On the theory of transition radiation in a nonstationary medium

V. L. Ginzburg and V. N. Tsytovich

P. N. Lebedev Physics Institute
(Submitted February 14, 1973)
Zh. Eksp. Teor. Fiz. **65**, 132–144 (July 1973)

A theory of transition radiation in a medium with a time-varying refractive index (which is assumed to vary in a steplike manner) is developed. Along with the radiation field and energy (including the relativistic case), calculations are performed for the work of the radiative friction on the radiating particle and also for the change in energy of the field dragged by the particle.

Transition radiation was first studied in a case when a uniformly moving charged particle crossed an abrupt interface between two media[1]. Subsequently, a number of other problems of transition-radiation theory was also investigated (see the reviews[2,3]), and transition radiation produced in a homogeneous medium when its properties change abruptly in time has recently attracted attention[4]. Such a possibility is clear already from the general considerations that lead to the conclusion that transition radiation exists. Indeed, in a medium with refractive index $N = \epsilon^{1/2}$ the radiation is produced when the parameter vN/c changes with time (v is the particle velocity and c is the speed of light in vacuum; the case of Cerenkov radiation is not considered at present). An important role is played here by the change of this parameter at a spot occupied by a charge (or by some other emitter). It is obvious that at constant velocity v the parameter vN/c varies both in a spatially inhomogeneous medium (the usual transition radiation) and in a spatially homogeneous medium but one in which N varies with time. This type of transition radiation has its own peculiarities and does not reduce by far to transition radiation in a spatially inhomogeneous medium. We note that transition radiation in a medium that varies in time and in space sinusoidally was considered in[5].

We discuss in this article the problem of transition radiation in a nonstationary medium with an abrupt variation of N with time, and examine the region of applicability of such an approximation for variation of the refractive index in a dispersive medium. We estimate the time of formation of the radiation. We discuss in considerable detail the radiation of an ultrarelativistic particle. We also calculate the work done by the radiation force on a radiating particle, and take into account the effect of particle-mass renormalization in a medium with variable parameters. Finally, we carry out a comparison with the usual transition radiation for an interface between two media.

1. We derive the results obtained by one of the authors[4] for the radiation field by a somewhat different method that lends itself to an uncomplicated extension to the case of arbitrary modes in a magnetoactive and spatially-dispersive medium. We assume that the dielectric properties of the medium, described by the tensor $\hat{\epsilon}_{\mathbf{k}} = \epsilon_{ij}(\mathbf{k})$, are instantaneously altered at t = 0, with $\hat{\epsilon}_{\mathbf{k}} = \epsilon^{1}_{ij}(\mathbf{k})$ at t < 0 and $\hat{\epsilon}_{\mathbf{k}} = \epsilon_{ij}(\mathbf{k})$ at t > 0. We neglect first the frequency dispersion, but this dispersion will be later taken into account and the region of applicability of the results to a nondispersive medium will be assessed.

We write down the equation for the spatial Fourier components of the electric field $\mathbf{E}_{\mathbf{k}}(t)$ and of the magnetic field $\mathbf{H}_{\mathbf{k}}(t)$ excited by a charged particle moving with constant velocity:

$$k^2\mathbf{E}_{\mathbf{k}}(t) - \mathbf{k}(\mathbf{k}\mathbf{E}_{\mathbf{k}}(t)) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\hat{\varepsilon}_{\mathbf{k}}\mathbf{E}_{\mathbf{k}}(t) = \frac{4\pi ie}{c^2(2\pi)^3}\mathbf{v}(\mathbf{k}\mathbf{v})e^{-it\mathbf{k}\mathbf{v}},$$
$$\frac{\partial \mathbf{H}_{\mathbf{k}}(t)}{\partial t} = -ic[\mathbf{k}\mathbf{E}_{\mathbf{k}}(t)]. \qquad (1)^*$$

The conditions for matching at t = 0[4]

$$\mathbf{D}^{(1)}(0) = \mathbf{D}^{(2)}(0), \quad \mathbf{H}^{(1)}(0) = \mathbf{H}^{(2)}(0) \qquad (2)$$

assume the following form for the spatial Fourier components

$$\varepsilon_{ij}^{(1)}(\mathbf{k})E_{j,\mathbf{k}}^{(1)}(0) = \varepsilon_{ij}^{(2)}(\mathbf{k})E_{j,\mathbf{k}}^{(2)}(0), \qquad (3)$$

$$\int_{-\infty}^{0} dt[\mathbf{k}\mathbf{E}_{\mathbf{k}}^{(1)}(t)] = \int_{-\infty}^{0} dt[\mathbf{k}\mathbf{E}_{\mathbf{k}}^{(2)}(t)]. \qquad (4)$$

We consider first an isotropic medium in which only two modes exist, longitudinal and transverse. We shall show that the considered transition radiation does not exist for longitudinal waves. Indeed, the longitudinal field of the charge is determined by the equation div **D** = $4\pi\rho$, while the continuity of the induction D (see (2)) forbids the emission of a longitudinal wave (the change in the energy of the particle's own longitudinal field will be discussed later on). The transverse field

$$\mathbf{E}_{\mathbf{k}}^{tr} = \mathbf{E}_{\mathbf{k}} - \mathbf{k}(\mathbf{k}\mathbf{E}_{\mathbf{k}})/k^2$$

is excited only by the particle velocity component $\mathbf{v}^{tr} = \mathbf{v} - \mathbf{k}(\mathbf{k}\cdot\mathbf{v})/k^2$ perpendicular to **k**. Therefore, without loss of generality, we can put

$$\mathbf{E}_{\mathbf{k}}^{tr} = E_{\mathbf{k}}^{tr}\frac{4\pi e}{c^2(2\pi)^3}\mathbf{v}_{\mathbf{k}}^{tr}, \qquad (5)$$

where $E_{\mathbf{k}}^{tr}$ is the field amplitude (normalized in accordance with (5)) and satisfies the equation

$$k^2E_{\mathbf{k}}^{tr} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varepsilon_{\mathbf{k}}^{tr}E_{\mathbf{k}}^{tr} = i\mathbf{k}\mathbf{v}e^{-i\mathbf{k}\mathbf{v}t}. \qquad (6)$$

