Anisotropic turbulent distributions for waves with a nondecay dispersion law

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A weakly turbulent wave system is considered which can be described by a kinetic equation in which scattering of waves by each other is taken into account. Stationary solutions are found by symmetry transformations of the collision integral in k space. These solutions are not power functions and are nonisotropic deviations from solutions of the Kolmogorov type. These transformations change the quadrangle expressing the law of conservation of momentum and energy in scattering into a similar one for constant energy and momentum of one of the quasiparticles. The general properties of nonisotropic solutions with flows along the spectrum are considered. It is shown that the drift terms describe the appearance of constant flows of conserved quantities (momentum, particle number, or energy) which are absent in the initial single-parameter distribution. Locality of the solutions is investigated. As an example, distributions in turbulent systems of gravitational waves on the surface of a liquid and of Langmuir waves are considered.

1. INTRODUCTION. FLUXES OF CONSERVED QUANTITIES IN THE TURBULENCE SPECTRUM

In weakly-turbulent systems of waves obeying a "nondecaying" dispersion law $\omega(\mathbf{k})$, the conservation laws associated with collisions do not allow decay (coalescence) processes involving the participation of three quasi-particles,[1)] and the four-particle scattering processes

$$\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4, \quad \omega_1+\omega_2=\omega_3+\omega_4. \qquad (1.1)$$

become the most probable. We consider isotropic media with the waves satisfying a power-law dispersion law $\omega(\mathbf{k}) \sim k^{\beta}$ ($\beta < 1$), which is essentially utilized below in connection with the symmetry transformations, and we also consider systems (of the Langmuir-wave type) having a small dispersion correction $\omega(\mathbf{k}) \sim k^{\beta}$ ($\beta \gtrless 1$) to the activation frequency ω_0 (which drops out of Eqs. (1.1). In the kinetic equation that describes the weakly-turbulent distributions $N(\mathbf{k})$,[1,2] the collision integral is cubic with respect to the distribution function $I_{\text{coll}}\{N\} \sim N^3$, and conserves the number of particles. Local turbulence spectra of the Kolmogorov type appear in the presence of a source which is localized in k-space, and correspond to nonvanishing fluxes of conserved quantities. For isotropic stationary distributions N_ω the energy flux (or the number of particles) remains constant in frequency space (which is natural in view of the one-dimensional nature of the problem in the isotropic case). As Zakharov showed,[3] these distributions N_ω can be found from the equation

$$I_{\text{coll}}\{N(\mathbf{k})\}=0 \qquad (1.2)$$

with the aid of a special transformation that takes the symmetry of the collision operator averaged over angles in ω-space into account.[2)]

Nonisotropic solutions of Eq. (1.2) are obtained in the present article with the aid of transformations which utilize the symmetry of I_{coll} in k-space[8] (see Secs. 2 and 3). The obtained distributions are sums of power-law functions and are no longer one-parameter distributions. In contrast to solutions of the Kolmogorov type, which correspond to a single nonvanishing flux in the turbulence spectrum, they correspond to nonvanishing fluxes of the three conserved quantities, namely, the number of particles, the energy, and the momentum (see Sec. 4). As a consequence of the anisotropy of the obtained distributions, in order to interpret them it is necessary to consider the fluxes of conserved quantities in k-space. Thus, we write the kinetic equation

$$\partial N(\mathbf{k})/\partial t - D(\mathbf{k}) = I_{\text{coll}}\{N(\mathbf{k})\} \qquad (1.3)$$

($D(\mathbf{k})$ denotes the strength of the source of particles) in the form of an equation of continuity:

$$\partial N(\mathbf{k})/\partial t + \operatorname{div}_{\mathbf{k}}\mathbf{Q} = D(\mathbf{k}), \qquad (1.4)$$

where $\mathbf{Q}(\mathbf{k})$ is the vector density of the particle flux in k-space, which is defined by the equation

$$\operatorname{div}_{\mathbf{k}}\mathbf{Q} = -I_{\text{coll}}\{N(\mathbf{k})\}. \qquad (1.5)$$

Here $N(\mathbf{k})$ is normalized by the condition

$$\mathscr{E}/\rho L^d = \int d\mathbf{k}\,\hat{\omega}(\mathbf{k})N(\mathbf{k}) \qquad (1.6)$$

(ρ denotes the density of the medium, d is the dimensionality of k-space, $\mathscr{E}$ is the energy of the disturbance, and $\hat{\omega}(\mathbf{k}) = \omega_0 + \omega(\mathbf{k})$).

The energy and momentum conservation laws correspond to similar equations (see Sec. 4) obtained from Eqs. (1.4) and (1.5) by replacing $N(\mathbf{k})$, $\mathbf{Q}(\mathbf{k})$, and $D(\mathbf{k})$ by $\omega N(\mathbf{k})$, $\mathbf{P}$, $\tilde{D}(=\omega D^{3)})$ and $k_i N(\mathbf{k})$, $\Pi_i(\mathbf{k})$, and $\mathbf{D}(=\mathbf{k}D)$ where the energy flux density $\mathbf{P}$ and the momentum flux density Π_i are determined by the following equations relating them to the distribution $N(\mathbf{k})$:

$$\operatorname{div}_{\mathbf{k}}\mathbf{P} = -\omega(\mathbf{k})I_{\text{coll}}\{N\}, \quad \partial\Pi_{ij}/\partial k_j = -k_i I_{\text{coll}}\{N\}. \qquad (1.7)$$

We note for an activational dispersion law, $\mathbf{P}$ represents only that part of the energy flux which is not connected with the particle flux.

