On tunneling of electron pairs in a sound field in superconductors

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The effect of sound vibrations on a stationary Josephson current is considered. The dependence of the sound-induced effective tunnel current on sound frequency is investigated at absolute zero temperature.

The present paper is devoted to a theoretical investigation of the influence of acoustic oscillations on the Josephson tunnel current. The acoustic oscillations in a superconductor-dielectric-superconductor tunnel structure modulate the tunnel current of appropriate frequency. If the electron-phonon interaction constants of the metals on the left and on the right of the barrier are different, then the influence of the acoustic oscillations leads to the onset of a certain differential value of the effective tunnel current, which is superimposed on the Josephson dc current that flows in the system in the absence of sound. It should be noted that this effective current is connected with phase difference α of the electron pairs by the usual relation $j_{eff}(\omega) = j_s(\omega)\sin\alpha$.

We investigate in this paper the singularity of the maximum value of the effective current $j_s(\omega)$ as a function of the frequency of the acoustic oscillations at absolute zero, under the assumption that the potential difference on the barrier is zero.

1. Following[1], we express the change in the number of electron pairs in the right-hand metal, due to their penetrating through the barrier, as follows:

$$\langle \dot{n}(x)\rangle = i\,\mathrm{Sp}_{\alpha\alpha'}\int d\mathbf{r}'\,d\mathbf{r}_1\,d\mathbf{r}_2\,T(\mathbf{r},\mathbf{r}')T(\mathbf{r}_1,\mathbf{r}_2) \times \int_{-\infty}^{+\infty} dt'\,\varepsilon(t-t')\langle[\varphi_\alpha^+(\mathbf{r}',t)\psi_\alpha(\mathbf{r},t)\varphi_{\alpha'}^+(\mathbf{r}_2,t')\psi_{\alpha'}(\mathbf{r}_1,t')]\rangle, \tag{1}$$

where the angle brackets represent averaging over the canonical ensemble, $T(\mathbf{r}, \mathbf{r}')$ is the amplitude of the passage of the electron through the barrier, $\epsilon(t - t')$ is the sign function, and $\varphi(\mathbf{r}, t)$ and $\psi(\mathbf{r}, t)$ are the electron-field operators on the left and on the right of the barrier in the Heisenberg representation.

$$\psi_\sigma(\mathbf{r},t) = e^{i\mathscr{H}t}\psi_\sigma(\mathbf{r})e^{-i\mathscr{H}t}.$$

The Hamiltonian $\mathscr{H}$ of the system is represented in the form of the sum of three terms, $\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2 + \mathscr{H}_{int}$, of which $\mathscr{H}_1$ and $\mathscr{H}_2$ are the Hamiltonians of the left- and right-side metals, and $\mathscr{H}_{int}$ is the Hamiltonian of the interaction of the acoustic field with the electrons. The latter is given by

$$\mathscr{H}_{int} = \mathrm{Sp}\int d\mathbf{r}\,\lambda_{\alpha\beta}^{(1)}u_{\alpha\beta}^{(1)}(x)\psi_\sigma^+(x)\psi_\sigma(x) + \mathrm{Sp}\int d\mathbf{r}\,\lambda_{\alpha\beta}^{(2)}u_{\alpha\beta}^{(2)}(x)\varphi_\sigma^+(x)\varphi_\sigma(x), \tag{2}$$

where $\lambda_{\alpha\beta}^{(i)}$ is a tensor characterizing the electron-phonon interaction in the metals, and $u_{\alpha\beta}^{(i)}(x)$ is the deformation tensor. The superscripts (1) and (2) pertain to the left-hand and the right-hand metals, and the trace is taken over the spin indices.

In expression (1) we have under the integral with respect to time contains a retarded commutator whose Fourier transform is an analytic continuation from the discrete set of frequencies to the entire upper half of the ω plane of the Fourier transform of the following quantity:

$$P^T = \langle \hat{T}_\tau(\varphi^+(\mathbf{r}',\tau)\psi(\mathbf{r},\tau)\varphi^+(\mathbf{r}_2,\tau')\psi(\mathbf{r}_1,\tau'))\rangle, \tag{3}$$

where $\tau = -it$ and $\hat{T}_T$ is the operator of time ordering with respect to τ. Bearing this circumstance in mind, we can express (1) with the aid of the temperature Green's functions $F(x, x')$ and $F^+(x, x')$. Since the electron-phonon interaction does not depend on the spin, we can sum immediately over the spin indices. We obtain