Here $\epsilon_{\mathbf{k}}^{tr}$ is the transverse dielectric constant, with

$$\varepsilon_{ij}(\mathbf{k}) = \left(\delta_{ij} - \frac{k_ik_j}{k^2}\right)\varepsilon_{\mathbf{k}}^{tr}.$$

Using the boundary conditions (3) and (4) and assuming no waves to be present at t < 0, we obtain the following solution of (6):

$$E_{\mathbf{k}}^{tr(1)}(t) = \frac{i\mathbf{k}\mathbf{v}}{k^2 - \varepsilon_{\mathbf{k}}^{tr(1)}(\mathbf{k}\mathbf{v})^2/c^2}e^{-i\mathbf{k}\mathbf{v}t}, \quad t < 0; \qquad (7)$$

$$E_{\mathbf{k}}^{tr(2)}(t) = \frac{i\mathbf{k}\mathbf{v}}{k^2 - \varepsilon_{\mathbf{k}}^{tr(2)}(\mathbf{k}\mathbf{v})^2/c^2}e^{-i\mathbf{k}\mathbf{v}t} + \frac{kc}{\varepsilon_{\mathbf{k}}^{tr(2)}}a_+^{tr}\exp\left(-i\frac{kc}{\varepsilon_{\mathbf{k}}^{tr(2)1/2}}t\right)$$
$$+ \frac{kc}{\varepsilon_{\mathbf{k}}^{tr(2)}}a_-^{tr}\exp\left(i\frac{kc}{(\varepsilon_{\mathbf{k}}^{tr(2)})^{1/2}}t\right), \quad t > 0; \qquad (8)$$

$$a_{\pm}{}^{tr}=\frac{i}{2k^2}\left\{\frac{\varepsilon_k^{tr(1)}\,\mathrm{kv}/kc\pm(\varepsilon_k^{tr(2)})^{1/2}}{1-\varepsilon_k^{tr(1)}(\mathrm{kv})^2/k^2c^2}-\frac{\varepsilon_k^{tr(2)}\,\mathrm{kv}/kc\pm(\varepsilon_k^{tr(2)})^{1/2}}{1-\varepsilon_k^{tr(2)}(\mathrm{kv})^2/k^2c^2}\right\}. \qquad (9)$$

Given **k**, two waves propagating in opposite directions with unequal amplitudes are produced at $t > 0$. Morgentaller[6] was first to note that a wave propagating in one direction splits into two oppositely directed waves if the properties of the medium are abruptly altered in time. In this case, the charge's own field is represented by a spectrum of waves with different **k**, and its restructuring at $t = 0$ is accompanied by a "disruption" of an entire spectrum of oppositely directed waves. From (7)—(9) we can obtain the earlier result of[4]. At the same time, by virtue of the spatial homogeneity of the problem, the solution (7)—(9) can be generalized to include the case of arbitrary normal waves.

Let $\mathbf{e}_k^\sigma$ be a normal unit vector of the wave σ of interest to us at $t > 0$. In the general case $\mathbf{e}_k^\sigma$ need not be a normal unit vector of some arbitrary wave at $t < 0$. We introduce the dielectric constant ϵ_k^σ and the square of the wave vector k_σ^2 defined by the relations

$$\varepsilon_k{}^\sigma=e_{k,i}^{\sigma^*}\varepsilon_{ij}(\mathbf{k})e_{k,j}^{\sigma},\qquad k_\sigma{}^2=k^2-(\mathbf{k}\mathbf{e}_k{}^{\sigma*})(\mathbf{k}\mathbf{e}_k{}^\sigma).$$

The values of ϵ_k^σ are different at $t < 0$ and $t > 0$, but those of k_σ^2 are the same. We seek a solution of (1) in the form

$$\mathbf{E}_k=\mathbf{e}_k{}^\sigma\frac{4\pi e}{c^2(2\pi)^3}(\mathbf{e}_k{}^{\sigma*}\mathbf{v})E_k{}^\sigma.$$

Multiplying (1) by $\mathbf{e}_k^{\sigma *}$, we obtain an equation that differs from (6) in the label ($\mathrm{tr}\to\sigma$). The same holds for the boundary condition. Thus, the solution for E_k^σ will be of the same form as (7)—(9), with the substitutions

$$tr\to\sigma,\qquad k^2\to k_\sigma{}^2,\qquad k\to(k_\sigma{}^2)^{1/2},$$

and $\mathbf{k}\cdot\mathbf{v}$ remains unchanged.

In the particular case of transverse waves, the values of ϵ_k^σ for the two possible polarizations coincide, i.e., the amplitudes E_k^σ are also equal, $\mathbf{k}\cdot\mathbf{e}_k^\sigma = 0$, and finally

$$\sum_{\sigma=1,2}\mathbf{e}_k{}^\sigma(\mathbf{e}_k{}^{\sigma*}\mathbf{v})=\mathbf{v}_k{}^{tr}=\mathbf{v}-\mathbf{k}(\mathbf{kv})/k^2.$$

Bearing in mind the indicated generalization, we shall henceforth use for simplicity the solutions (7)—(9).

