a_1(\lambda_2, i\nu_2)$.

The quantities $b_1^-(t)$ and $b_2^-(t)$ determine the positions of the solitons as $t \to -\infty$ (see formulas (32) and (36)). We introduce analogously the quantities $b_1^+(t)$ and $b_2^+(t)$, we determine the positions of the solitons as $t \to \infty$. Reasoning as before, we obtain

$$b_1^+(t) = b_1(t) / a_2(\lambda_1, i\nu_1), \quad b_2^+(t) = b_2(t) a_1^*(\lambda_2, i\nu_2).$$

From (32) and (36) we have for the soliton shifts

$$\delta x_1 = \frac{1}{2\nu_1} \ln \frac{b_1^+}{b_1^-} = \frac{1}{2\nu_1} \ln \frac{1}{|a_2(\lambda_1, i\nu_1)|^2} = \frac{1}{2\nu_1} \ln \frac{(\lambda_1 - \lambda_2)^2 + (\nu_1 + \nu_2)^2}{(\lambda_1 - \lambda_2)^2 + (\nu_1 - \nu_2)^2},$$
$$\delta x_2 = \frac{1}{2\nu_2} \ln \frac{b_2^+}{b_2^-} = \frac{1}{2\nu_2} \ln |a_1(\lambda_2, i\nu_2)|^2 = -\frac{1}{2\nu_2} \ln \frac{(\lambda_1 - \lambda_2)^2 + (\nu_1 + \nu_2)^2}{(\lambda_1 - \lambda_2)^2 + (\nu_1 - \nu_2)^2}. \quad (38)$$

Thus, the soliton 1, which has the larger velocity, acquires a positive shift, and soliton 2 a negative shift. Solitons behave like repelling one-dimensional particles. From (38) we get the relation

$$\nu_1 \delta x_1 + \nu_2 \delta x_2 = 0. \quad (39)$$

This relation was obtained by Tsuzuki[3] directly from Eq. (1) by analyzing the motion of the mass center of a Bose gas. It can be interpreted as a condition for the conservation of the mass center of the solitons during the collision.

We proceed now to the case of N solitons. Reasoning just as in[11], we can show that as $t \to \pm\infty$ an arbitrary N-soliton solution breaks up into individual solitons. The sequence of the solitons as the prime changes from $-\infty$ to $+\infty$ is reversed, so that each soliton collides with each other soliton. Repeating the argument given above, we verify that the total shift of a soliton, regardless of the detailed picture of the collisions, is equal to the sum of the shifts in individual collisions:

$$\delta_i = x_i^+ - x_i^- = \sum_{j \neq i} \delta_{ij},$$
$$\delta_{ij} = \operatorname{sign}(\lambda_i - \lambda_j) \frac{1}{2\nu_i} \ln \frac{(\lambda_i - \lambda_j)^2 + (\nu_i + \nu_j)^2}{(\lambda_i - \lambda_j)^2 + (\nu_i - \nu_j)^2}. \quad (40)$$

An analogous fact was established earlier for the KdV equation[11,12] and for Eq. (1) at $x < 0$[8].

4. REFLECTION OF A SOLITON FROM A BOUNDARY

The entire preceding reasoning pertained to the case of an infinite condensate. It can be extended, however, to include the case of a semi-infinite ($0 < x < \infty$) condensate with a reflecting boundary on which zero boundary conditions are specified. We note for this purpose that the soliton solutions of (1) include a singular "standing" soliton, for which $\lambda = 0$ and $\nu = 1$. The standing solitons, with its center at the origin, is given by

$$q(x) = \operatorname{th} x. \quad (41)$$

Since the wave function of the condensate vanishes at the reflecting boundary, the standing soliton describes (at $x > 0$) the state of the condensate in the presence of a boundary. Formula (41) continues it in antisymmetrical fashion into the region $x < 0$. Such a continuation can be effected for any solution of Eq. (1) satisfying the zero condition on the plane boundary.