On the other hand, under stationary conditions ($\partial N/\partial t = 0$) the continuity equations enable us, as is clear from Eq. (1.4), to express the flux densities in terms of the moments of the sources (or else in terms of the moments of the sinks if they are localized near $k = 0$ whereas the sources are located at infinity). Let us consider, for example, the equation of continuity for the energy in the stationary case:

$$\operatorname{div}_{\mathbf{k}}\mathbf{P} = \tilde{D}(\mathbf{k}). \qquad (1.8)$$

Imposing the condition curl $\mathbf{P} = 0$, we arrive at a convenient electrostatic analogy, where $\tilde{D}/4\pi$ plays the role of the charge density, and $\mathbf{P}$ plays the role of the field. We obtain the following result from Eq. (1.8) in the three-dimensional case:

$$\mathbf{P} = -\nabla\tilde{\varphi}, \quad \tilde{\varphi}(\mathbf{k}) = \int d\mathbf{k}' \frac{\tilde{D}(\mathbf{k}')}{4\pi|\mathbf{k}-\mathbf{k}'|}. \tag{1.9}$$

For a point source located at the origin of coordinates, $\tilde{D}(\mathbf{k})/4\pi = P\delta(\mathbf{k})$ with P = const ("charge"), we have

$$\tilde{\varphi}(k) = P/k, \quad \mathbf{P}(\mathbf{k}) = P\mathbf{k}/k^3. \tag{1.10}$$

According to Gauss' theorem the "charge" P has the meaning of a constant energy flux in the space of the moduli k (or, what amounts to the same thing, in the space of the frequencies ω). The constancy of this quantity is an essential attribute of the theory of isotropic local turbulence. In similar fashion the particle-flux density $\mathbf{Q}(\mathbf{k})$ can be expressed in terms of its own "charge" (the particle-flux Q): $\mathbf{Q}(\mathbf{k}) = Q\mathbf{k}/k^3$. The electrostatic analogy can be extended even further. Thus, for example, a point source with a "dipole moment" $\mathbf{B}$ leads to a particle-flux density given by

$$\mathbf{Q}(\mathbf{k}) = \frac{3\varkappa(\varkappa\mathbf{B}) - \mathbf{B}}{4\pi k^3}, \quad \varkappa = \frac{\mathbf{k}}{k}, \tag{1.11}$$

and $\mathbf{P}(\mathbf{k})$ is thus exactly expressed in terms of the dipole moment of the energy source $\tilde{\mathbf{B}}$ (for more details, see Sec. 4).

Thus, in the general case the distribution is characterized by the multipole moments of the source, where constant fluxes (scalar quantities) correspond to point "charges," and the fluxes are not constant for higher multipoles. A dipole source of particles leads (together with the particle-flux (1.11)) to the appearance of a nonvanishing momentum flux density

$$\Pi_{ij} = B_i k_j / 4\pi k^3. \tag{1.12}$$

Here the quantity $k^2(\Pi_i \cdot \kappa) = B_i/4\pi$, having the meaning of momentum flux, is constant. We note that a point scalar charge leads to $\Pi_i = 0$. The dipole moment of the source generates a "vector charge" in the equation of continuity for the momentum.

The stationary solutions derived below for the kinetic equation (1.3) and causing the collision integral to vanish (Sec. 3) represent the deviations from isotropic distributions. We show that the anisotropic deviations describe the appearance of a constant momentum flux in the turbulence spectrum (Sec. 4). The isotropic deviations also have a similar meaning. Thus, in the solutions corresponding to a constant energy flux, a term arises describing a small particle-flux, and so forth. The localizability of these distributions is investigated (Sec. 5). The deviations from the equilibrium distribution, corresponding to the formation of fluxes of energy, momentum, and number of particles in the spectrum, are also derived (Sec. 3).

In the same way as the one-parameter isotropic distributions, the found turbulence spectra can be obtained from dimensional considerations. However, for a system of waves with dispersion, it is still necessary to use the connection between the flux and the distribution which is dictated by the kinetic equation (for more details, see[9]). According to Eqs. (1.1) and (1.7) we obtain $N \sim Q^{1/3}$ and $N \sim P^{1/3}$ (for a decaying spectrum $I_{coll} \sim N^2$ and $N \sim P^{1/2}$), from which it follows (see Eq. (2.16) that the distributions are given by

$$N = Q^{1/3}\omega^{s_0}, \quad N = P^{1/3}\omega^{s_1}. \tag{1.13}$$

Writing down the multiple-flux distribution in terms of a dimensionless function F of the flux ratios,

$$N(\mathbf{k}) = Q^{1/3}\omega^{s_0}F_0\left(\frac{\delta P}{\omega Q}, \frac{\mathbf{k}\delta\mathbf{B}}{k^2 Q}\right), \; N(\mathbf{k}) = P^{1/3}\omega^{s_1}F_1\left(\frac{\omega\delta Q}{P}, \frac{\omega(\mathbf{k}\delta\mathbf{B})}{k^2 P}\right), \tag{1.14}$$

in the approximation linear in δQ, δP, and δB we arrive at the distributions given by Eqs. (3.8) and (3.9). It is important, however, that these distributions are found below as the exact solutions of the kinetic equation.