2. For sufficiently large times (an estimate is given below), the radiation intensity is determined by the energy W^R of the radiation field, which combines additively with the energy of the self-field carried by the particle, with

$$W^R=\int\frac{\hat{\varepsilon}^{tr(2)}E^2+H^2}{8\pi}d\mathbf{r}=\pi^2\int(\varepsilon_k{}^{tr(2)}|E_k{}^{tr}(\infty)|^2+|H_k{}^{tr}(\infty)|^2)d\mathbf{k}$$

$$=\int W_k{}^R\,d\mathbf{k}=\int_0^\infty d\omega\int d\Omega\,W_{\omega,\Omega}^R, \qquad (10)$$

where Ω is the solid angle of the radiated waves, $\omega = kc/(\epsilon_k^{tr(2)})^{1/2}$ is their frequency, and $d\Omega = 2\pi\sin\vartheta\, d\vartheta$, where $\cos\vartheta = \mathbf{k}\cdot\mathbf{v}/kv$.

To obtain the correct angular dependence of the radiation it is necessary to represent the quadratic combinations of the fields E^2 and H^2 in a form containing only $\cos^2(\mathbf{k}\cdot\mathbf{r} - \omega t)$ or $\sin^2(\mathbf{k}\cdot\mathbf{r} - \omega t)$, for which purpose **k** must be replaced by $-\mathbf{k}$ in the expressions containing $\cos^2(\mathbf{k}\cdot\mathbf{r} + \omega t)$. Then waves with positive **k** will propagate at an acute angle to the particle velocity, and those with negative **k** at an obtuse angle. As a result we get

$$W_{\omega,\Omega}^R=\frac{e^2v^2\sin^2\vartheta}{\pi^2c^3(\varepsilon_k^{tr(2)})^{1/2}}|k^2a_+{}^{tr}|^2$$

or finally, using (9)

$$W_{\omega,\Omega}^R=\frac{e^2v^4\sin^2\vartheta\cos^2\vartheta}{4\pi^2c^5(\varepsilon_k^{tr(2)})^{1/2}}\frac{(\varepsilon_k^{tr(1)}-\varepsilon_k^{tr(2)})^2}{(1-\varepsilon_k^{tr(1)}v^2\cos^2\vartheta/c^2)^2(1-(\varepsilon_k^{tr(2)})^{1/2}v\cos\vartheta/c)^2}. \qquad (11)$$

We call attention to a number of features of this radiation in the case of ultrarelativistic particles. The radiation is always asymmetrical; in particular, the radiation intensity is higher in the half-plane in the direction of particle motion. For ultrarelativistic energies

$$E/mc^2=(1-v^2/c^2)^{-1/2}\gg1$$

this asymmetry is most clearly pronounced, and the bulk of the radiation is concentrated in a small solid angle $\Delta\Omega \approx \pi\vartheta^2 \sim (mc^2/E)^2$. Also radiated into this solid angle are rather high frequencies for which the dielectric constant of any medium can be approximated by the plasma formula (the possibility of taking dispersion into account here is discussed later on):

$$\varepsilon^{(1,2)}\approx1-\frac{\omega_{p1,2}^2}{\omega^2},\qquad \omega_{p1,2}^2=\frac{4\pi e^2}{m_e}n_{1,2}.$$

Retaining the most significant terms that contain in the denominators the small parameters ω_p^2/ω^2, ϑ^2, and $(mc^2/E)^2$, we obtain for the forward radiation

$$W_{\omega,\Omega}^R\approx\frac{e^2\vartheta^2\omega_{p1}^4}{\pi^2c\omega^4}\left(\frac{n_2-n_1}{n_1}\right)^2\left(\frac{m^2c^4}{E^2}+\frac{\omega_{p1}^2}{\omega^2}+\vartheta^2\right)^{-2}\left(\frac{m^2c^4}{E^2}+\frac{\omega_{p2}^2}{\omega^2}+\vartheta^2\right)^{-2}. \qquad (12)$$

The frequency dependence of the intensity of the total radiation at high frequencies can be obtained from

$$W_\omega{}^R=\int_0^\infty W_{\omega,\Omega}^R\pi\,d\vartheta^2.$$

The integration in (12) with respect to ϑ^2 from 0 to ∞ is possible because the main contribution to the radiation is made by small angles ϑ^2 on the order of $(mc^2/E)^2$. When

$$\frac{\Delta n}{n}=\left|\frac{n_1-n_2}{\max(n_1,n_2)}\right|\ll1$$

we obtain

$$W_\omega{}^R=\frac{e^2}{6\pi c}\left(\frac{\Delta n}{n}\right)^2\frac{\omega_p{}^4}{\omega^4}\left(\frac{m^2c^4}{E^2}+\frac{\omega_p{}^2}{\omega^2}\right)^{-2}.$$

The spectrum is flat down to frequencies lower than

$$\omega_*=\omega_pE/mc^2,$$

and then falls off like $1/\omega^4$. The main contribution to the integrated radiation is made by frequencies of the order of ω_*. Further,

$$W^R=\int_0^\infty W_\omega{}^R\,d\omega=\frac{e^2\omega_p}{24c}\frac{E}{mc^2}\left(\frac{\Delta n}{n}\right)^2. \qquad (13)$$

A different numerical factor is obtained in the limit at $\Delta n/n = 1$, for example when the particle goes from a medium to a vacuum or vice versa (in this case it is convenient to integrate in (12) first with respect to ω and then with respect to ϑ^2):

$$W^R=\frac{e^2\omega_p}{3c}\frac{E}{mc^2},\qquad\frac{E}{mc^2}\gg1. \qquad (14)$$

This result coincides exactly with the intensity of ordinary transition radiation of an ultrarelativistic particle crossing the interface between a vacuum and a medium[2,7,8]. For the transition radiation considered here (as, incidentally, for the usual transition radiation), the integrated backward radiation is much smaller than (14) when $E/mc^2 \gg 1$, since the factor E/mc^2 is missing, although the angular distribution has two narrow maxima in the range $\pi - \vartheta \lesssim mc^2/E$.

