# Nonlinear interaction of high-amplitude Langmuir waves in a collisionless plasma

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The behavior in time and space of high-amplitude Langmuir waves in a collisionless plasma is considered for the case when the wave energy density greatly exceeds the plasma pressure. It is shown that under the action of the high-frequency force the electrons and ions escape from that part of space in which the wave amplitude is larger. This results in the formation of a wave of the simple Riemann-wave type. Under certain conditions the wave may overturn and thus lead to ion heating.

Papers have recently appeared (see, for example,[1]), in which the temporal behavior of Langmuir turbulence is studied for the case in which the wave energy density $W = E^2/8\pi$ is not too small: $(m_e/m_i)^{1/2} < W/nT < 1$ ($m_e$ and $m_i$ are the masses of electron and ion, nT the plasma pressure). It has been shown that the plasma oscillations exhibit a tendency to bunch at isolated points in space, which leads, in the one-dimensional plane case, to the formation of solitary nonlinear waves —solitons, and in the case of spherical symmetry, to a collapse.

Not only is this problem one of theoretical interest, but it also has an applied value, since it allows us to ascertain, for example, where the energy of the Langmuir oscillations goes; these oscillations are pumped into the plasma upon stationary injection of the relativistic electron beam.[2,3] Because of the small group velocity of the Langmuir waves, they can accumulate near the input of the beam into the plasma, up to such a level for which the ratio W/nT becomes much greater than unity.[4,5] The study of the evolution of such waves in a plasma forms the subject of the present work.

For a description of the system under consideration, we make use of the equations of two-fluid hydrodynamics of a cold plasma ($W/nT \gg 1$), which, in the one-dimensional plane case, are of the form

$$m_{e,i}(\partial v_{e,i}/\partial t + v_{e,i}\partial v_{e,i}/\partial x) = \mp eE, \tag{1}$$

$$\partial n_{e,i}/\partial t + \partial(n_{e,i}v_{e,i})/\partial x = 0, \tag{2}$$

$$\partial E/\partial x = 4\pi e(n_i - n_e). \tag{3}$$

The notation in (1)—(3) is standard.

The fundamental component of the field E is the field of the Langmuir oscillations generated in the plasma with frequency $\omega_0 = (4\pi n_0 e^2/m_e)^{1/2}$ ($n_0$ is the equilibrium plasma density). Moreover, the field E has a component $\bar{E}$ that is slowly varying with time (the bar over the E denotes averaging over the high-frequency oscillations), and a component that oscillates with frequencies that are multiples of $\omega_0$. However, we shall take into consideration only the second harmonic, inasmuch as the higher harmonics give negligibly small contributions to the fundamental mode, as will be seen from what follows. The amplitudes of the oscillating field components, as well as that of E, depend slowly on time T in comparison with $\omega_0$.

Thus the field E has the structure

$$E = \bar{E} + \tilde{E}_1 + \tilde{E}_2;$$
$$\tilde{E}_1 = E_1(x, t)\cos\phi, \quad \tilde{E}_2 = E_2(x, t)\cos 2\phi, \quad \phi = \omega_0 t + \varphi(x, t), \tag{4}$$

and upon differentiation of all the amplitudes and of the phase $\varphi$,

$$\partial/\partial x \sim 1/L, \quad \partial/\partial t \sim \gamma \ll \omega_0, \tag{5}$$

where L and $1/\gamma$ are the characteristic lengths and times for slow motion.

We shall seek the electron velocity and density in forms similar to (4):

$$v_e = \bar{v}_e + \tilde{v}_1 + \tilde{v}_2, \quad n_e = \bar{n}_e + \tilde{n}_1 + \tilde{n}_2.$$

We shall neglect the high-frequency oscillations of the ions, assuming that

$$v_i = \bar{v}_i, \quad n_i = \bar{n}_i.$$

For applicability of the averaging, not only must (5) be satisfied, but also the condition

$$a/L = \varepsilon \ll 1, \tag{6}$$

where $a = eE_1/m_e\omega_0^2$ is the amplitude of the oscillations of the electrons in the field $\tilde{E}_1$.