2. Transformation of the collision integral

Only processes involving the scattering of the particles by one another are allowed for a non-decaying dispersion law, so the collision integral has the form

$$I_{coll}\{N(\mathbf{k})\} = \int d\tau_k W_{\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3} f(\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3), \tag{2.1}$$

where $d\tau_k \equiv d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$, $W_k \equiv W_{\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3}$ is the transition probability,

$$W_{\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3} = \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3)U_{\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3}, \tag{2.2}$$

and the function f_k is given by

$$f_k \equiv f(\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3) = N_1N_2N_3 + NN_2N_3 - NN_1N_2 - NN_1N_3, \quad N \equiv N(\mathbf{k}), \quad N_1 \equiv N(\mathbf{k}_1) \text{ etc.} \tag{2.3}$$

Here both U_k and W_k have the following symmetry properties:

$$U_{\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3} = U_{\mathbf{k}_1\mathbf{k}|\mathbf{k}_2\mathbf{k}_3} = U_{\mathbf{k}_2\mathbf{k}_3|\mathbf{k}\mathbf{k}_1}; \tag{2.4}$$

they are invariant under rotations $\hat{g}$ in virtue of the isotropy of the medium, and they are homogeneous functions of their arguments in view of the assumed self-similarity:

$$U_{\lambda\hat{g}k} = \lambda^m U_{\hat{g}k} = \lambda^m U_k. \tag{2.5}$$

As is clear from Eq. (2.3), the function f_k is symmetric under interchange of the arguments in each of the pairs and is antisymmetric with respect to the interchange of pairs of arguments

$$f(\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3) = f(\mathbf{k}_1\mathbf{k}|\mathbf{k}_2\mathbf{k}_3) = -f(\mathbf{k}_2\mathbf{k}_3|\mathbf{k}\mathbf{k}_1). \tag{2.6}$$

The integral I_{coll} vanishes for the equilibrium distribution

$$N_0(\mathbf{k}) = (\omega_\mathbf{k} - \mathbf{k}\mathbf{u} - \mu)^{-1} \tag{2.7}$$

because the function

$$f(\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3) \equiv N_1N_2N_3N(N^{-1} + N_1^{-1} - N_2^{-1} - N_3^{-1}) \tag{2.8}$$

vanishes as a consequence of the conservation of energy, momentum, and number of particles during collisions. The parameters $\mathbf{u}$ and μ represent the drift velocity and the chemical potential of the quasi-particles, and here we assume the activation term in the spectrum to be included in μ.

We shall utilize the symmetry properties of the collision integral in order to discover nonequilibrium solutions with $f_k \neq 0$. Temporarily denoting the integration variables in (2.1) by $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ so that $\mathbf{k} + \mathbf{q}_1 = \mathbf{q}_2 + \mathbf{q}_3$, we consider the transformations determined by the conservation laws (2.2) which leave the region of integration invariant:

$$G_1: \; \mathbf{q}_1 = G_1^2\mathbf{k}_1, \; \mathbf{q}_2 = G_1\mathbf{k}_3, \; \mathbf{q}_3 = G_1\mathbf{k}_2, \tag{2.9}$$

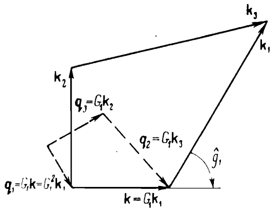
where the operation G_1 is determined by the condition (see the figure)

$$G_1\mathbf{k}_1 = \mathbf{k}, \tag{2.10}$$

and analogously

$$G_2: \; \mathbf{q}_1 = G_2\mathbf{k}_3, \; \mathbf{q}_2 = G_2^2\mathbf{k}_2, \; \mathbf{q}_3 = G_2\mathbf{k}_1 \; (G_2\mathbf{k}_2 = \mathbf{k}), \tag{2.11}$$

$$G_3: \; \mathbf{q}_1 = G_3\mathbf{k}_2, \; \mathbf{q}_2 = G_3\mathbf{k}_1, \; \mathbf{q}_3 = G_3^2\mathbf{k}_3 \; (G_3\mathbf{k}_3 = \mathbf{k}). \tag{2.12}$$



The transformation $G_1 = \lambda_1 g_1 (\lambda_1 = k/k_1)$ transforms the quadrangle $\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 = 0$ which expresses the law of momentum conservation during the scattering process, into a similar quadrangle $\mathbf{k} + \mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3 = 0$ associated with the invariant energy $\omega(\mathbf{k})$ and momentum $\mathbf{k}$ of one of the quasi-particles.