3. We consider the work W^F of the forces on the charge:

$$\frac{dW^F}{dt} = e\mathbf{v}\mathbf{E}^{tr}|_{\mathbf{r}=\mathbf{v}t}, \quad W^F = \int_0^\infty e\mathbf{v}\mathbf{E}^{tr}|_{\mathbf{r}=\mathbf{v}t}\, dt. \tag{15}$$

In spite of the fact that for transverse waves we have

$$\mathbf{E}^{tr} = -\frac{1}{c}\frac{\partial \mathbf{A}^{tr}}{\partial t},$$

it is obvious that the integral (15) is not equal to $e\mathbf{V}\cdot\mathbf{A}^{tr}(0)/c$, since $\partial \mathbf{A}^{tr}/\partial t$ contains also a time dependence via $\mathbf{r} = \mathbf{v}t$.

We write down the work of the forces in a form analogous to that used for the radiation intensity

$$W^F = \int_0^\infty d\omega \int d\Omega\, W^F_{\omega,\Omega},$$

$$W^F_{\omega,\Omega} = \frac{e^2v^2\sin^2\vartheta}{\pi^2c^3}\frac{k^2a_+^{tr}}{i}\left(1 - (\varepsilon_{\mathbf{k}}^{tr(2)})^{1/2}\frac{v}{c}\cos\vartheta\right)^{-1}.$$

Substituting here a_+^{tr} from (9), we obtain

$$W^F_{\omega,\Omega} = \frac{e^2v^3\sin^2\vartheta\cos\vartheta(\varepsilon_{\mathbf{k}}^{tr(1)} - \varepsilon_{\mathbf{k}}^{tr(2)})}{2\pi^2c^4(1 - \varepsilon_{\mathbf{k}}^{tr(1)}v^2\cos^2\vartheta/c^2)(1 - (\varepsilon_{\mathbf{k}}^{tr(2)})^{1/2}v\cos\vartheta/c)^2}. \tag{16}$$

Unlike the radiation power, the work forces depends on the sign

$$\varepsilon_{\mathbf{k}}^{tr(1)} - \varepsilon_{\mathbf{k}}^{tr(2)}.$$

For an ultrarelativistic particle, the bulk of the work is performed by radiation emitted forward at a small angle $\vartheta \sim mc^2/E$. Therefore

$$W^F_{\omega,\Omega} \approx \frac{2e^2\vartheta^2\omega_{p1}^2(n_2 - n_1)}{\pi^2c\omega^2n_1}\left(\frac{m^2c^4}{E^2} + \vartheta^2 + \frac{\omega_{p1}^2}{\omega^2}\right)^{-1}\left(\frac{m^2c^4}{E^2} + \vartheta^2 + \frac{\omega_{p2}^2}{\omega^2}\right)^{-2}.$$

In the limiting case $\Delta n/n \ll 1$, the work of the forces greatly exceeds the radiation power (13), which is proportional to $(\Delta n/n)^2$:

$$W^F = \int_0^\infty d\omega \int_0^\infty \pi\, d\vartheta^2\, W^F_{\omega,\Omega} \approx \pm\frac{e^2\omega_p}{2c}\frac{\Delta n}{n}.$$

For a transition from vacuum into a medium ($\Delta n/n = 1$, $n_1 = 0$) we have

$$W^F = \frac{2}{3}\frac{e^2\omega_p}{c}\frac{E}{mc^2}, \tag{17}$$

whereas for a transition from a medium into vacuum ($\Delta n/n = 1$, $n_2 = 0$) we have

$$W^F = -\frac{4}{3}\frac{e^2\omega_p}{c}\frac{E}{mc^2}. \tag{18}$$

4. To explain the differences between the work of the forces and the radiation power, we consider first the energy balance in general form. We write down Poynting's theorem

$$w = \int_{-\infty}^{t}\frac{1}{4\pi}\left(\mathbf{E}\frac{\partial\mathbf{D}}{\partial t} + \frac{1}{2}\frac{\partial}{\partial t}H^2\right)dt = -\int_{-\infty}^{t}\mathbf{jE}\,dt. \tag{19}$$

For the sake of simplicity, assuming ϵ to be a scalar quantity and using the equality

$$\mathbf{E}\frac{\partial\mathbf{D}}{\partial t} = \frac{\partial}{\partial t}\frac{D^2}{2\varepsilon} - D^2\frac{\partial}{\partial t}\frac{1}{2\varepsilon},$$

we rewrite (19) in the form

$$w = \frac{D^2(t)}{8\pi\varepsilon} + \frac{H^2(t)}{8\pi} - \frac{1}{8\pi}\int_{-\infty}^{t} D^2\frac{\partial}{\partial t}\frac{1}{\varepsilon}\,dt.$$

In our problem $\mathbf{E}$ and ϵ become discontinuous at $t = 0$ (see (2)), but by virtue of the continuity of D at $t = 0$ we can take D(0) in the integral term outside the integral sign and obtain the expression[1)]

$$w = \frac{D^2(t)}{8\pi\varepsilon} + \frac{H^2(t)}{8\pi} - \frac{D^2(0)}{8\pi}\left(\frac{1}{\varepsilon^{(2)}} - \frac{1}{\varepsilon^{(1)}}\right)\theta(t), \tag{20}$$

where

$$\theta(t) = \begin{cases}1, & t>0\\ 0, & t<0.\end{cases}$$

The last term in (20) ensures continuity of the electric energy w at $t = 0$. Indeed, at $t = -0$ this term is equal to zero and the electric energy is equal to $D^2(0)/8\pi\epsilon^{(1)}$, while at $t = +0$ the last term of (2) cancels completely the jump in the first term. We put

$$w = \frac{D^2(t)}{8\pi\varepsilon} + \frac{H^2(t)}{8\pi} + \Delta w, \quad \Delta w = \frac{\varepsilon^{(2)} - \varepsilon^{(1)}}{\varepsilon^{(2)}\varepsilon^{(1)}}\frac{D^2(0)}{8\pi}\theta(t). \tag{21}$$

