On the statistical theory of reflection of light in randomly inhomogeneous media

V. I. Klyatskin

Institute of the Physics of the Atmosphere, USSR Academy of Sciences

(Submitted January 5, 1973)
Zh. Eksp. Teor. Fiz. **65**, 54–60 (July 1973)

The problem of reflection of light waves in a randomly inhomogeneous medium by an infinite mirror is considered in the parabolic-equation approximation. An expression for the mean field strength of the reflected wave is obtained on the basis of the solution in the form of a continual integral. By employing the probability interpretation of the solution in the continual form, a relation is derived between the mean reflected field strength and the statistical characteristics of the wave incident on the mirror.

1. The present status of the theory of light propagation in a randomly inhomogeneous medium has been surveyed in[1,2]. The authors of these papers started from a description of the propagation of light in a randomly inhomogeneous medium on the basis of a stochastic wave (or parabolic) equation and analyzed the equations obtained for the mean values when the stochastic equations are averaged over the ensemble of realizations of the field of the dielectric constant.

If large-angle scattering is immaterial, light propagation can be described by the parabolic equation[2]

$$\frac{\partial u}{\partial x}=\frac{i}{2k}\Delta_\perp u+\frac{ik}{2}\varepsilon(x,\rho)u(x,\rho),\qquad u(0,\rho)=u_0(\rho),\tag{1}$$

where the x axis coincides with the direction of the incident wave, $\boldsymbol{\rho}=\{y, z\}$ represents the transverse coordinates, $\Delta_\perp=\partial^2/\partial y^2+\partial^2/\partial z^2$, and $\epsilon(x, \boldsymbol{\rho})$ is the fluctuating part of the field of the dielectric constant, which we consider to be a Gaussian random field with zero mean value.

It was shown in[2] that the process of light propagation in a randomly inhomogeneous medium is well described by the approximation of a diffuse random process when the correlation function of the field ϵ is approximated by the expression

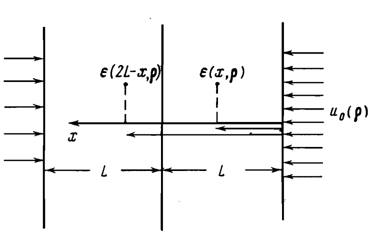
$$\langle\varepsilon(x_1,\rho_1)\varepsilon(x_2,\rho_2)\rangle=\delta(x_1-x_2)A(\rho_1-\rho_2),\tag{2}$$

where

$$A(\rho)=2\pi\int d\varkappa\,\Phi_\varepsilon(\varkappa)\exp\{i\varkappa\rho\},$$

and $\boldsymbol{\Phi}_\epsilon(\boldsymbol{\kappa})$ is a three-dimensional spectral function of the field $\epsilon(\mathbf{r})$ of the two-dimensional vector $\boldsymbol{\kappa}$. The proof of formula (2) and a discussion of the considered model of the fluctuations of ϵ is contained in[2]. We note that representation of the correlation function ϵ in the form (2) in the statistical problem of the propagation of light constitutes an expansion in terms of a small parameter, the ratio of the correlation of the field ϵ to other characteristic scales.

The diffuse random process approximation cannot be used directly to describe the process of reflection of a light wave from an obstacle. The reason is the reflected wave passes over the same inhomogeneities as the incident wave.



We consider the problem of light-wave reflection in an randomly inhomogeneous medium from an infinite mirror located in the plane x = L. The reflected wave is described by the stochastic equation (see the figure)

$$\frac{\partial u}{\partial x}=\frac{i}{2k}\Delta_\perp u+i\frac{k}{2}\tilde\varepsilon(x,\rho)u,\qquad u(0,\rho)=u_0(\rho),\tag{3}$$

where

$$\tilde\varepsilon(x,\rho)=\\=\begin{cases}\varepsilon(x,\rho), & x\leqslant L\\ \varepsilon(2L-x,\rho), & L\leqslant x\leqslant 2L\end{cases}.\tag{4}$$