Before undertaking the rigorous solution of the problem, we compare the order of magnitude of the quantities that are being sought. We are interested in a regime in which the kinetic energy of the ion is, in order of magnitude, equal to the energy of the high-frequency field incident on a single particle:

$$\tfrac{1}{2}m_i\bar{v}_i^2 \sim E_1^2/16\pi n_0.$$

The quantity $E_1^2/16\pi n_0$ is identical with the Miller potential $U = e^2E_1^2/4m_e\omega_0^2$ of the field of the Langmuir wave, which is the average kinetic energy of the oscillations of the electron $\overline{\tfrac{1}{2}m_0\tilde{v}_1^2}$. It then follows that the velocity $\tilde{v}_1 \sim (m_e/m_i)^{1/2}v_1$. But, by virtue of the quasineutrality of the plasma, the ions and electrons in slow motion have velocities of the same order, such that

$$\bar{v}_e \sim \bar{v}_i \sim (m_e/m_i)^{1/2}v_1. \tag{7}$$

On the other hand, $\tilde{v}_1 \sim \gamma L$ and $v_1 \sim a\omega_0 \sim \epsilon\omega_0 L$. Consequently,

$$\gamma \sim \varepsilon(m_e/m_i)^{1/2}\omega_0. \tag{8}$$

The remaining quantities can now be estimated from Eqs. (1)—(3). We have

$$\bar{E} \sim E_2 \sim \varepsilon E_1, \quad v_2 \sim \varepsilon v_1, \quad n_2 \sim \varepsilon n_1 \sim \varepsilon^2 n_0, \quad \bar{n}_e \sim \bar{n}_i \sim n_0. \tag{9}$$

The field $E_1$, which in the final analysis determines the dynamics of the system, can be found from the nonlinear equation

$$\frac{\partial^2 E_1}{\partial t^2} = -\omega_0^2\left\{\frac{n_e}{n_0}E + \frac{m_e}{e}\frac{\partial}{\partial x}\left(\frac{n_e}{n_0}v_e^2\right)\right\}_1, \tag{10}$$

which is obtained after double differentiation of Eq. (3)

with respect to t, with account of the equations of motion and continuity for electrons, and a single integration with respect to x with zero integration constant. The subscript one of the curly brackets at the right of the equation means here and below that only terms containing the first harmonic are used in the expression in curly brackets. In the given case, the contribution to the first harmonic from the first term in the brackets is equal to

$$\left\{\frac{n_e}{n_0}E\right\}_1=\frac{1}{n_0}\{\tilde n_e\tilde E_1+\tilde n_1\bar E+\tilde n_2\tilde E_1+\tilde n_1\tilde E_2\}_1.$$

By using Eq. (3), this expression can be represented in the form

$$\left\{\frac{n_e}{n_0}E\right\}_1=\frac{\bar n_i}{n_0}\tilde E_1-\frac{1}{4\pi n_0 e}\frac{\partial}{\partial x}\{\tilde E_1(\bar E+\tilde E_2)\}_1. \tag{11}$$

The second component in the curly brackets of Eq. (10) gives the contribution to the first harmonic due to the terms

$$\left\{\frac{n_e}{n_0}v_e^2\right\}_1=\left\{2\tilde v_1(\bar v_e+\tilde v_2)+\frac{\tilde n_1}{n_0}\tilde v_1^2\right\}_1. \tag{12}$$

In the derivation of Eqs. (11) and (12), we have retained the nonlinear terms proportional to the maximum of the third power of $E_1$, which gives the first nonvanishing correction to the linear equation

$$\partial^2\tilde E_1/\partial t^2=-\omega_0^2\tilde E_1.$$

The relative correction, according to (6)—(9), has the order of $\epsilon^2$ here. Account of higher harmonics would have yielded terms of higher order of smallness.

The first harmonics of the electron velocity and density $\tilde v_1$ and $\tilde n_1$ are determined by the linear equations

$$m_e\partial\tilde v_1/\partial t=-e\tilde E_1, \tag{13}$$

$$\partial\tilde n_1/\partial t+n_0\partial\tilde v_1/\partial x=0 \tag{14}$$

and are equal to

$$\tilde v_1=-\frac{e}{m_e\omega_0}E_1\sin\phi,\qquad \tilde n_1=-\frac{en_0}{m_e\omega_0^2}\frac{\partial}{\partial x}(E_1\cos\phi). \tag{15}$$