As is clear from Eqs. (2.10)—(2.12), the geometrical meaning of the transformations $G_i \equiv \lambda_i g_i$ consists in rotations g_i and dilatations $\lambda_i = k/k_i$, successively converting k_i into k while preserving the similarity of the quadrangle that expresses the law of momentum conservation (see the figure). The power-law nature of the dispersion law $\omega_k = k^\beta$, guarantees that under the transformations (2.9), (2.11), and (2.12) the law of energy conservation will be satisfied in terms of the new variables, provided that it was satisfied in terms of the old variables:

$$\omega(\mathbf{k}) + \omega(\mathbf{q}_1) - \omega(\mathbf{q}_2) - \omega(\mathbf{q}_3) = \pm\lambda^\beta[\omega(\mathbf{k}) + \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2) - \omega(\mathbf{k}_3)].$$

Taking the symmetry properties (2.4) and (2.5) into account, the transition probability is transformed into itself to within the scale factor $\lambda_i = k/k_i$:

$$W_{\mathbf{kq}_1|\mathbf{q}_2\mathbf{q}_3} = W_{G_1\mathbf{k}_1,\, G_1\mathbf{k}|G_1\mathbf{k}_3,\, G_1\mathbf{k}_2} = \lambda_1^{m-d-\beta} W_{\mathbf{k}_1\mathbf{k}|\mathbf{k}_3\mathbf{k}_2}, \tag{2.13}$$

$$W_{\mathbf{kq}_1|\mathbf{q}_2\mathbf{q}_3} = W_{G_2\mathbf{k}_2,\, G_2\mathbf{k}_3|G_2\mathbf{k},\, G_2\mathbf{k}_1} = \lambda_2^{m-d-\beta} W_{\mathbf{k}_2\mathbf{k}_3|\mathbf{kk}_1} \text{ etc.}$$

Thus, with the aid of the symmetry transformations (2.9)—(2.12) the collision integral is brought to the form

$$I_{\text{coll}}\{N(\mathbf{k})\} = {}^1/_4 \int d\tau_k W_k[f_k + \lambda_1{}^r f_{G_1k} + \lambda_2{}^r f_{G_2k} + \lambda_3{}^r f_{G_3k}], \quad r = m + 3d - \beta \tag{2.14}$$

or (in detailed notation)

$$I_{\text{coll}}\{N(\mathbf{k})\} = {}^1/_4 \int d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3\, W_{\mathbf{kk}_1|\mathbf{k}_2\mathbf{k}_3}[f(\mathbf{kk}_1|\mathbf{k}_2\mathbf{k}_3) + (k/k_1)^r f(G_1\mathbf{k}_1, G_1\mathbf{k}|G_1\mathbf{k}_3, G_1\mathbf{k}_2) + (k/k_2)^r f(G_2\mathbf{k}_2, G_2\mathbf{k}_3|G_2\mathbf{k}, G_2\mathbf{k}_1) + (k/k_3)^r f(G_3\mathbf{k}_3, G_3\mathbf{k}_2|G_3\mathbf{k}_1, G_3\mathbf{k})]. \tag{2.14'}$$

For isotropic and homogeneous distributions, $N(\mathbf{k}) = N_\omega = \omega^s$, the function $f_k = f_k^0$ is invariant under rotations, $f_{gk}^0 = f_k^0$, and is also homogeneous. As a result the integrand in Eq. (2.14′) can be factorized:

$$I_0\{N_\omega\} = \frac{\omega^\nu}{4}\int d\tau_k W_k f_k^0 (\omega^{-\nu} + \omega_1^{-\nu} - \omega_2^{-\nu} - \omega_3^{-\nu}), \tag{2.15}$$

$$f_k^0 = (\omega\omega_1\omega_2\omega_3)^s(\omega^{-s} + \omega_1^{-s} - \omega_2^{-s} - \omega_3^{-s}),$$

$$\nu \equiv \nu(s) = \alpha r + 3s = \alpha(m + 3d) - 1 + 3s, \ \alpha \equiv 1/\beta.$$

As is evident from Eq. (2.15), the collision integral vanishes either because $f_k^0 = 0$ for $s = 0, -1$ (which corresponds to the limiting cases of the equilibrium distribution $(\omega - \mu)^{-1}$ for $\mu \gg \omega$ and $\mu \ll \omega$), or else because the last factor vanishes at values of $s = s_0, s_1$ such that $\nu(s_{0,1}) = 0, -1$. Here the vanishing occurs as a consequence of energy conservation ($s = -1, \nu = -1$) or conservation of the number of particles ($s = 0, \nu = 0$) during collisions. The nonequilibrium distributions of the Kolmogorov type

$$N_\omega = \omega^{s_0}, \quad s_0 = \frac{1}{3} - \frac{\alpha}{3}(m + 3d), \tag{2.16}$$

$$N_\omega = \omega^{s_1}, \quad s_1 = -\frac{\alpha}{3}(m + 3d)$$

correspond to the constancy of the particle flux ($s = s_0$) and of the energy flux ($s = s_1$) in the turbulence spectrum (compare Eq. (2.16) with Eqs. (4.5) and (4.6)).