For a medium described by the field $\tilde\epsilon$, the longitudinal correlation radius is of the order of L, i.e., of the same order as the path traversed by the wave. The diffusion-process approximation can therefore not be applied to (3). Equation (3) can be rewritten in the form

$$\frac{\partial u_{\rm ref}}{\partial x}=\frac{i}{2k}\Delta_\perp u_{\rm ref}+\frac{ik}{2}\varepsilon(2L-x,\rho)u_{\rm ref},\quad L\leqslant x\leqslant 2L,\\ u_{\rm ref}(L,\rho)=u_{\rm inc}(L,\rho).\tag{5}$$

In this case the field ϵ has already a small longitudinal correlation radius in comparison with the path length traversed by the wave, but it is impossible to obtain an equation, say, for the average reflected field, since the boundary condition for (5) is itself functionally dependent on the field ϵ.

2. We can, however, write the solution of (1) in operator form[3]

$$u(x,\rho)=\exp\left\{\frac{i}{2k}\int_0^x d\xi\frac{\delta^2}{\delta\tau^2(\xi)}\right\}\left\{u_0\left(\rho+\int_0^x d\xi\,\tau(\xi)\right)\right.\\ \left.\times\exp\left[i\frac{k}{2}\int_0^x d\xi\,\varepsilon\left(\xi,\rho+\int_\xi^x d\eta\,\tau(\eta)\right)\right]\right\}\Bigg|_{\tau=0}\tag{6}$$

or in the form of a continual integral

$$u(x,\rho)=\int D\mathrm{v}\exp\left\{\frac{ik}{2}\int_0 d\xi\left[\mathrm{v}^2(\xi)+\right.\right.\\ \left.\left.+\varepsilon\left(\xi,\rho+\int_\xi d\eta\,\mathrm{v}(\eta)\right)\right]\right\}u_0\left(\rho+\int_0 d\xi\,\mathrm{v}(\xi)\right),\\ D\mathrm{v}=\prod_{\xi=0}^{x}d\mathrm{v}(\xi)\Big/\int\ldots\int\prod_{\xi=0}^{x}d\mathrm{v}(\xi)\exp\left\{\frac{ik}{2}\int_0^x d\xi\,\mathrm{v}^2(\xi)\right\}.$$

Expression (7) admits of a probabilistic interpretation, namely, it can be formally expressed in the form of the average quantity

$$u(x,\rho)=\left\langle u_0\left(\rho+\int_0^x d\xi\, \mathbf{v}(\xi)\right)\exp\left\{\frac{ik}{2}\int_0^x d\xi\, \varepsilon\left(\varepsilon,\rho+\int_\xi^x d\eta\, \mathbf{v}(\eta)\right)\right\}\right\rangle_{\mathbf{v}}, \quad (8)$$

where the averaging is over the ensemble of the field $\mathbf{v}(\xi)$ $(\xi \le x)$, which can be regarded as a random Gaussian field with zero mean value and with a complex "correlation" function

$$\langle v_\alpha(\xi_1)v_\beta(\xi_2)\rangle_{\mathbf{v}}=ik^{-1}\delta_{\alpha,\beta}\delta(\xi_1-\xi_2), \quad \alpha,\ \beta=y,\ z. \quad (9)$$

It is easily seen here that all the formulas that are valid for the usual Gaussian random fields are valid in this case, too.

We represent $u_0(\boldsymbol{\rho})$ in the form $u_0(\boldsymbol{\rho}) = \int d\boldsymbol{\kappa} u_0(\boldsymbol{\kappa}) \times \exp\{i\boldsymbol{\kappa\rho}\}$. Then expression (8) can be rewritten in the form

$$u(x,\rho)=\int d\varkappa\, u_0(\varkappa)\exp\{i\varkappa\rho\}\left\langle \exp\left\{i\varkappa\int_0^x d\xi\, \mathbf{v}(\xi)+\frac{ik}{2}\int_0^x d\xi\, \varepsilon\left(\xi,\rho+\int_\xi^x d\eta\, \mathbf{v}(\eta)\right)\right\}\right\rangle_{\mathbf{v}} \quad (8')$$