For the second harmonics, the system (1)—(3) takes the form

$$m_e\{\partial\tilde v_2/\partial t+\tilde v_1\partial\tilde v_1/\partial x\}_2=-e\tilde E_2, \tag{16}$$

$$\frac{\partial\tilde n_2}{\partial t}+\frac{\partial}{\partial x}\{n_0\tilde v_2+\tilde n_1\tilde v_1\}_2=0, \tag{17}$$

$$\partial\tilde E_2/\partial x=-4\pi e\tilde n_2. \tag{18}$$

Eliminating $\tilde n_2$ from (17), (18), and integrating the resultant equation with respect to x, we find an expression for $\partial\tilde E_2/\partial t$, which we must then substitute in Eq. (16), first differentiating it with respect to t, with account of (13). As a result, we obtain the following inhomogeneous linear equation for $\tilde v_2$:

$$\frac{\partial^2\tilde v_2}{\partial t^2}+\omega_0^2\tilde v_2=\left\{\frac{e}{m_e}\frac{\partial}{\partial x}(\tilde v_1\tilde E_1)-\omega_0^2\frac{\tilde n_1}{n_0}\tilde v_1\right\}_2. \tag{19}$$

This equation describes the stimulated oscillations of the electrons at the frequency $2\omega_0$.

After substitution of the expressions for $\tilde E_1, \tilde v_1, \tilde n_1$ on the right side of (19), and isolation of the second harmonic, we get the equation

$$\frac{\partial^2\tilde v_2}{\partial t^2}+\omega_0^2\tilde v_2=-\frac{3e^2}{4m_e^2\omega_0}\frac{\partial}{\partial x}(E_1^2\sin 2\phi).$$

The particular solution of this equation is

$$\tilde v_2=\frac{e^2}{4m_e^2\omega_0^3}\frac{\partial}{\partial x}(E_1^2\sin 2\phi).$$

We then get from Eq. (16)

$$\tilde E_2=-\frac{e}{4m_e\omega_0^2}\frac{\partial}{\partial x}(E_1^2\cos 2\phi)$$

and, furthermore, from (18),

$$\tilde n_2=\frac{1}{16\pi m_e\omega_0^2}\frac{\partial^2}{\partial x^2}(E_1^2\cos 2\phi).$$

Thus, all the oscillating components of E, $v_e$, and $n_e$ are expressed in terms of $E_1$ and $\varphi$.

We now turn to the equations for quantities averaged over the fast oscillations. These equations have the form

$$m_e\left(\frac{\partial\bar v_e}{\partial t}+\bar v_e\frac{\partial\bar v_e}{\partial x}+\overline{\tilde v_1\frac{\partial\tilde v_1}{\partial x}}\right)=-e\bar E, \tag{20}$$

$$\frac{\partial\bar n_e}{\partial t}+\frac{\partial}{\partial x}(n_0\bar v_e+\overline{\tilde n_1\tilde v_1})=0, \tag{21}$$

$$m_i(\partial\bar v_i/\partial t+\bar v_i\partial\bar v_i/\partial x)=e\bar E, \tag{22}$$

$$\partial\bar n_i/\partial t+n_0\partial\bar v_i/\partial x=0, \tag{23}$$

$$\partial\bar E/\partial x=4\pi e(\bar n_i-\bar n_e). \tag{24}$$

In Eq. (20), by taking (3)—(9) into account, we can neglect the first two terms in parentheses, which are connected with the inertia of the electrons in slow motion. Then, after substitution of $\tilde v_1$ from (15) in it, we obtain an expression for $\bar E$ in terms of the Miller potential:

$$\bar E=-\frac{1}{e}\frac{\partial U}{\partial x}=-\frac{1}{16\pi n_0 e}\frac{\partial E_1^2}{\partial x}. \tag{25}$$

When averaged over time, with account of (15), Eq. (21) takes the form

$$\frac{\partial\bar n_e}{\partial t}+n_0\frac{\partial}{\partial x}\left(\bar v_e-\frac{E_1^2}{8\pi n_0 m_e\omega_0}\frac{\partial\varphi}{\partial x}\right)=0. \tag{26}$$

We now differentiate Eq. (24) with respect to t and, first substituting $\partial\bar n_i/\partial t$, $\partial\bar n_e/\partial t$ and $\bar E$ from (23), (26), (25) in it, we integrate once more over x. As a result, we obtain the equation which connects $v_e$ and $v_i$:

$$\bar v_e=\bar v_i-\frac{1}{16\pi n_0 m_e\omega_0^2}\frac{\partial}{\partial t}\left(\frac{\partial E_1^2}{\partial x}\right)+\frac{E_1^2}{8\pi n_0 m_e\omega_0}\frac{\partial\varphi}{\partial x}. \tag{27}$$