$$\langle \dot{n}(x)\rangle = 2\,\mathrm{Im}\int d\tau\int d\mathbf{r}'\,d\mathbf{r}_1\,d\mathbf{r}_2\,T(\mathbf{r},\mathbf{r}')T(\mathbf{r}_1,\mathbf{r}_2)F_1^+(\mathbf{r},\tau;\mathbf{r}_2\tau')F_2(\mathbf{r},\tau;\mathbf{r}_1\tau')e^{i\alpha},$$

$$F_1(x,x') = \langle T(\psi(x)\psi(x'))\rangle,\quad F_2(x,x') = \langle T(\varphi(x)\varphi(x'))\rangle, \tag{4}$$

$$F_1^+(x,x') = \langle T(\psi^+(x)\psi^+(x'))\rangle,\quad F_2^+(x,x') = \langle T(\varphi^+(x)\varphi^+(x'))\rangle.$$

To find the functions $F_1^+(x, x')$ and $F_2(x, x')$ we can derive the Gor'kov equation with allowance for (2) (see e.g.,[3]). Following[3] we write down the expressions for those increments to the Green's functions $F^{(0)}(x, x')$, $F^{(0)+}(x, x')$, $G^{(0)}(x, x')$ of the unperturbed problem which are linear in the deformation tensor. We have

$$\delta F(x,x') = \int \varphi(x'')[F^{(0)}(x'',x')G^{(0)}(x,x'') - G^{(0)}(x'',x)F^{(0)}(x',x'')]d^4x'',$$

$$\delta F^+(x,x') = \int \varphi(x'')[G^{(0)}(x'',x')F^{(0)+}(x,x'') - F^{(0)+}(x'',x')G^{(0)}(x,x'')]d^4x'',$$

$$\varphi(x) = \lambda_{\alpha\beta}u_{\alpha\beta}(x).$$

We substitute $F_1^+(x, x') = F_1^{(0)+}(x, x') + \delta F_1^+(x, x')$, $F_2(x, x') = F_2^{(0)}(x, x') + \delta F_2(x, x')$ in (4) and, taking (5) into account, represent (4) in the form of a sum of two terms, of which the first describes the stationary Josephson current[1] and the second describes the pair currents due to the acoustic field. The latter, after changing over to the Fourier representation and omitting the index of the functions $G(x, x')$ and $F(x, x')$, is expressed by the following formula

$$j(\omega) = \frac{2eT}{\hbar}\mathrm{Im}\sum_{\omega'} e^{i\alpha} \times \int\frac{d\mathbf{p}_1\,d\mathbf{p}_2}{(2\pi)^6}T^2(\mathbf{p}_1,\mathbf{p}_2)\{\varphi^{(1)}(\mathbf{p}_1,\omega)[G_1(\mathbf{p}_1-\mathbf{q},\omega')F_1^+(\mathbf{p}_1,\omega'+\omega) - G_1(\mathbf{p}_1,\omega'+\omega)F_1^+(\mathbf{p}_1-\mathbf{q},\omega')]F_2(\mathbf{p}_2,-\omega') - \varphi^{(2)}(\mathbf{p}_2,\omega) \times[G_2(\mathbf{p}_2-\mathbf{q},\omega')F_2^+(\mathbf{p}_2,\omega'+\omega) - G_2(\mathbf{p}_2,\omega'+\omega)F_2^+(\mathbf{p}_2-\mathbf{q},\omega')]F_1(\mathbf{p}_1,-\omega')\}, \tag{6}$$

where $\varphi^{(i)}(\mathbf{p}, \omega) = u_\alpha^{(i)}q_\beta^{(i)}\lambda_{\alpha\beta}^{(i)}$, $u_\alpha^{(i)}$ is the amplitude of the displacement vector, $\mathbf{q}$ and ω are the wave vector and the frequency of the sound, and T is the absolute temperature. The summation extends over the discrete set of frequencies $\omega' = (2n + 1)\pi T$, $n = 0, \pm 1, \pm 2, \ldots$