The equations (33) of the inverse problem yield an antisymmetrical potential q(x) if the function $F_1^{(1)}(z)$ is odd and the function $F_2^{(1)}(z)$ is even. In addition, there should exist an eigenvalue $\lambda = 0$, $\mu = 1$. The remaining discrete spectrum should be symmetrical, namely, to each λ_n there should correspond $\lambda_{-n} = -\lambda_n$ and $\mu_{-n} = 1/\mu_n$.

This reasoning enables us to calculate the shift of the soliton as it is reflected from the boundary. Such a reflection can be regarded as a double scattering, viz., by a soliton with opposite velocity and by a standing soliton. For a soliton with velocity 2λ, calculating with the aid of formulas (40), we obtain

$$\delta = \frac{1}{\nu} \ln \frac{1}{1 - \nu}.$$

The shift of the soliton when it is reflected from the boundary is positive—the soliton is repelled by the boundary.

5. DIFFRACTION BY A BAND IN A NONLINEAR DEFOCUSING MEDIUM

The problem of Fraunhofer diffraction of a plane monochromatic wave by a band in a nonlinear defocusing medium leads to Eq. (1) with initial conditions

$$u|_{t=0} = 0 \text{ at } |x| < a, \quad u|_{t=0} = 1 \text{ at } |x| > a. \quad (42)$$

Here t and x are the longitudinal and transverse coordinates, and a is the half-width of the band; the amplitude of the infinite wave is set equal to unity.

In accordance with the results of Sec. 1, it is necessary to solve the eigenvalue problem (5) with the function $q = u|_{t=0}$. We have at $|x| < a$

$$v = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i\lambda x} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{i\lambda x}, \quad (43)$$

at $x > a$

$$v = c_3 \begin{bmatrix} i(1-\lambda^2)^{1/2} - \lambda \\ 1 \end{bmatrix} \exp\{-(1-\lambda^2)^{1/2} x\}, \quad (44)$$

and finally, at $x < -a$

$$v = c_4 \begin{bmatrix} 1 \\ i(1-\lambda^2)^{1/2} - \lambda \end{bmatrix} \exp\{(1-\lambda^2)^{1/2} x\}. \quad (45)$$

Making the solution (43)—(45) continuous at the point $x = \pm a$, we obtain after elementary transformations the eigenvalue equation

$$\cos 2\lambda a = \lambda. \quad (46)$$

Equation (46) has a set of symmetrically disposed zeros $\pm\lambda_n$, and, as expected, $|\lambda_n| < 1$.

At sufficiently small a, there is only one pair of zeros. At large a, the number of pairs of zeros N can be estimated from the formula $N \sim 2a/\pi$.

The zeros of Eq. (46) correspond to solitons that have in the given case the meaning of partial-shadow bands propagating along straight lines in the (x, t) plane. The soliton with eigenvalue λ_n makes an angle $\varphi_n = \tan^{-1} 2\lambda_n$ with the wave direction. The minimum number of such bands is equal to two.

Thus, the diffraction of a plane wave by a band in a nonlinear medium differs in principle from the diffraction in a linear medium in that it can be observed at an arbitrarily large distance from the band, whereas in a linear medium the diffraction picture becomes "smeared out" at large distances.

CONCLUSION

Soliton solutions are only particular solutions of Eq. (1), inasmuch as the general case $b(\lambda) \neq 0$, and the practical reconstruction of q(x, t) from the scattering data is a very complicated problem. It can be assumed, however, that the N-soliton solution "contained" in any initial condition determines its asymptotic behavior as $t \to \infty$. This conclusion is based on the fact that in the linear approximation there can propagate in the condensate waves whose minimal group velocity (speed of sound) coincides with the maximum velocity of the soliton. This circumstance leads to a separation of the soliton in the non-soliton parts of the solution, with the non-soliton part going outside the limit of the "sound cone" $|x| \leq 2|t|$, in which the solitons are contained, and its amplitude decreases like $t^{-1/2}$, owing to the dispersion spreading.

The entire preceding theory was one-dimensional and in this sense strongly idealized. One can hope, however, that it will be useful also in the solution of non-one-dimensional problems, at least such in which the dependence on the transverse coordinates is weak. This pertains primarily to the problem of the stability of a soliton with respect to long transverse perturbations.

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