The last term Δw in (21) must be taken into account as a constant in order to obtain the correct reference from which the energy is reckoned. We note that in the case of longitudinal waves, which do not radiate, the change of the self-energy of the longitudinal field of the charge reduces to the indicated energy change Δ_W. For the transverse field, integrating Δw over the entire volume, we obtain

$$\Delta W = \int \Delta w\, d\mathbf{r} = \int \pi^2|D_{\mathbf{k}}(0)|^2\frac{(\varepsilon_{\mathbf{k}}^{tr(2)} - \varepsilon_{\mathbf{k}}^{tr(1)})}{\varepsilon_{\mathbf{k}}^{tr(2)}\varepsilon_{\mathbf{k}}^{tr(1)}}d\mathbf{k}$$

$$= \pi^2\int\frac{\varepsilon_k^{tr(1)}}{\varepsilon_{\mathbf{k}}^{tr(2)}}(\varepsilon_{\mathbf{k}}^{tr(2)} - \varepsilon_{\mathbf{k}}^{tr(1)})|E_{\mathbf{k}}^{tr(1)}(0)|^2\,d\mathbf{k} = \int_0^\infty d\omega\int d\Omega\,\Delta W_{\omega,\Omega}.$$

The concrete expression for $\Delta W_{\omega,\Omega}$ is obtained in this case directly from (7):

$$\Delta W_{\omega,\Omega} = \frac{e^2v^4\sin^2\vartheta\cos^2\vartheta\varepsilon_{\mathbf{k}}^{tr(1)}(\varepsilon_{\mathbf{k}}^{tr(2)} - \varepsilon_{\mathbf{k}}^{tr(1)})}{4\pi^2c^5(\varepsilon_{\mathbf{k}}^{tr(2)})^{1/2}(1 - \varepsilon_{\mathbf{k}}^{tr(1)}v^2\cos^2\vartheta/c^2)^2}. \tag{22}$$

For ultrarelativistic particles, ΔW is negligibly small in comparison with W^F.

We now find the change in the energy of the field carried together with the particle, the so-called energy of macroscopic mass renormalization[7,9]. It is necessary here first to compare the values of the energy before and after the instant $t = 0$ (more accurately, compare the energy at $t = +\infty$). This comparison is meaningful only when the "constant" that determines the energy reference point is one and the same. Thus, ΔW should be included in the energy of the self-field of the particle at $t > 0$. The macroscopic mass renormalization energy W^M is determined by the difference between the values

$$W = \int\left(\frac{D^2}{8\pi\varepsilon} + \frac{H^2}{8\pi}\right)d\mathbf{r}$$

for the self-field (see (7) and the first term of (8)), added to ΔW. In other words, it is necessary to use for the energy density the expression (20), which follows from the Poynting theorem (19) and consequently from the field equations. We have

$$W^M = \int_0^\infty d\omega\int d\Omega\, W^M_{\omega,\Omega},$$

$$W^M_{\omega,\Omega} = \Delta W_{\omega,\Omega} + \frac{e^2v^2\sin^2\vartheta(\varepsilon_{\mathbf{k}}^{tr(2)})^{1/2}}{4\pi^2c^3}$$

$$\times\left\{\frac{1 + \varepsilon_{\mathbf{k}}^{tr(2)}v^2\cos^2\vartheta/c^2}{(1 - \varepsilon_{\mathbf{k}}^{tr(2)}v^2\cos^2\vartheta/c^2)^2} - \frac{1 + \varepsilon_{\mathbf{k}}^{tr(1)}v^2\cos^2\vartheta/c^2}{(1 - \varepsilon_{\mathbf{k}}^{tr(1)}v^2\cos^2\vartheta/c^2)^2}\right\} \tag{23}$$

$$= \frac{e^2v^4\sin^2\vartheta\cos^2\vartheta}{4\pi^2c^5(\varepsilon_{\mathbf{k}}^{tr(2)})^{1/2}}\frac{[3\varepsilon_{\mathbf{k}}^{tr(2)} + \varepsilon_{\mathbf{k}}^{tr(1)} - \varepsilon_{\mathbf{k}}^{tr(2)}(3\varepsilon_{\mathbf{k}}^{tr(1)} + \varepsilon_{\mathbf{k}}^{tr(2)})v^2\cos^2\vartheta/c^2)}{(1 - \varepsilon_{\mathbf{k}}^{tr(1)}v^2\cos^2\vartheta/c^2)^2(1 - \varepsilon_{\mathbf{k}}^{tr(2)}v^2\cos^2\vartheta/c^2)^2}$$

It is easy to verify that the energy balance is satisfied, by using the general relations (11), (16), and (23), from which we get

$$W_\omega^M + W_\omega^R = -W_\omega^F,$$

as it should be, since $-W^F$ is the work done by the particle on the field (W^F is the work of the field on the particle). We see that the balance is obtained already after integrating with respect to the angle variables, from which it is naturally obvious that the total balance $W^M + W^R = -W^F$ holds.

It follows from the foregoing, in particular, that at $\Delta n/n \ll 1$ the work of the forces practically coincides with the energy connected with the renormalization of the mass, and the radiation intensity is smaller by a factor $\Delta n/n$.

When a particle enters a medium from vacuum it radiates and becomes accelerated, because the work performed is positive (see (17)). This possibility is connected with the fact that in this case the energy of the macroscopic mass renormalization is negative (see (14) and (17)):

$$W^M = -W^F - W^R = -\left(\frac{2}{3}+\frac{1}{3}\right)\frac{e^2\omega_p}{c}\frac{E}{mc^2} = -\frac{e^2\omega_p}{c}\frac{E}{mc^2}. \qquad (24)$$

The deceleration of a particle emerging from a medium into vacuum exceeds the radiation energy. This difference is due to the positive value of the macroscopic mass renormalization (see (14) and (18)):

$$W^M = \left(\frac{4}{3}-\frac{1}{3}\right)\frac{e^2\omega_p}{c}\frac{E}{mc^2} = \frac{e^2\omega_p}{c}\frac{E}{mc^2}. \qquad (25)$$

It is natural that W^M has different signs on entering and leaving the medium. The concrete expression (24) and (25) for W^M in the case of an ultrarelativistic particle is obtained directly also from (23). The mass renormalization energy (24), (25) coincides with that obtained earlier in[9] by the Green's function method (see also also[7], where the renormalization was carried out by another method, but one that must be generalized if the longitudinal field is taken into account).