2. We substitute in (6) the expressions for the temperature Green's functions of a pure superconductor[4]

and integrate with respect to the momenta and the angles. We consider the most interesting case, when the inequality $qv \gg \Delta$, where v is the fermi energy of the electrons and Δ is the gap in the energy spectrum of the superconductor, is satisfied for both superconductors. We then obtain for the effective value of the tunnel current

$$j(\omega)=2\sqrt{2}\left(\frac{u_1\Theta_1}{\varepsilon_1}-\frac{u_2\Theta_2}{\varepsilon_2}\right)\zeta R^{-1}\Delta_2\,\mathrm{Im}\,e^{i\alpha}\Sigma(\omega), \qquad (7)$$

where $R^{-1} = (2\pi e/\hbar)N_1N_2$ is the conductivity of the normal tunnel current; Δ_1, Δ_2, ϵ_1, ϵ_2, Θ_1, Θ_2 are the gaps in the energy spectrum, the Fermi energies, and the Debye energies respectively for the left and right superconductors, $u_i = u^{(i)}/a_i$ are the amplitude of the displacement vector relative to the lattice constant, ζ is a constant on the order of unity, and

$$\Sigma(\omega)=T\sum_{\omega'}\frac{\Delta_1}{\sqrt{\omega'^2+\Delta_1^2}\sqrt{(\omega'+\omega)^2+\Delta_2^2}}. \qquad (8)$$

The total pair tunnel current flowing through the barrier is obtained by adding to the current $j(\omega)$ the quantity $j_0 \sin\alpha$, where j_0 is the maximum possible pair current flowing through the barrier in the absence of sound. The quantity j_0 was obtained in[1] and is equal to $j_0 = R^{-1}\Delta_2 K(\sqrt{1-(\Delta_2/\Delta_1)^2})$, where $K(x)$ is a complete elliptic integral of the first kind. Taking this circumstance into account as well as formula (7), we obtain for the total pair current $j(\omega)$

$$j(\omega)=[j_0+j_s(\omega)]\sin\alpha, \qquad (9)$$

where

$$\frac{2j_s(\omega)}{j_0}=\frac{\beta\,\mathrm{Re}\,\Sigma(\omega)}{K(\mu)}+\frac{[\beta\,\mathrm{Im}\,\Sigma(\omega)/K(\mu)]^2}{1+[\beta/2K(\mu)]\,\mathrm{Re}\,\Sigma(\omega)},$$
$$\beta=4\sqrt{2}\zeta\left(\frac{u_1\Theta_1}{\varepsilon_1}-\frac{u_2\Theta_2}{\varepsilon_2}\right),\qquad \mu=(1-\Delta_2/\Delta_1)^{1/2}. \qquad (10)$$

We represent formula (8) in the form of a contour integral and calculate the integrals with corresponding contours. We regard the resultant expression formally as a function of the complex variable $\omega = i\omega'$, which is analytic in the upper complex half-plane. Assuming $\Delta_1 > \Delta_2$ and performing the necessary calculation, we obtain

$$\frac{j_s(\omega)}{j_0}=\frac{\beta\Delta_1}{K(\mu)\sqrt{(\Delta_1+\Delta_2)^2-\omega^2}}\times\left[F(\lambda_1,r_1)+\frac{F^2(\nu_1,q^{-1})}{K(\mu)\sqrt{(\Delta_1+\Delta_2)^2-\omega^2}/\beta\Delta_1+F(\lambda_1,r_1)}\right] \qquad (11)$$

at $0 \le \omega \le \Delta_1 - \Delta_2$, and

$$\frac{j_s(\omega)}{j_0}=\frac{\beta}{2K(\mu)}\sqrt{\frac{\Delta_1}{\Delta_2}}\left[F(\lambda_2,r_2)+K(r_2)+\frac{F^2(\nu_1,q^{-1})}{2\sqrt{\Delta_2/\Delta_1}\,K(\mu)/\beta+F(\lambda_2,r_2)+K(r_2)}\right]$$

at $\Delta_1 - \Delta_2 \le \omega \le \Delta_1 + \Delta_2$, where $F(\lambda, r)$ is an incomplete elliptic integral of the first kind, and