The second term on the right in (27), by virtue of condition (5), is much smaller than the third, so that we have, with sufficient accuracy,

$$\bar v_e=\bar v_i+\frac{E_1^2}{8\pi n_0 m_e\omega_0}\frac{\partial\varphi}{\partial x}.$$

The results obtained above permit us to return to Eq. (10). Differentiating with respect to time on the left side of (10), with account of the condition (5), selecting the first harmonic on the right side and equating the summed coefficients of sin $\varphi$ and cos $\varphi$ to zero, we obtain the equations

$$\frac{\partial E_1}{\partial t}+\frac{\partial}{\partial x}(\bar v_i E_1)=0, \tag{28}$$

$$\partial\varphi/\partial t+\bar v_i\partial\varphi/\partial x={}^1\!/_2\,\omega_0(\bar n_i/n_0-1). \tag{29}$$

By taking Eq. (25) into account, we can write the equation of motion of the ions (22) in the form

$$m_i\left(\frac{\partial\bar v_i}{\partial t}+\bar v_i\frac{\partial\bar v_i}{\partial x}\right)=-\frac{1}{16\pi n_0}\frac{\partial E_1^2}{\partial x}. \tag{30}$$

Then Eqs. (23), (28)—(30) form the complete set of equations for the four unknown functions $E_1$, $\varphi$, $\tilde v_i$, and $\tilde n_i$.

We write down Eq. (30) in the form

$$\frac{\partial\bar v_i}{\partial t}+\bar v_i\frac{\partial\bar v_i}{\partial x}=-\frac{1}{E_1}\frac{\partial}{\partial x}\left(\frac{E_1^3}{24\pi n_0 m_i}\right). \tag{31}$$

It is easy to see that (28) and (31) are similar to the equations of hydrodynamics of an ideal fluid with an adiabatic exponent equal to three, while (28) plays the role of the equation of continuity and (31) that of Euler's equation. But in ideal hydrodynamics, the front of any initial disturbance increases its curvature with passage of time up to the moment of breaking, when a shock wave is formed. Consequently, even in the case considered here of a high-amplitude Langmuir wave in a cold plasma, strengthening of the front of the initial disturbance with passage of time should occur. In the final analysis, this should lead to breaking of the wave, dissipation of the energy of the Langmuir oscillations, and heating of the ions. It is true that in the last stage of development of the shock wave, when the parameter $\epsilon$ becomes of the order of unity, the inequality (6) is violated—one of the conditions for applicability of what has been considered. Therefore, the relaxation length of the Langmuir wave, on the basis of the theory developed above, can be calculated only with accuracy to within the amplitude of oscillation of the electron in the field of the wave.

For the special case in which $\bar{v}_i = \bar{v}_i(E_1)$, the Riemann solution can be obtained by analogy with ordinary hydrodynamics, a solution which describes a simple wave of arbitrary amplitude. We write down Eqs. (28), (31) for this case in the form

$$\frac{\partial E_1}{\partial t}+\frac{d(\bar{v}_i E_1)}{dE_1}\frac{\partial E_1}{\partial x}=0,$$

$$\frac{\partial \bar{v}_i}{\partial t}+\left(\bar{v}_i+\frac{1}{16\pi n_0 m_i}\frac{dE_1^2}{d\bar{v}_i}\right)\frac{\partial \bar{v}_i}{\partial x}=0.$$

From these equations we get, respectively,

$$(\partial x/\partial t)_{E_1}=\bar{v}_i+E_1 d\bar{v}_i/dE_1,$$
$$\left(\frac{\partial x}{\partial t}\right)_{\bar{v}_i}=\bar{v}_i+\frac{1}{16\pi n_0 m_i}\frac{dE_1^2}{d\bar{v}_i}. \tag{32}$$

But, since $\bar{v}_i$ = const for $E_1$ = const, then $(\partial x/\partial t)_{E_1} = (\partial x/\partial t)_{v_i}$ and, of course,

$$E_1\frac{d\bar{v}_i}{dE_1}=\frac{E_1}{8\pi n_0 m_i}\frac{dE_1}{d\bar{v}_i}.$$

We then get the equation

$$d\bar{v}_i/dE_1=\pm\frac{1}{(8\pi n_0 m_i)^{1/2}},$$

which, after integration, gives the desired function

$$\bar{v}_i(E_1)=\pm E_1(8\pi n_0 m_i)^{-1/2}. \tag{33}$$

Further, integrating Eq. (32) we find the dependence of $\bar{v}_i$ on time and the coordinates (in implicit form):