Using formula (A.11) of the Appendix, we can rewrite (8′) in the form

$$u(x,\rho)=\int d\varkappa\, u_0(\varkappa)\exp\left\{i\varkappa\rho-\frac{i\varkappa^2}{2k}x\right\}\psi(x,\rho,\varkappa), \quad (10)$$

where

$$\psi(x,\rho,\varkappa)=\left\langle\exp\left\{\frac{ik}{2}\int_0^x d\xi\varepsilon\left(\xi,\rho+\int_\xi^x d\eta\left[\mathbf{v}(\eta)-\frac{\varkappa}{k}\right]\right)\right\}\right\rangle_{\mathbf{v}}.$$

Returning to the operator form, we can express $\psi(x, \boldsymbol{\rho}, \boldsymbol{\kappa})$ in the form

$$\psi(x,\ \rho,\ \varkappa)=\exp\left\{\frac{i}{2k}\int_0^x d\xi\frac{\delta^2}{\delta\tau^2(\xi)}\right\}\times\exp\left\{\frac{ik}{2}\int_0^x d\xi\,\varepsilon\left(\xi,\ \rho+\int_\xi^x d\eta\left[\tau(\eta)-\frac{\varkappa}{k}\right]\right)\right\}\Bigg|_{\tau=0}. \quad (11)$$

Expression (11) is a solution of the differential equation

$$\frac{\partial\psi}{\partial x}=\frac{i}{2k}\Delta_\perp\psi+\frac{ik}{2}\varepsilon\psi-\frac{\varkappa}{k}\nabla_\perp\psi, \quad \psi(0,\rho,\varkappa)=1. \quad (12)$$

Expressions (10)–(12) are the expansion of the solution of (1) in plane waves. Expression (10) and Eq. (12) for ψ can be obtained, of course, also directly from (1). We emphasize, however, that the foregoing conclusion is based on a probabilistic analogy for the solution of Eq. (1). In addition, it remains in force also in the case when there is no corresponding differential equation (see Sec. 4).

We can write down the solution of (5) in operator form for the reflected wave in analogy with (6):

$$u_{\rm ref}(2L,\ \rho)=\exp\left\{\frac{i}{2k}\int_L^{2L}d\xi\,\frac{\delta^2}{\delta\tau^2(\xi)}\right\}\left\{u_{\rm inc}\left(L,\ \rho+\int_L^{2L}d\xi\tau(\xi)\right)\times\exp\left[i\frac{k}{2}\int_L^{2L}d\xi\,\varepsilon\left(2L-\xi,\ \rho+\int_\xi^{2L}d\eta\tau(\eta)\right)\right]\right\}\Bigg|_{\tau=0}$$
$$=\exp\left\{\frac{i}{2k}\int_0^L d\xi\,\frac{\delta^2}{\delta\tau^2(2L-\xi)}\right\}\left\{u_{\rm inc}\left(L,\ \rho+\int_0^L d\xi\,\tau(2L-\xi)\right)\times\exp\left[i\frac{k}{2}\int_0^L d\xi\,\varepsilon\left(\xi,\ \rho+\int_0^\xi d\eta\,\tau(2L-\eta)\right)\right]\right\}\Bigg|_{\tau=0}. \quad (13)$$

Recognizing that (13) contains in all terms the quantity $\tau(2L-\xi)$, we can replace it by a new functional variable $\tau_1(\xi)$ and rewrite (13) in the form

$$u_{\rm ref}(2L,\ \rho)=\exp\left\{\frac{i}{2k}\int_0^L d\xi\,\frac{\delta^2}{\delta\tau_1^2(\xi)}\right\}\left\{u_{\rm inc}\left(L,\ \rho+\int_0^L d\xi\,\tau_1(\xi)\right)\times\exp\left[i\frac{k}{2}\int_0^L d\xi\,\varepsilon\left(\xi,\ \rho+\int_0^\xi d\eta\,\tau_1(\eta)\right)\right]\right\}\Bigg|_{\tau_1=0}. \quad (14)$$

