On the superfluidity of the Fermi component in a Fermi-Bose gas and in He^3-He^4 solutions

V. A. Andrianov and V. N. Popov
Leningrad Section of the V. A. Steklov Mathematical Institute, USSR Academy of Sciences
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The possibility of superfluidity of the Fermi component of a Fermi-Bose gas is demonstrated for a model in which the pure Fermi system is not superfluid even at T = 0. Extrapolation of the formulas obtained to $He^3 - He^4$ solutions yields an estimate of $10^{-3} - 10^{-4}$°K for the upper limit of the transition temperature at which the Fermi component becomes superfluid.

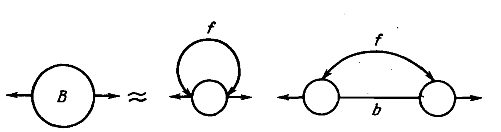
1. It is well known that the model of a non-ideal Fermi gas with repulsive interaction between the particles does not possess the property of superfluidity, even at zero temperature.[1,2] In the presence of a Bose component, an additional interaction of an attractive type arises between the Fermi particles. In Sec. 2 of this paper we consider a model of a low-density Fermi-Bose gas with repulsive interaction between the particles. Assuming that the Bose component is already superfluid, we find that superfluidity of the Fermi component appears at sufficiently low temperatures and low concentrations of the Fermi particles. In Sec. 3 the possibility of superfluidity of the Fermi component in $He^3 - He^4$ solutions is discussed. Extrapolation of the formulas obtained in Sec. 2 gives estimates of the corresponding transition temperature. Of course, the reliability of these estimates is considerably reduced because of the exponential dependence of the transition temperature on the parameters of the system. The upper bound for the transition temperature is found to be of the order of $10^{-3} - 10^{-4}$°K. But the most reasonable estimate appears to be T $10^{-6} - 10^{-7}$°K, in agreement with the estimate in[3].

2. In the formalism of temperature Green functions,[4] the superfluidity of the Fermi component is associated with the appearance of an anomalous Fermi-particle Green function. This function, denoted below by G_1, can be expressed in terms of the normal and anomalous self-energy parts A and B by the formula

$$G_1(p) = \frac{-B(p)}{(i\omega + k^2/2m_f - \mu_f + A(-p))(i\omega - k^2/2m_f + \mu_f - A(p)) - B(p)B(-p)} \quad (2.1)$$

Here, $p = (\mathbf{k}, \omega)$ is the four-momentum, m_f is the mass of a Fermi particle and μ_f is the Fermi chemical potential.

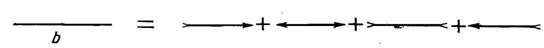
For a low-density Fermi-Bose system, the principal contribution to the anomalous self-energy part B(p) is made by the following two perturbation-theory graphs:



, (2.2)

in which the index b corresponds to a Bose line and the index f to a Fermi line. The first of the graphs (2.2) contains the anomalous Fermi Green function and a four-point function describing the scattering of two Fermi particles with opposite spins. In the limit of low density, the four-point function corresponds to the constant t_{ff}, i.e., the t-matrix for scattering of two Fermi particles at zero energy.

The second of the graphs (2.2) describes the Fermi-Bose interaction. It is actually the sum of the four graphs obtained by arranging arrows at the ends of the b-line in all possible ways:



(2.3)

The expressions for the normal G_b and anomalous G_{b1} Bose Green functions can be written in the form

$$G_b = \frac{i\omega + k^2/2m_b + m_b c_b^2}{(i\omega)^2 - \varepsilon^2(\mathbf{k})}, \quad G_{b1} = \frac{-m_b c_b^2}{(i\omega)^2 - \varepsilon^2(\mathbf{k})}, \quad (2.4)$$

where m_b is the mass of a Bose particle, c_b^2 is the square of the sound velocity, and

$$\varepsilon^2(k) = (k^2/2m_b)^2 + c_b^2 k^2 \quad (2.5)$$

is the square of the energy spectrum. Corresponding to each three-point function occurring in the second graph of (2.2) is a factor $2t_{bf}\sqrt{\rho_0}$, where t_{bf} is the t-matrix component for scattering of a Fermi and a Bose particle at zero energy and ρ_0 is the condensate density. In the low-density limit, we have

$$\rho_0 \approx \rho \approx m_b c_b^2 / t_{bb}. \quad (2.6)$$

We shall consider Eq. (2.6) for T = 0. The principal part of the contribution of the first graph (2.2) can be written in the form

$$-\frac{t_{ff} k_f m_f}{(2\pi)^2} \int \frac{B\, d\xi}{\sqrt{\xi^2 + B^2}}, \quad (2.7)$$

where $\xi = (k^2 - k_f^2)/2m_f$ and the integral can be assumed to be taken over a layer near the Fermi surface. The contribution of the second graph equals

$$\frac{\rho_0 t_{bf}^2}{(2\pi)^4 i m_b} \int \frac{B(k_1, E_1)(\mathbf{k} - \mathbf{k}_1)^2 d^3k_1\, dE_1}{(E_1^2 - \xi_1^2 - B^2(\mathbf{k}_1, E_1))((E - E_1)^2 - \varepsilon^2(\mathbf{k} - \mathbf{k}_1))}. \quad (2.8)$$