$$x=2\bar{v}_i t+f(\bar{v}_i).$$

Here $f(\bar{v}_i)$ is a function determined by the initial conditions. If $f(\bar{v}_i) = x(0, \bar{v}_i)$ is known, then the place $(x_*)$ and time $(t_*)$ of formation of the shock wave adjacent to an undisturbed plasma can be found from the condition

$$(\partial x/\partial \bar{v}_i)_t|_{\bar{v}_i=0}=0.$$

with accuracy to within the amplitude of oscillations of the electron. As a result, we get

$$x_*=f(0),\quad t_*=-{}^1\!/_2 f'(0).$$

Intersection of the ion trajectories takes place at the point $x_*$, as well as formation of multistream motion and, consequently, heating of the ions. The energy of the Langmuir oscillations is thus transformed into thermal energy of the ions.

In conclusion, we consider the problem of the applicability of these results. A competing process in the given case is the intersection of the electron trajectories due to the inhomogeneity of the density, leading to an increase in the electron temperature and, in principle, to a violation of the condition $W/nT \gg 1$. This effect was considered by Vedenov et al.,[6] who showed that by neglecting the motion of the ions, one can obtain a solution of the problem of Langmuir oscillations of arbitrary amplitude. For a given initial gradient of the electron density, for there is an intersection of the trajectories of the electrons for some amplitude of the wave, since the amplitude of the oscillations of the electrons is different at different points of space and electrons can be found at a certain point x with different velocities.

Mathematically, this circumstance is reflected in the fact that the phase $\varphi$ in our solution will change very rapidly for some amplitude of the field $E_1$ and some gradient of the inhomogeneity $1/L$. In this case, all the quantities are proportional not to $\sin\omega_0 t$ but to

$$\sin\int_0^t \omega[x(t'),t']dt'.$$

For our solution this would mean that the correction to the frequency

$$\Delta\omega\sim\frac{\partial\varphi}{\partial x}a\omega_0\sim\omega_0,\ \text{i.e.}\ \frac{\partial\varphi}{\partial x}a\sim 1.$$

In other words, when the phase changes over a length of the amplitude of the oscillation, due to the inhomogeneity of the density, by an amount of the order of unity, randomization of the phase and of the motion of the electrons takes place.

We now find the time at which the intersection of the electron trajectories takes place. In our case,

$$a\frac{\partial\varphi}{\partial x}\sim\frac{a}{L}\Delta\varphi\sim 1,$$

and, from Eq. (29),

$$\Delta\varphi\sim\omega_0\left(\frac{\bar{n}_i-n_0}{n_0}\right)\Delta t\sim\omega_0\varepsilon^2\Delta t.$$

The time at which intersection of the electron trajectories occurs will be of the order of $1/\omega_0\epsilon^3$. The breaking time is

$$t_*\sim\frac{L}{\bar{v}_i}\sim\frac{1}{\omega_0}\left(\frac{m_i}{m_e}\right)^{1/2}\frac{1}{\varepsilon}.$$

Thus, for the condition $\epsilon < (m_e/m_i)^{1/4}$, the breaking takes place earlier. The latter criterion is the condition for the applicability of the solutions that have been obtained.


[1] V. E. Zakharov, Preprint IYaF (Institute of Nuclear Physics, Siberian Department, Academy of Sciences USSR, No. 22–72, Novosibirsk, 1972.

[2] L. I. Rudakov, Zh. Eksp. Teor. Fiz. 59, 2091 (1970) [Sov. Phys.-JETP 32, 1134 (1971)].

[3] B. N. Breĭzman and D. D. Ryutov, Zh. Eksp. Teor. Fiz. 60, 408 (1971) [Sov. Phys.-JETP 33, 220 (1971)].

[4] Ya. B. Faĭnberg, V. D. Shapiro, Zh. Eksp. Teor. Fiz. 47, 1389 (1964) [Sov. Phys.-JETP 20, 935 (1965)]; V. N. Tsytovich and V. D. Shapiro, Zh. Tech. Fiz. 35, 1925 (965) [Sov.Phys.-Tech.Phys. 10, 1485 (1966)].

[5] A. A. Ivanov and L. I. Rudakov, Zh. Eksp. Teor. 51, 1522 (1966) [Sov. Phys.-JETP 24, 1027 (1967)].

[6] A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Nuclear Fusion, No. 1, 82 (1961).



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