Using now the operator form for the incident wave, we obtain the final expression

$$u_{\rm ref}(2L,\ \rho)=\exp\left\{\frac{i}{2k}\int_0^L d\xi\left[\frac{\delta^2}{d\tau_1^2(\xi)}+\frac{\delta^2}{\delta\tau_2^2(\xi)}\right]\right\}\times\left\{u_0\left(\rho+\int_0^L d\xi\,[\tau_1(\xi)+\tau_2(\xi)]\right)\exp\left[i\frac{k}{2}\int_0^L d\xi\varepsilon\left(\xi,\ \rho+\int_0^\xi d\eta\,\tau_1(\eta)\right)+i\frac{k}{2}\int_0^L d\xi\,\varepsilon\left(\xi,\ \rho+\int_\xi^L d\eta\,\tau_2(\eta)+\int_0^L d\eta\,\tau_1(\eta)\right)\right]\right\}\Bigg|_{\tau_1=\tau_2=0}. \quad (15)$$

3. In (14) and (15) we can already use the diffusion random process approximation, since the quantity $u_{\rm inc}(L, \boldsymbol{\rho})$ depends functionally only on the preceding values of $\epsilon(\xi, \boldsymbol{\rho})$ at $\xi \le L$. Therefore, averaging (15) over the ensemble of the field ϵ and using the correlation function of ϵ in the form (2), we obtain an expression for the mean field of the reflected wave:

$$\langle u_{\rm ref}(2L,\ \rho)\rangle=\exp\left\{\frac{i}{2k}\int_0^L d\xi\left[\frac{\delta^2}{\delta\tau_1^2(\xi)}+\frac{\delta^2}{\delta\tau_2^2(\xi)}\right]\right\}\times\left[u_0\left(\rho+\int_0^L d\xi\,[\tau_1(\xi)+\tau_2(\xi)]\right)\exp\left\{-\frac{k^2}{4}A(0)L-\frac{k^2}{4}\int_0^L d\xi\,A\left(\int_\xi^L d\eta\,[\tau_1(\eta)+\tau_2(\eta)]\right)\right\}\right]\Bigg|_{\tau_1=\tau_2=0}. \quad (16)$$

Introducing the new functional variables $\tau_1 - \tau_2 = \tau$ and $\tau_1 + \tau_2 = \tau$, we can rewrite (16) in the form $(D(\boldsymbol{\rho}) = A(0) - A(\boldsymbol{\rho}))$:

$$\langle u_{\rm ref}(2L,\ \rho)\rangle=\exp\left\{\frac{i}{k}\int_0^L d\xi\,\frac{\delta^2}{\delta\tau^2(\xi)}\right\}\left[u_0\left(\rho+\int_0^L d\xi\,\tau(\xi)\right)\times\exp\left\{-\frac{k^2}{4}A(0)L-\frac{k^2}{4}\int_0^L d\xi\,A\left(\int_\xi^L d\eta\,\tau(\eta)\right)\right\}\right]\Bigg|_{\tau=0}$$
$$=\exp\left\{-\frac{k^2}{2}A(0)L\right\}\exp\left\{\frac{i}{k}\int_0^L d\xi\,\frac{\delta^2}{\delta\tau^2(\xi)}\right\}\times\left[u_0\left(\rho+\int_0^L d\xi\,\tau(\xi)\right)\exp\left\{\frac{k^2}{4}\int_0^L d\xi\,D\left(\int_\xi^L d\eta\,\tau(\eta)\right)\right\}\right]\Bigg|_{\tau=0}. \quad (17)$$

The factor $\exp\{-\frac{1}{2}k^2A(0)L\}$ describes the damping of the reflected wave in the absence of diffraction, and the operator-type equation in (17) is connected with the diffraction of the light wave by small-scale inhomogeneities of the dielectric-constant field.