5. The limits of applicability of our results are seen most clearly from an estimate of the necessary abruptness of the discontinuity of the dielectric constant in time, taking the frequency dependence of ϵ_k^{tr} into account. We introduce the following quantities that characterize the temporal scales of the processes: the characteristic times t_f of formation of the considered transition radiation, and the characteristic times $t_d^{(1,2)}$ for "memorizing" the past electromagnetic state of the medium before and after the jump. The absence of dispersion of the dielectric constant, which was assumed above, means that $t_d^{(1,2)} = 0$ or, more accurately, $t_d^{(1,2)} \ll t_0$. At the same time, it was assumed above, in fact, that t_0 is shorter than t_f. Thus, the conditions for the applicability of the results obtained above take the form

$$t_d^{(1,2)} \ll t_0 \ll t_f. \qquad (26)$$

We now estimate t_f and see how the results are altered if t_0 is respectively shorter or longer than the two times $t_d^{(1,2)}$ and t_f. The time of radiation formation can be estimated from the condition for separating the radiation field from the self-field of the particle; more accurately, t_f corresponds to the fact that at $t \gg t_f$ the field energy is made up additively of the mass renormalization energy and the radiation energy. This corresponds to the condition that the interference terms in the energy (20) are neglected, or

$$\left(\mathbf{kv} \pm \frac{kc}{(\varepsilon_k^{tr(2)})^{1/2}}\right) t \gg 1 \qquad (27)$$

i.e.,

$$t_f = \max\left|\mathbf{kv} \pm \frac{kc}{(\varepsilon_k^{tr(2)})^{1/2}}\right|^{-1}. \qquad (28)$$

For a relativistic particle, the forward radiation takes longest to form ($\vartheta \sim mc^2/E \ll 1$) when

$$t_f = \frac{2}{\omega}\left(\frac{m^2c^4}{E^2}+\vartheta^2+\frac{\omega_{p2}^2}{\omega^2}\right)^{-1} \approx \frac{1}{\omega}\left(\frac{m^2c^4}{E^2}+\frac{\omega_{p2}^2}{\omega^2}\right)^{-1}.$$

At $\omega \ll \omega_* = \omega_{p2}E/mc^2$ we have $t_f \approx \omega/\omega_{p2}^2$, and when $\omega \gg \omega_{p2}E/mc^2$ we have $t_f \approx E^2/\omega m^2c^4$, i.e., the time of formation of radiation with frequency $\omega \approx \omega_{p2}E/mc^2$ is maximal, and in this case

$$t_f \approx \frac{1}{\omega_p}\frac{E}{mc^2}.$$

At the same time, if $\epsilon = 1 - \omega_p^2/\omega^2$, then $t_d \approx 1/\omega \approx mc^2/\omega_p E$, and the condition (26) can thus be satisfied at $E/mc^2 \gg 1$ in a range of values that increases rapidly with increasing particle energy. For $\vartheta^2 \gg m^2c^4/E^2$ and $\vartheta^2 \gg \omega_p^2/\omega^2$, the formation time $t_f \approx 1/\omega\vartheta^2$ is much larger than t_d so long as $\vartheta \ll 1$. The backward radiation (+ sign in (28)) does not satisfy these requirements.

The foregoing does not hold for nonrelativistic particles in a condensed medium, since it cannot be assumed that $t_d \approx 1/\omega$. The backward radiation of relativistic particles depends on the details of the frequency distributions and on the ratio of t_0 to t_d in the different models. This follows from the fact that, according to (28), $t_f \approx 1\omega$ for nonrelativistic particles and backward radiation, and consequently, according to the inequalities (26), the results pertain only to the frequency region in which $t_d \ll 1/\omega$, i.e., to the region where the dielectric constant is approximately constant (for example, far from the spectral lines).

Only the forward radiation of an ultrarelativistic particle is universal. We examine therefore the change occurring in the results if

$$t_0 \ll t_d^{(1,2)}, \qquad t_0 \ll t_f.$$

In the general case (we assume for simplicity that the dielectric constant ϵ is a scalar)

$$\mathbf{D}_k(t) = \int_{-\infty}^{t} \varepsilon_k(t, t-t')\mathbf{E}_k(t')\,dt',$$

where the dependence of ϵ_k on $t - t'$ characterizes the memory of the electric state (i.e., ϵ_k as a function of $t - t'$ decreases within a characteristic time interval on the order of t_d), whereas the dependence on t characterizes the nonstationary character of the medium (its change within a time $t \sim t_0$). When $t_0 \gg t_d$, the approximation $\epsilon_k = \epsilon_k(t)\delta(t - t')$ leads to the results already discussed. When $t_0 \ll t_d$, and particularly as $t_0 \to 0$, the concrete results depend on the form of the function $\epsilon_k(t, t - t')$. In particular, if only the density of the medium experiences a jump at $t = 0$, then (see the Appendix)

$$\varepsilon_k(t, t-t') = \begin{cases} \varepsilon_k^{(1)}(t-t'), & t<0, \quad t>t' \\ \varepsilon_k^{(2)}(t-t'), & t>0, \quad t>t' \end{cases}. \qquad (29)$$