For isotropic distributions one can also obtain the solution from the collision integral averaged over angles, by using the symmetry properties in ω-space. In fact, averaging over the angles in Eqs. (2.15) and (2.1) only affects the transition probability W_k. In this connection the averaged transition probability[4-6] $T_{\omega\omega_1|\omega_2\omega_3}$ is homogeneous and in frequency space it possesses the same symmetry properties (2.4), which permits us to factorize the integral (2.1) with the aid of the linear-fractional transformations of the frequencies proposed by Zakharov:

$$\omega_1 \to \left(\frac{\omega}{\omega_1}\right)^2 \omega_1, \quad \omega_2 \to \frac{\omega}{\omega_1}\omega_2, \quad \omega_3 \to \frac{\omega}{\omega_1}\omega_3;$$

$$\omega_1 \to \frac{\omega}{\omega_2}\omega_3, \quad \omega_2 \to \left(\frac{\omega}{\omega_2}\right)^2 \omega_2, \quad \omega_3 \to \frac{\omega}{\omega_2}\omega_1;$$

$$\omega_1 \to \frac{\omega}{\omega_3}\omega_2, \quad \omega_2 \to \frac{\omega}{\omega_3}\omega_1, \quad \omega_3 \to \left(\frac{\omega}{\omega_3}\right)^2 \omega_3.$$

It is obvious that these transformations correspond to a reduction of the vector transformations (2.9)—(2.12) in k-space to frequency space. But whereas the Zakharov transformations are only applicable in the case of isotropic distribution functions, the transformations (2.9)—(2.12) enable us to also determine the nonisotropic distributions.

3. DRIFTING STATIONARY DISTRIBUTIONS

In the same way that the more general distribution $N_\omega^0(\mathbf{k}) = (\omega_k - \mathbf{k}\cdot\mathbf{u} - \mu)^{-1}$ given by Eq. (2.7) corresponds to the Rayleigh-Jeans distribution $N_\omega^0 = \omega^{-1}$, it is natural to seek analogous distributions which correspond to the turbulence spectra (2.16). In view of the nonequilibrium nature of the distributions (2.16), it is impossible to obtain the drifting distributions corresponding to them by the simple replacement $\omega \to \omega - \mathbf{k}\cdot\mathbf{u} - \mu$. However, if the drift parameters are small, then one can seek the distribution $N(\mathbf{k})$ in the form

$$N(\mathbf{k}) = N_\omega(1 + \omega^t\delta\mu + \omega^p(\varkappa\delta\mathbf{u})), \quad N_\omega = \omega^s, \quad \varkappa = \mathbf{k}/k, \tag{3.1}$$

in the approximation linear in the drift parameters $\delta\mu$ and $\delta\mathbf{u}$. For the distribution (3.1) one can rewrite the linearized collision integral (2.14′) and the function f_k in the form

$$I = I_0 + I_u\delta\mu + I_u(\varkappa\delta\mathbf{u}), \quad f_k = f_k^0 + f_k^\mu\delta\mu + (\mathbf{f}_k\delta\mathbf{u}), \tag{3.2}$$

where f_k^0 is given in Eq. (2.15),

$$f_k^\mu = \omega^t\xi + \omega_1^t\xi_1 - \omega_2^t\xi_2 - \omega_3^t\xi_3,$$

$$\mathbf{f}_k = \varkappa\omega^p\xi + \varkappa_1\omega_1^p\xi_1 - \varkappa_2\omega_2^p\xi_2 - \varkappa_3\omega_3^p\xi_3, \tag{3.3}$$

and the quantities ξ are given by the equations

$$\xi \equiv \xi(\omega, \omega_1, \omega_2, \omega_3) = (\omega\omega_1\omega_2\omega_3)^s(\omega_1^{-s} - \omega_2^{-s} - \omega_3^{-s}), \tag{3.4}$$

$$\xi_1 \equiv \xi(\omega_1, \omega, \omega_2, \omega_3), \quad \xi_2 \equiv \xi(\omega_2, \omega_3, \omega, \omega_1), \quad \xi_3 \equiv \xi(\omega_3, \omega_2, \omega, \omega_1).$$

It follows from Eq. (2.14′) that by using the properties of the symmetry transformations G_i given by Eqs. (2.9)—(2.12) the integrals I_μ and I_u reduce to the following factorized form:

$$I_\mu = \frac{\omega^{\nu_t}}{4}\int d\tau_k W_{\mathbf{kk}_1|\mathbf{k}_2\mathbf{k}_3} f^\mu(\mathbf{kk}_1|\mathbf{k}_2\mathbf{k}_3)(\omega^{-\nu_t} + \omega_1^{-\nu_t} - \omega_2^{-\nu_t} - \omega_3^{-\nu_t}), \quad \nu_t \equiv \nu + t; \tag{3.5}$$

$$I_u = \frac{\omega^{\nu_p}}{4}\int d\tau_k W_{\mathbf{kk}_1|\mathbf{k}_2\mathbf{k}_3}\mathbf{f}(\mathbf{kk}_1|\mathbf{k}_2\mathbf{k}_3)[\varkappa\omega^{-\nu_p} + \varkappa_1\omega_1^{-\nu_p} - \varkappa_2\omega_2^{-\nu_p} - \varkappa_3\omega_3^{-\nu_p}], \quad \nu_p \equiv \nu + p. \tag{3.6}$$