4. We now obtain the connection between the average reflected field and the statistical characteristics of the wave incident on the mirror. Representing $u_0(\boldsymbol{\rho})$ again in the form of a Fourier expansion and using the probabilistic analogy for the operator form (17), we can rewrite (17) in a form similar to (10):

$$\langle u_{\rm ref}(2L,\rho)\rangle=\int d\varkappa\, u_0(\varkappa)\exp\left\{i\varkappa\rho-\frac{i\varkappa^2}{k}L\right\}\times\exp\left\{-\frac{k^2}{2}A(0)L\right\}\exp\left\{\frac{i}{k}\int_0^L d\xi\,\frac{\delta^2}{\delta\tau^2(\xi)}\right\}\times\exp\left\{\frac{k^2}{4}\int_0^L d\xi\,D\left(\int_\xi^L d\eta\left[\tau(\eta)-\frac{2\varkappa}{k}\right]\right)\right\}\Bigg|_{\tau=0}. \quad (18)$$

From this we see readily that formula (18) can be rewritten in the form

$$\langle u_{\rm ref}(2L,\rho)\rangle=\int d\varkappa\, u_0(\varkappa)\exp\left\{i\varkappa\rho-\frac{i\varkappa^2}{k}L\right\}\langle\psi_{\rm inc}(L,\rho,\varkappa)\psi_{\rm inc}(L,\rho,-\varkappa)\rangle, \quad (19)$$

where the function $\psi_{\rm inc}(x, \boldsymbol{\rho}, \boldsymbol{\kappa})$ is described by Eq. (12), the solution of which is (11).

In the case of a plane incident wave, when $u_0(\boldsymbol{\rho}) \equiv 1$, expression (19) becomes much simpler, namely,

$$\langle u_{\text{ref}}(2L,\rho)\rangle = \langle u^2_{\text{inc}}(L,\rho)\rangle. \qquad (20)$$

We note that in this problem we cannot write the corresponding differential equation for the statistical characteristics of the reflected wave.

APPENDIX

MEAN VALUE OF A PRODUCT OF FUNCTIONALS

Let z(t) be a random process[1)]. Its statistical processes are completely described by its characteristic functional

$$\Phi[v(\tau)] = \left\langle \exp\left\{ i\int d\tau\, z(\tau)v(\tau)\right\}\right\rangle, \qquad (A.1)$$

where the angle brackets denote averaging over the ensemble of the realizations of the process z(t). The expansion of the functional $\Phi[v]$ in a functional Taylor series is determined by the n-point moments of the process z(t). It is convenient to represent $\Phi[v]$ as

$$\Phi[v] = \exp\{\Theta[v]\}, \qquad (A.2)$$

where the expansion of the functional $\Theta[v(\tau)]$ in a functional Taylor series is determined by the n-point cumulants (semi-invariants) of the process.

For a Gaussian stationary random process with a zero mean value, which is usually considered in statistical theory of light propagation, the only nonzero cumulant function is $\langle z(t)z(t')\rangle = B(t-t')$, and the functional $\Theta[v]$ takes the form

$$\Theta[v] = -\frac{1}{2}\iint d\tau_1\, d\tau_2\, B(\tau_1-\tau_2)v(\tau_1)v(\tau_2). \qquad (A.3)$$

Let us examine the mean value of a product of two functionals, $\langle F[z(t)]R[z(\tau)]\rangle$, where F[z] is specified in explicit form, and R[z] can depend on the random process either implicitly or explicitly. To calculate this mean value, we introduce a determined function $\eta(\tau)$ and consider the mean value of the product

$$\langle F[z]R[z(\tau)+\eta(\tau)]\rangle = \left\langle F[z]\exp\left\{\int d\tau\, z(\tau)\frac{\delta}{\delta\eta(\tau)}\right\}\right\rangle R[\eta(\tau)], \qquad (A.4)$$

where the operator in the right-hand side of (A.4) is the functional-shift operator.