Another example may be the change of the nonlinear response of a medium that is stationary in time when an external field is turned on sufficiently rapidly. Then (see the Appendix)

$$\varepsilon_k(t,t-t')=\begin{cases}\varepsilon_k^{(1)}(t-t'), & t'>0, \quad t>t'>0\\ \varepsilon_k^{(2)}(t-t'), & t'<0, \quad t>t', \quad t\gtrless 0\end{cases}. \qquad (30)$$

In both cases, the solution at $t<0$ retains the form (7). The boundary conditions (2), however, lead to a condition different from (3). Indeed, from (2) and (29) we obtain

$$\mathbf{E}_k^{(1)}(0)-\mathbf{E}_k^{(2)}(0)=\int_{-\infty}^{0}(\varepsilon_k^{(2)}(-t')-\varepsilon_k^{(1)}(-t'))\mathbf{E}_k^{(1)}(t')dt'. \qquad (31)$$

It is important that the right-hand side of (31) contains only the field at $t<0$, i.e., the field $E_k^{(1)}$. Using furthermore

$$\varepsilon_k^{(1,2)}(-t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\,\varepsilon_{k,\omega}^{(1,2)}e^{i\omega t'}, \quad \mathbf{E}_k^{(1)}(t)=\mathbf{E}_k^{(1)}(0)e^{-i\mathbf{kv}t},$$

we obtain

$$\mathbf{E}_k^{(1)}(0)-\mathbf{E}_k^{(2)}(0)=\int_{-\infty}^{\infty}d\omega\frac{\varepsilon_{k,\omega}^{(2)}-\varepsilon_{k,\omega}^{(1)}}{2\pi i(\omega-\mathbf{kv}+i\delta)}\mathbf{E}_k^{(1)}(0)$$

The dielectric constants $\epsilon_{k,\omega}^{(1,2)}$ have no poles in the upper half-plane of complex ω under equilibrium conditions[10]. Closing the contour with respect to ω in the upper half-plane, we obtain

$$\mathbf{E}_k^{(1)}(0)-\mathbf{E}_k^{(2)}(0)=(\varepsilon_{k,\mathbf{kv}}^{(2)}-\varepsilon_{k,\mathbf{kv}}^{(1)})\mathbf{E}_k^{(1)}(0). \qquad (32)$$

This boundary condition differs from (3):

$$\varepsilon_k^{(1)}\mathbf{E}_k^{(1)}(0)=\varepsilon_k^{(2)}\mathbf{E}_k^{(2)}(0).$$

In exactly the same manner we obtain in the case (3) another boundary condition:

$$\mathbf{E}_k^{(1)}(0)=\mathbf{E}_k^{(2)}(0).$$

By way of illustration, we present the solution obtained with the aid of condition (32). At $t<0$ we obtain formula (7), in which $\epsilon_k^{tr(1)}\to\epsilon_{k,\mathbf{k}\cdot\mathbf{v}}^{tr(1)}$. At $t>0$ it is convenient to express the solution in the form

$$E_k^{tr(2)}(t)=\frac{i\mathbf{kv}}{k^2-\varepsilon_{k,\mathbf{kv}}^{tr(2)}(\mathbf{kv})^2/c^2}e^{-i\mathbf{kv}t}+cb_k^+e^{-i\omega_k t}+cb_k^-e^{i\omega_k t},$$

where ω_k is a solution of the equation $k^2-\epsilon_{k,\omega}^{tr(2)}\omega^2/c^2=0$ and

$$b_k^{\pm}=\frac{i}{2k}\left\{\frac{(1+\varepsilon_{k,\mathbf{kv}}^{tr(1)}-\varepsilon_{k,\mathbf{kv}}^{tr(2)})\mathbf{kv}/kc\pm\omega_k/kc}{1-\varepsilon_{k,\mathbf{kv}}^{tr(1)}(\mathbf{kv})^2/k^2c^2}-\frac{\mathbf{kv}/kc\pm\omega_k/kc}{1-\varepsilon_{k,\mathbf{kv}}^{tr(2)}(\mathbf{kv})^2/k^2c^2}\right\} \qquad (33)$$

The radiation intensity is then calculated in the following manner:

$$W^R=\pi^2\int\frac{1}{\omega_k}\frac{\partial}{\partial\omega_k}\omega^2\varepsilon_{k,\omega}^{tr(2)}|E_k^{tr}(\infty)|^2dk=\int_0^{\infty}dk\int d\Omega\,W_{k,\Omega}^R,$$

$$W_{k\Omega}^R=\frac{e^2v^2\sin^2\vartheta}{2\pi^2c^2\omega_k}\frac{\partial}{\partial\omega_k}\omega^2\varepsilon_{k,\omega}^{tr(2)}|kb_k^+|^2. \qquad (34)$$

For ultrarelativistic particles $\epsilon^{tr}\approx 1$, $\omega_k\approx kc$, and we have in (34)

$$\frac{1}{\omega_k}\frac{\partial}{\partial\omega_k}\omega^2\varepsilon^{tr}\approx 2,$$

while in (33) we can put in the numerator of the first term $\epsilon^{tr(1)}-\epsilon^{tr(2)}\approx 0$. The result for W^R coincides with that obtained above. Thus, for relativistic particles the forward radiation depends in fact very little on the ratio of t_0 and t_d, and remains the same if $t_0\ll t_d$. What is important is the ratio of t_0 to t_f. When $t_0\gg t_f$ the forward radiation should decrease sharply. This effect is analogous to the effect of the smeared boundary for ordinary transition radiation,[11,12], when the thinness of the interface in comparison with the spatial zone of radiation formation is important. The work connected with the mass renormalization remains the same also at $t_0\gg t_f$ (cf.[12]).