The principal part of the expression (2.8) for $k \approx k_f$ is given by the formula

$$\frac{\rho_0 t_{bf}^2 m_f m_b}{k_f (2\pi)^2} \int \frac{B\, d\xi}{\sqrt{\xi^2 + B^2}} \ln\left(1 + \left(\frac{m_f v_f}{m_b c_b}\right)^2\right). \quad (2.9)$$

The transformation from (2.8) to (2.9) is analogous to that used in the paper by Éliashberg.[5] Equation (2.2) acquires the form

$$B = \frac{t_{bf}^2 k_f m_f}{t_{bb}(2\pi)^2}\left(\frac{1}{z}\ln(1 + z) - \frac{t_{ff} t_{bb}}{t_{bf}^2}\right) \int \frac{B\, d\xi}{\sqrt{\xi^2 + B^2}}, \quad (2.10)$$

where

$$z = (m_f v_f / m_b c_b)^2. \quad (2.11)$$

In going from (2.7) and (2.9) to (2.10), formula (2.6) has been used. Superfluidity of the Fermi component is possible if

$$\frac{1}{z}\ln(1+z) - \frac{t_{ff}t_{bb}}{t_{bf}^2} > 0. \quad (2.12)$$

If we assume the Bose and Fermi particles to be hard spheres of the same radius r, then

$$t_{bb}t_{ff} / t^2_{bf} = 4m_b m_f / (m_b + m_f)^2 \leqslant 1, \quad (2.13)$$

since, in this case, $t = 4\pi a/m^*$, where $m^* = m_1 m_2/(m_1 + m_2)$ is the reduced mass and a is the diameter of the sphere.[1)] Thus, for a system of hard spheres, the condition (2.12) is fulfilled if the masses of the Fermi and Bose particles are different. For example, if $m_f/m_b = 3/4$, then

$$t_{bb}t_{ff} / t^2_{bf} = {}^{48}/_{49},$$

and superfluidity is possible when $z < 2/49 \sim 1/25$, i.e., for a sufficiently low concentration of Fermi particles.

The energy gap Δ at T = 0 and, with it, the temperature of the transition of the Fermi component into the superfluid state are given in order of magnitude by the formula

$$\Delta \sim T_c \sim \varepsilon_f \exp\left\{ -\frac{2\pi^2 t_{bb} t_{bf}^{-2} k_f^{-1} m_f^{-1}}{\frac{1}{z}\ln(1+z) - t_{bb}t_{ff}/t_{bf}^2} \right\} \equiv \varepsilon_f \exp\left(-\frac{1}{\alpha}\right). \quad (2.14)$$

We note that, for a system with repulsive interaction between all the particles, the exponent in this formula is at least two orders of magnitude greater than for a system with attractive interaction between the Fermi particles. Nevertheless, we can conclude that superfluidity of the Fermi component in a Fermi-Bose system is possible, even when superfluidity is not observed in the pure Fermi system.

3. We shall discuss the possibility of superfluidity in $He^3 - He^4$ solutions, extrapolating the formula (2.14) to the real system. We have seen that, for the case of repulsive interaction (namely, for a system of hard spheres), the absolute value of the exponent in (2.14) is of the order of 10^{-2}, which leads to very low transition temperatures 10^{-40}°K. However, the recently detected[6] phase transiton of He^3 (apparently into a superfluid state) at $T = 2.65 \times 10^{-3}$°K gives grounds for assuming the quantity t_{ff} to be negative. With the assumption that all the t-matrix components t_{bb}, t_{bf} and t_{ff} are of the same order in magnitude, we find that the left-hand side of the inequality (2.12) is a quantity of order two. The formula for the transition temperature takes the form

$$T_c \sim \varepsilon_f \exp\{-\pi^2 / |t_{ff}| k_f m_f\}. \quad (3.1)$$

The quantity t_{ff} can be determined in order of magnitude from the formula for the transition temperature in the pure Fermi system:

$$T_c \sim \varepsilon_f \exp\{-2\pi^2 / |t_{ff}| k_f m_f\}. \quad (3.2)$$

Putting $T_c = 2.65 \times 10^{-3}$°K in this formula and knowing the quantity k_f for pure He^3, we determine $k_f |t_{ff}|$. We substitute this quantity into (3.1), taking into account that the Fermi momentum k_f is smaller by a factor of 2.5 in a saturated 6% $He^3 - He^4$ solution than in pure He^3. As a result, we obtain for the T_c of the solution a quantity of the order of 10^{-4}°K.

This, however, is an overestimate, for the following reason. In deriving the formula (2.10), we replaced $m_b c_b^2 \rho^{-1}$ by t_{bb}, which is permissible only for a low-density system. But for the real system $m_b c_b^2 \rho^{-1} \gg t_{bb}$, and so

$$| m_b c_b^2 \rho^{-1} t_{ff} t_{bf}^{-2} | \gg 1 > z^{-1}\ln(1+z).$$

This means that, in reality, the contribution of the second graph to Eq. (2.2) is small compared with that of the first. In this case, the estimate formula for the transition temperature coincides with (3.2) and gives $T_c \sim 10^{-6} - 10^{-7}$°K, which agrees with the estimate obtained in [3].

Thus, we arrive at the conclusion that the estimate $10^{-6} - 10^{-8}$°K for the transition temperature is the most realistic. That the transition temperature is somewhat higher is not ruled out, however. The "optimistic" estimate ($10^{-3} - 10^{-4}$°K) is an upper bound above which superfluidity of the Fermi component of $He^3 - He^4$ solutions is scarcely possible.

[1)]In the general case, a has the meaning of the scattering length.

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