$$q=\left[\frac{4\Delta_1\Delta_2}{(\Delta_1+\Delta_2)^2-\omega^2}\right]^{1/2},\qquad r_1=\left[\frac{(\Delta_1-\Delta_2)^2-\omega^2}{(\Delta_1+\Delta_2)^2-\omega^2}\right]^{1/2},$$
$$r_2=\left[\frac{\omega^2-(\Delta_1-\Delta_2)^2}{4\Delta_1\Delta_2}\right]^{1/2},\qquad \sin\lambda_1=\left[\frac{\omega(\Delta_1+\Delta_2+\omega)}{(\omega+2\Delta_2)(\omega+\Delta_1+\Delta_2)}\right]^{1/2}, \qquad (12)$$
$$\sin\lambda_2=\left[\frac{2\omega\Delta_1}{(\Delta_1+\Delta_2)(\Delta_1-\Delta_2+\omega)}\right]^{1/2},\qquad \sin\nu_1=\left[\frac{\omega(\Delta_1+\Delta_2+\omega)}{2\Delta_1(\Delta_1-\Delta_2)}\right]^{1/2},$$
$$\sin\nu_2=\left[\frac{2\Delta_1(\Delta_1-\Delta_2)}{\omega(\Delta_1+\Delta_2+\omega)}\right]^{1/2}.$$

3. We consider tunneling between impurity superconductors. The impurities are assumed to be nonmagnetic and, using the results of[3,4], expression (6) is averaged over them. We obtain

$$j(\omega)=\frac{\pi\zeta}{\sqrt{2}}R^{-1}\omega^2\,\mathrm{Im}\left[\frac{\tau_{tr}^{(2)}\varepsilon_2u_2}{\Theta_2}\Sigma_2(\omega)-\frac{\tau_{tr}^{(1)}\varepsilon_1u_1}{\Theta_1}\Sigma_1(\omega)\right]e^{i\alpha} \qquad (13)$$

where

$$\Sigma_1(\omega)=T\sum_{\omega'}\frac{\Delta_1\Delta_2}{\{(\Delta_1^2+\omega'^2)(\Delta_2^2+\omega'^2)[\Delta_2^2+(\omega'+\omega)^2]\}^{1/2}},$$
$$\Sigma_2(\omega)=T\sum_{\omega'}\frac{\Delta_1\Delta_2}{\{(\Delta_1^2+\omega'^2)(\Delta_2^2+\omega'^2)[\Delta_1^2+(\omega'+\omega)^2]\}^{1/2}},$$
$$\frac{1}{\tau_{tr}^{(i)}}=\frac{n_imp_{0i}}{(2\pi)^2}\int\left(1-\frac{\varphi^{(i)}(\Theta,\varphi)}{\overline{\varphi^{(i)}(\Theta,\varphi)}}\right)U_i^2(\Theta)\,d\Omega, \qquad (14)$$
$$\overline{\varphi^{(i)}(\Theta,\varphi)}=\int\varphi^{(i)}(\Theta,\varphi)\frac{d\Omega}{4\pi},\qquad i=1,2;$$

n is the impurity concentration, and $U(\Theta, \varphi)$ is the amplitude of the volume scattering of the electrons by the impurity.

Calculating the sums $\Sigma_1(\omega)$ and $\Sigma_2(\omega)$, we have

$$2\pi\,\mathrm{Re}\,\Sigma_1(\omega)=\frac{\Delta_2\omega K(\omega/2\Delta_1)}{\Delta(\omega/2,0)},$$
$$\frac{\pi}{\Delta_1\Delta_2\omega}\mathrm{Im}\,\Sigma_1(\omega)=\frac{K((\Delta_1-\Delta_2)/(\Delta_1+\Delta_2))}{(\Delta_1+\Delta_2)\Delta(0,{}^1/_2(\Delta_1-\Delta_2)-\omega)}$$
$$+\frac{1}{\Delta(0,{}^1/_2(\Delta_1-\Delta_2-\omega))}\left[\varepsilon(\Delta_1-\Delta_2-\omega)\frac{K(r_1)}{\Delta(\Delta_2,\omega-\Delta_2)}-\varepsilon(\omega-\Delta_1+\Delta_2)\frac{K(r_2)}{(\Delta_1\Delta_2)^{1/2}}\right], \qquad (15)$$
$$\frac{\pi}{\Delta_1\Delta_2\omega}\mathrm{Re}\,\Sigma_2(\omega)=\frac{1}{\Delta(\omega-{}^1/_2(\Delta_1-\Delta_2),0)}\left[\frac{K((\Delta_1-\Delta_2)/(\Delta_1+\Delta_2))}{\Delta_1+\Delta_2}+\varepsilon(\Delta_1-\Delta_2-\omega)\frac{K(r_1)}{\Delta(\Delta_2,\omega-\Delta_2)}\right],$$
$$2\pi\,\mathrm{Im}\,\Sigma_2(\omega)=\frac{\Delta_1\omega K(\omega/2\Delta_2)}{\Delta(0,-\omega/2)};\qquad \Delta(x,y)=[(\Delta_1+x)^2-(\Delta_2+y)^2]^{1/2}.$$