The integral I_0 and also $\nu = \nu(s)$ are given in (2.15). In deriving Eq. (3.6) we have used the fact that the angles transform one into the other under the transformations (2.9)—(2.12):

$$(\varkappa, g_i\varkappa_j) = (g_i\varkappa_i, g_i\varkappa_j) = (\varkappa_i, \varkappa_j). \tag{3.7}$$

The vanishing of the collision integral (3.2) indicates

Turbulent systems	Parameters and distributions															
	$\omega = k^{\beta}$, $\alpha \equiv \beta - 1$	Dimensionality of k-space	$U_{\lambda k} = \lambda^m U_k$	$U_k \sim (k_1 k_3)^{m_1/2}\ k_{1,3} \to 0$	Turbulent distributions $N=\omega^s(1+\omega^t\,\delta\mu+\omega^{p}\varkappa\delta u)$						Deviations from equilibrium distributions $N=\omega^s(1+\omega^t\,\delta\mu+\omega^{\tilde{t}}\,\delta\tilde{\mu}+\omega^{p}\varkappa\delta u)$					
					$s=s_1$			$s=s_0$			$s=-1$			$s=0$		
	β	d	m	m_1	s_1	p_1	t_1	s_0	p_0	t_0	t_2	$\tilde{t}_2$	p_2	t_3	$\tilde{t}_3$	p_3
Gravitational waves	1/2	2	6	4	−8 [4]	−1	1	−23/3 [5]	−2	−1	−20	−21	−22	−23	−24	−25
Plasma waves	2	3	0	0	−3/2 [6] *	1/2	1	−7/6 [6]	−1/2*	−1	−1/2	−3/2	−1	−7/2	−9/2	−4

*Turbulent distributions–nonlocal.

that each of the terms I_0, I_μ, and I_u vanishes independently, which determines the exponents s, p, and t in the distribution (3.1). It follows from the condition $I_0 = 0$ that $s = 0, -1, s_0, s_1$ (2.16).

Let us consider turbulent distributions. As is clear from Eq. (3.5), $I_\mu = 0$ if $\nu_t = 0, -1$. A solution which violates the homogeneity of the distribution arises when ν and ν_t are determined by different conservation laws ($\nu = 0$, $\nu_t = -1$ and $\nu = -1$, $\nu_t = 0$). For $\nu_t = \nu$ ($t = 0$) the normalization constant in the distribution N_ω is simply changed. The vanishing of I_u occurs when $\nu_p = -\alpha$ because of the law of momentum conservation. Finally we obtain two solutions. They correspond to small deviations from the distributions with a constant flux of particles (3.8) and a constant flux of energy (3.9) in the spectrum:

$$N(k) = \omega^{s_0}[1+\omega^{-1}\delta\mu + \omega^{-2\alpha}(\mathbf{k}\delta\mathbf{u})],\quad t_0=-1,\quad p_0=-\alpha, \tag{3.8}$$

$$N(k) = \omega^{s_1}[1+\omega\delta\mu + \omega^{1-2\alpha}(\mathbf{k}\delta\mathbf{u})],\quad t_1=1,\quad p_1=1-\alpha. \tag{3.9}$$

It is clear that these solutions cannot be obtained by expanding the quantities $(\omega_{\mathbf{k}} - \mathbf{k}\cdot\delta\mathbf{u} - \delta\mu)^s$ at $s = s_0$, s_1. This is due to the nonequilibrium nature of the initial isotropic distribution (see[9]). Whereas the drift distribution $N^0(\mathbf{k}) = (\omega_{\mathbf{k}} - \mathbf{k}\cdot\mathbf{u} - \mu)^{-1}$ corresponds to the equilibrium associated with nonvanishing total momentum of the quasiparticles, here we are dealing with essentially nonequilibrium distributions. Nevertheless, they are also determined by a small number of "drift" macroscopic parameters, formed in accordance with the conservation laws associated with the presence of flux in the spectrum (a source at the origin of k-space). We shall return to a discussion of the physical meaning of the distributions (3.8) and (3.9) in the next section, after a detailed investigation of fluxes in the turbulence spectrum.

Now let us dwell on the stationary, nonequilibrium deviations from the isotropic Rayleigh-Jeans distribution. It turns out that the equation (1.2) together with the equilibrium drifting distribution

$$N^0(\mathbf{k}) = (\omega_{\mathbf{k}} - \mathbf{k}\delta\mathbf{u} - \delta\mu)^{-1} \approx \omega^{-1}[1+\mathbf{k}\delta\mathbf{u}/\omega_k + \delta\mu/\omega_k] \tag{3.10}$$

also leads to nonequilibrium deviations from the distribution $N^0_\omega = \omega^{-1}$. Let us return to formulas (3.2)–(3.6) for $s = -1$. It is easy to verify that the exponents t and p, which lead to the vanishing of f^μ and f, correspond to an expansion of the locally equilibrium distribution $N^0_{\omega-\mathbf{k}\cdot\delta\mathbf{u}-\delta\mu}$ and to its renormalization, i.e., they correspond to a change of the temperature. This is also clear from the fact that f^μ and f represent derivatives of f_k which vanish for the equilibrium distribution (2.7). However, other solutions of Eq. (1.2) exist and can be obtained by choosing the exponents t and p in such a way that the last factors in I_μ and I_u vanish. This gives the following values for the exponents in the distribution (3.1):

$$\begin{aligned} t_2 &= -\nu(-1) = 4-\alpha(m+3d),\\ \tilde{t}_2 &= -1-\nu(-1) = 3-\alpha(m+3d)\quad (s=-1),\\ p_2 &= -\nu(-1) - \alpha = 4-\alpha(m+3d+1). \end{aligned} \tag{3.11}$$

Thus, we obtain modified Rayleigh-Jeans distributions of the form (3.1); the final expressions for these distributions and the values of the exponents for specific systems are given in the table.