We introduce also the functional

$$\Omega[v(\tau)] = \left\langle F[z(\tau)]\exp\left\{ i\int d\tau\, z(\tau)v(\tau)\right\}\right\rangle \Big/ \left\langle \exp\left\{ i\int d\tau\, z(\tau)v(\tau)\right\}\right\rangle. \qquad (A.5)$$

Then (A.4) can be rewritten as

$$\langle F[z(\tau)]R[z(\tau)+\eta(\tau)]\rangle = \left\langle F[z]\exp\left\{\int d\tau\, z(\tau)\frac{\delta}{\delta\eta(\tau)}\right\}\right\rangle \left\langle \exp\left\{\int d\tau\, z(\tau)\frac{\delta}{\delta\eta(\tau)}\right\}\right\rangle^{-1} \qquad (A.6)$$
$$\times\left\langle \exp\left\{\int d\tau\, z(\tau)\frac{\delta}{\delta\eta(\tau)}\right\}\right\rangle R[\eta(\tau)] = \Omega\left[\frac{\delta}{i\delta\eta(\tau)}\right]\langle R[z(\tau)+\eta(\tau)]\rangle.$$

Putting $\eta \equiv 0$ in (A.6), we obtain the final expression

$$\langle F[z(\tau)]R[z(\tau)]\rangle = \left\langle \Omega\left[\frac{\delta}{i\delta z(\tau)}\right]R[z(\tau)]\right\rangle. \qquad (A.7)$$

The case of a linear functional F(z) = z(t), when

$$\Omega[v] = \left\langle z(t)\exp\left\{ i\int d\tau\, z(\tau)v(\tau)\right\}\right\rangle \Big/ \left\langle \exp\left\{ i\int d\tau\, z(\tau)v(\tau)\right\}\right\rangle \qquad (A.8)$$
$$= \frac{1}{\Phi[v]}\frac{\delta\Phi[v]}{i\delta v(t)} = \frac{\delta\Theta[v]}{i\delta v(t)},$$

was considered in detail in[4]. We consider now the case when the functional F(z) is given by

$$F[z] = \exp\left\{ i\int d\tau\, z(\tau)\varkappa(\tau)\right\}.$$

In this case

$$\Omega[v] = \exp\{\Theta[v+\varkappa] - \Theta[v]\}. \qquad (A.9)$$

For a Gaussian random process described by the functional (A.3) we have

$$\Omega[v] = \exp\left\{-\frac{1}{2}\int d\tau_1\, d\tau_2\, B(\tau_1-\tau_2)[\varkappa(\tau_1)\varkappa(\tau_2) + 2v(\tau_1)\varkappa(\tau_2)]\right\} \qquad (A.10)$$

and therefore, according to (A.7),

$$\left\langle \exp\left\{ i\int d\tau\, z(\tau)\varkappa(\tau)\right\}R[z]\right\rangle = \left\langle \exp\left\{ i\int d\tau\, z(\tau)\varkappa(\tau)\right\}\right\rangle$$
$$\times\left\langle \exp\left\{ i\int d\tau_1\, d\tau_2\, B(\tau_1-\tau_2)\varkappa(\tau_2)\frac{\delta}{\delta z(\tau_1)}\right\}R[z]\right\rangle$$
$$= \Phi[\varkappa(\tau)]\left\langle R\left[z(\tau) + i\int d\tau_1\, B(\tau-\tau_1)\varkappa(\tau_1)\right]\right\rangle, \qquad (A.11)$$

i.e., a determined imaginary component is added to the random process $z(\tau)$ in the right-hand side of (A.11) under the averaging sign.

1)We confine ourselves to a one-dimensional random process. The generalization to many dimensions is obvious.

1 Yu. N. Barabanenkov, Yu. A. Kravtsov, S. M. Rytov and V. I. Tatarskiĭ, Usp. Fiz. Nauk **102**, 3 (1970) [Sov. Phys.-Uspekhi **13**, 551 (1971)].

2 V. I. Klyatskin and V. I. Tatarskiĭ, Izv. VUZov Radiofizika **15**, 1437 (1972).

3 V. I. Klyatskin and V. I. Tatarskiĭ, Zh. Eksp. Teor. Fiz. **58**, 624 (1970) [Sov. Phys.-JETP **31**, 335 (1970)].

4 V. I. Klyatskin and V. I. Tatarskiĭ, Teor. Mat. Fiz. **17** (1973).

Translated by J. G. Adashko
6