Using (13) and (15), we can write for the total pair current flowing through the barrier an expression analogous to (9), and for the maximum current $j_s(\omega)$ we now have

$$\frac{j_s(\omega)}{j_0}=\mathrm{Re}\,j(\omega)+\frac{[\mathrm{Im}\,j(\omega)]^2}{1+\mathrm{Re}\,j(\omega)}, \qquad (16)$$

$$j(\omega)=\frac{\sqrt{2}\pi\zeta}{K(\mu)}\frac{\omega^2}{2\Delta_2}\left[\frac{\tau_{tr}^{(2)}\varepsilon_2u_2}{\Theta_2}\Sigma_2(\omega)-\frac{\tau_{tr}^{(1)}\varepsilon_1u_1}{\Theta_1}\Sigma_1(\omega)\right]. \qquad (17)$$

Starting with formulas (5), we can easily conclude that at the frequencies $\omega < 2\Delta_2$ the second term in (16) is much less than the first, and we therefore can write

$$\frac{j_s(\omega)}{j_0}=\frac{\sqrt{2}\zeta}{K(\mu)}\frac{\tau_{tr}^{(1)}\Delta_1\varepsilon_1u_1}{\Theta_1}\left(\frac{\omega}{2\Delta_1}\right)^2\frac{f(\omega)}{[(1+\omega/2\Delta_1)^2-(\Delta_2/\Delta_1)^2]^{1/2}}, \qquad (18)$$

where

$$f(\omega)=4\nu\frac{\Delta(\omega/2,0)}{\Delta(\Delta_2+2\omega,\Delta_2)}\left[\frac{K((\Delta_1-\Delta_2)/(\Delta_1+\Delta_2))}{1+\Delta_2/\Delta_1}+\varepsilon(\Delta_1-\Delta_2-\omega)\frac{\Delta_1}{\Delta(\Delta_2,\omega-\Delta_2)}K\left(\frac{\Delta(-\Delta_2,\omega-\Delta_2)}{\Delta(\Delta_2,\omega-\Delta_2)}\right)\right]-K\left(\frac{\omega}{2\Delta_1}\right),$$
$$\nu=\tau_{tr}^{(2)}\varepsilon_2\Theta_1u_2/\tau_{tr}^{(1)}\varepsilon_1\Theta_2u_1.$$

4. Before we use formulas (11) and (18) for specific superconductors, the following circumstance should be noted. We have investigated the maximum Josephson current, due to modulation of the electron-state density by acoustic oscillations in superconductors, as a function of the sound frequency. However, the frequency dependence of the maximum current is influenced also by another mechanism, which we did not take into account, the decay of Cooper pairs into elementary excitations. At absolute zero temperature this mechanism

causes a change in the maximum tunnel current at sound frequencies $\omega \geq 2\Delta_2$. It can be easily seen from (11) and (18) that the tunnel current reaches its maximum value near the frequency $\Delta_1 - \Delta_2$. Therefore, to exclude the influence of the indicated mechanism on the frequency dependence of the tunnel current it is necessary to use in the experiment superconductors with energy gaps satisfying the inequality $\Delta_1/3\Delta_2 < 1$.

We present an estimate of the tunnel current in our case, where the tunnelling is between the superconductors lead and tin. Using the tables given in[5,6], we have for Pb the values $T_c = 7.2°K$, $\Theta = 94°K$, $\epsilon = \epsilon_0/2.1$, $\epsilon_0 = 10^4 °K$ and for Sn the values $T_c = 3.7°K$, $\Theta = 212°K$, $\epsilon = \epsilon_0/1.2$. Putting furthermore $\omega(\Delta_1 - \Delta_2) = 0.1$, $u_1 = u_2 = 0.2$ and $\tau_{tr}^{(1)} = \tau_{tr}^{(2)}$, we obtain with the aid of formulas (18) $j_s/j_0 = 30\%$.

The corresponding estimates in accordance with formulas (11) and (12) for pure superconductors give a negligibly small value for the ratio j_s/j_0, but for metals of the transition group (vanadium, niobium) this quantity can reach 15–20%.

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