Similar nonequilibrium drifting deviations from the solution $N_\omega = 1$ ($s = 0$) are found in a similar manner; the corresponding exponents in (3.1) are given by the expressions

$$\begin{aligned} t_3 &= -\nu(0) = 1-\alpha(m+3d),\\ \tilde{t}_3 &= -1-\nu(0) = -\alpha(m+3d)\quad (s=0),\\ p_3 &= -\nu(0) - \alpha = 1-\alpha(m+3d+1). \end{aligned} \tag{3.12}$$

We note that one can also obtain the deviations from the equilibrium distributions by dimensional considerations, by resolving the dimensionless function F with respect to small fluxes (cf. (1.14)):

$$N = \frac{T}{\rho\omega} F\left(\frac{\delta P}{x}, \frac{\omega\delta Q}{x}, \frac{\omega(\mathbf{k}\delta\mathbf{B})}{k^2 x}\right),\quad x = k^{3d+m}\left(\frac{T}{\rho\omega}\right)^3,\quad s=-1, \tag{3.13}$$

where x is a quantity having the dimensions of an energy flux and T is the temperature. Thus, we arrive at the distribution corresponding to (3.11), and by replacing $T/\rho\omega$ by $T/\rho\mu$ in (3.13) we arrive at expression (3.12).

4. THE CONNECTION BETWEEN THE DISTRIBUTIONS AND THE PROPERTIES OF THE SOURCE AND THE PHYSICAL INTERPRETATION OF THE OBTAINED SOLUTIONS

To clarify the meaning of distributions of the form (3.1), let us turn first to the equations for the fluxes of the conserved quantities. Under steady-state conditions they take the form

$$\operatorname{div}_{\mathbf{k}}\mathbf{Q} = D(\mathbf{k}),\quad \operatorname{div}_{\mathbf{k}}\mathbf{P} = \tilde{D}(\mathbf{k}),\quad \operatorname{div}_{\mathbf{k}}\mathbf{\Pi}_i = D_i(\mathbf{k}), \tag{4.1}$$

where the sources appear on the right-hand side. In analogy to (1.9) we express the solutions of Eqs. (4.1) in terms of potentials:

$$\begin{pmatrix}\varphi(\mathbf{k})\\ \tilde{\varphi}(\mathbf{k})\end{pmatrix} = \int d\mathbf{k}'\, g(\mathbf{k}-\mathbf{k}') \begin{pmatrix} D(\mathbf{k}')\\ \tilde{D}(\mathbf{k}')\end{pmatrix},\quad \boldsymbol{\varphi}(\mathbf{k}) = \int d\mathbf{k}'\, g(\mathbf{k}-\mathbf{k}')\mathbf{D}(\mathbf{k}'), \tag{4.2}$$

where q(k), which is equal to $1/4\pi k$ in the case of three dimensions and is equal to $(2\pi)^{-1}\ln k^{-1}$ in the two-dimensional case, is the Green's function for Poisson's equation. The particle-flux and energy-flux densities are given by $\mathbf{Q} = -\nabla\varphi$ and $\mathbf{P} = -\nabla\tilde{\varphi}$, and the momentum flux density is given by $\mathbf{\Pi}_i = -\nabla\varphi_i$.

In the inertial interval where, by definition, the distance to the source is large, the solutions (4.2) are expressed in terms of multipole moments:

$$\begin{pmatrix}\mathbf{Q}\\ \mathbf{P}\end{pmatrix} = -\begin{pmatrix}A\\ \tilde{A}\end{pmatrix}\nabla g + \begin{pmatrix}B_i\\ \tilde{B}_i\end{pmatrix}\nabla_i(\nabla g) - \begin{pmatrix}C_{ij}\\ \tilde{C}_{ij}\end{pmatrix}\nabla_i\nabla_j(\nabla g) + \dots,$$

$$\Pi_{ij} = -A_i\nabla_j g + B_{il}\nabla_l\nabla_j g - \dots$$

Here A, B, and C_{ij} denote the charge, dipole moment, and quadrupole moment of the particle source; $\tilde{A}$, $\tilde{B}$, and $\tilde{C}_{ij}$ denote the corresponding moments for the energy source; **A** denotes the "vector charge" of the momentum source, B_i is its "dipole moment," etc.