6. We also compare in conclusion a number of singularities of the considered transition radiation and the usual one. Although the radiation of an ultrarelativistic particle forward is approximately the same in both cases (accurate to terms of order mc^2/E), the backward radiation and the radiation of nonrelativistic particles differ appreciably. In the "nonrelativistic limit $|v(\epsilon^{(1,2)})^{1/2}\cos\vartheta|\ll c$ the radiation in question is proportional to v^4 rather than to v^2 as in the usual transition radiation, as was already noted earlier[4]. The reason for this difference is that the longitudinal electrostatic field of the particle experiences no change in this case at the discontinuity, and consequently only the transverse component, due to the current of the particle, is changed.

The criterion $|v(\epsilon^{(1,2)})^{1/2}\cos\vartheta|\ll c$ means that the particle velocity is low in comparison with the velocity of electromagnetic waves in the medium. Unlike the usual transition radiation, in our case the radiation intensity increases strongly if the particle velocity v in the final state is close to (but smaller than) $v_0=c/(\epsilon_k^{tr(2)})^{1/2}$, which corresponds to the threshold of Cerenkov radiation. Indeed, according to (11), when $(v-v_0)/v_0\ll 1$ and $v<v_0$, the radiation intensity is proportional to $(v_0-v+\vartheta^2/2)^{-2}$. In particular, at $|\epsilon_k^{tr(2)}|\gg 1$, $v<v_0\ll c$, and $\vartheta\ll 1$ we obtain

$$W_{\omega,\Omega}^R\approx\frac{e^2v^4\vartheta^2}{4\pi^2c^2v_0^3(v_0-v+\vartheta^2/2)^2}. \qquad (35)$$

The radiation described by (35) is strongly peaked forward. The spectral intensity integrated over the angles at $v\to v_0$ and $v<v_0\ll c$ is

$$W_\omega^R=\frac{e^2v_0}{2\pi^2c^2}\left(\frac{26}{3}+\ln\frac{4}{1-v^2/v_0^2}\right). \qquad (36)$$

We note in conclusion that for a particle bunch with a characteristic dimension l we can obtain analogous results, and the spectral density of the radiated energy, at $\lambda=2\pi c/N\omega\gg l$, is obtained simply by replacing e with eN_0, where N_0 is the number of charges in the bunch. It is obviously the same token that it is easier to observe the effect for bunches than for individual particles.

APPENDIX

For the simplest example of an oscillator in an external field

$$m\frac{d^2}{dt^2}x+m\omega_0^2x+m\nu\frac{d}{dt}x=eE$$

we obtain

$$x_\omega=-\frac{e}{m}\frac{E_\omega}{\omega^2-\omega_0^2+i\nu\omega}$$

or

$$p=ex(t)=-\frac{e^2}{2\pi m}\int_{-\infty}^{\infty}\frac{d\omega\,e^{-i\omega(t-t')}}{\omega^2-\omega_0^2+i\nu\omega}E(t')dt'.$$

Let the oscillator density change at $t=0$ from a value $n^{(1)}$ to a value $n^{(2)}$. Then

$$D(t)=E(t)+4\pi n^{(1,2)}p(t)=\int\varepsilon^{(1,2)}(t-t')E(t')dt',$$

$$\varepsilon^{(1,2)}(t-t')=1-\frac{2n^{(1,2)}e^2}{m}\int_{-\infty}^{\infty}\frac{d\omega\,e^{-i\omega(t-t')}}{\omega^2-\omega_0^2+i\nu\omega}.$$

We consider another example, when the change of the electromagnetic properties of the medium occurs in a nonlinear response

$$(j_{\mathbf{k},\omega})_i = \int S^{ijl}_{\mathbf{k},\omega;\mathbf{k}_1\omega_1} E_{\mathbf{k}_1,\omega_1;j} E_{\mathbf{k}-\mathbf{k}_1,\omega-\omega_1;l},$$

where S^{ijl} are components describing the nonlinear properties of a homogeneous isotropic medium. Assume that a field E_0 directed along z (along the 3 axis) is turned on at t = 0:

$$E_3(t) = \begin{cases} E_0, & t > 0 \\ 0, & t < 0 \end{cases},$$

i.e.,

$$E_{\mathbf{k},\omega,3} = E_0 \delta(\mathbf{k}) \frac{i/2\pi}{\omega + i\delta}.$$

Hence

$$j_{\mathbf{k},\omega,i} = \frac{1}{\pi} \int S^{ij3}_{\mathbf{k},\omega;\mathbf{k},\omega_1} E_{\mathbf{k},\omega_1,j} \frac{iE_0}{\omega - \omega_1 + i\delta} d\omega_1 \, d\omega$$

or for the nonlinear correction to the dielectric constant

$$\delta\varepsilon_{ij,\mathbf{k}}(t, t-t') = -\frac{4}{\omega} E_0 \int S^{ij3}_{\mathbf{k},\omega;\mathbf{k},\omega_1} e^{-i\omega(t-t')} d\omega \frac{e^{it'(\omega_1-\omega)}}{\omega - \omega_1 + i\delta} d\omega_1.$$

Integrating this expression under the assumption that $S_{\mathbf{k},\omega;\mathbf{k},\omega_1}$ has no poles in the upper complex ω_1 half-plane, we obtain

$$\delta\varepsilon_{ij\mathbf{k}}(t, t-t') = \begin{cases} -\frac{8\pi i E_0}{\omega} \int S^{ij3}_{\mathbf{k},\omega;\mathbf{k},\omega} e^{-i\omega(t-t')} d\omega, & t' > 0 \\ 0, & t' < 0 \end{cases}.$$

*[$\mathbf{k}\mathbf{E}_\mathbf{k} \equiv \mathbf{k} \times \mathbf{E}_\mathbf{k}$.

1)We note that the same results can be obtained without considering in greater detail the variation of ϵ at t = 0, if it is recognized that at the discontinuity of ϵ the external system causes the energy to change by an amount $D^2(0)(1/\epsilon^{(1)} - 1/\epsilon^{(2)})/8\pi$.

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Translated by J. G. Adashko
15