$$\begin{pmatrix}A\\ \tilde{A}\end{pmatrix} = \int d\mathbf{k}\begin{pmatrix}D(\mathbf{k})\\ \tilde{D}(\mathbf{k})\end{pmatrix},\quad \begin{pmatrix}\mathbf{B}\\ \tilde{\mathbf{B}}\end{pmatrix} = \int d\mathbf{k}\,\mathbf{k}\begin{pmatrix}D(k)\\ \tilde{D}(k)\end{pmatrix},$$

$$\begin{pmatrix}C_{ij}\\ \tilde{C}_{ij}\end{pmatrix} = \int d\mathbf{k}\left(k_i k_j - \frac{k^2}{3}\delta_{ij}\right)\begin{pmatrix}D(k)\\ \tilde{D}(k)\end{pmatrix},$$

$$\mathbf{A}=\int d\mathbf{k}\,\mathbf{D}(\mathbf{k}),\qquad B_{il}=\int d\mathbf{k}\,k_i D_l(\mathbf{k})\ \text{and so forth.}\tag{4.4}$$

If the sources are related ($\widetilde{D} = \omega D$, $\mathbf{D} = \mathbf{k}D$, see footnote 3), then $\mathbf{A} = \mathbf{B}$, that is, the "charge" source of the momentum is equal to the dipole moment of the particle source, and so forth.

When only one of the multipole moments does not vanish, we have a power-law distribution that can be determined (to within its angular dependence) from dimensional considerations. Here the functional dependence of the distribution on the multipole moment $\mathfrak{M}$ is determined by the kinetic equation $\mathfrak{M} \sim I_{\text{coll}} \sim N^3$ so that $N \sim \mathfrak{M}^{1/3}$. For example, the distribution corresponding to constant energy flux $P \equiv (\mathbf{P}\cdot\mathbf{k})k^{d-2} \sim A = \text{const}$ (when the particle flux vanishes) is given by the expression

$$N = P^{1/3}k^{-d-m/3}.\tag{4.5}$$

In analogous fashion the constant particle flux $Q \equiv (\mathbf{Q}\cdot\mathbf{k})k^{d-2} \sim A$ associated with zero energy flux corresponds to the following distribution (compare expressions (4.5) and (4.6) with (2.16)):

$$N = Q^{1/3}[\omega(k)]^{1/3}k^{-d-m/3}.\tag{4.6}$$

Here we note that the degree of homogeneity of the matrix element can also be determined from dimensional considerations under the conditions for complete self-similarity

$$m = 10 - 2d,\quad N = P^{1/3}k^{-(10+d)/3},\quad N = Q^{1/3}[\omega(k)]^{1/3}k^{-(10+d)/3},\tag{4.7}$$

when characteristic parameters having the dimensions of length and time are not present in the system. Such parameters exist for long waves on the surface of a liquid and for Langmuir waves, leading to the dimensionless combinations kh and $\omega(k)/\omega_{p,e}$, where h is the depth of the liquid, $\omega_{p,e}$ denote the ion and electron plasma frequencies, $\omega(k) \sim k^2$ is a small dispersion correction, and consequently these systems are not completely self-similar.

Now let us discuss the physical meaning of the distributions found in the preceding section. First let us consider the deviations from the solutions (4.5) and (2.16) corresponding to an energy flux $P = (\mathbf{P}\cdot\mathbf{k})k^{d-2}$, $\mathbf{Q} = 0$, $\Pi_i = 0$. From Eqs. (1.5) and (1.7) we obtain the following results for the corrections to the fluxes:

$$\operatorname{div}\delta\mathbf{Q} \equiv -\delta I = 0,\quad \operatorname{div}\delta\mathbf{P} \equiv -\omega\delta I = 0,\quad \operatorname{div}\delta\mathbf{\Pi}_i \equiv -k_i\delta I = 0,\tag{4.8}$$

where

$$\delta I = I_\mu\delta\mu + I_u(\varkappa\delta\mathbf{u}).$$

Let us show that the solution (3.9) corresponds to the appearance of a particle flux $\delta Q \sim \delta\mu$ and a momentum flux $\sim\delta u$. On the strength of the linearity of Eqs. (4.8), let us first consider only the scalar correction. Then, according to Eq. (3.5), $k^d\delta I \sim \omega^{\nu+t}\delta\mu = \delta\mu$, so that we obtain $\operatorname{div}\delta\mathbf{Q} \sim k^{-d}\delta\mu$ from (4.8). By comparing this result with the field of a point charge, $\delta\mathbf{Q} = -\delta Q\nabla g \sim \delta Qk^{-d+1}$, we see that $\delta Q \sim \delta\mu$, whereas the remaining equations in (4.8) can only be satisfied for $\delta\mathbf{P} = \delta\Pi_i = 0$. Thus, we have verified that a small isotropic deviation from the distribution with a constant energy flux corresponds to the appearance of a constant particle flux in the turbulence spectrum. In analogous fashion, an anisotropic deviation does not give any contribution to the particle and energy fluxes, leading instead to the appearance of a constant momentum flux $\mathbf{A}$: $\delta\mathbf{Q} = \delta\mathbf{P} = 0$, $\delta\mathbf{\Pi}_i = -A_i\nabla g \sim k^{-d+1}$ (compare with expression (1.12)) where $\mathbf{A} \sim \delta\mathbf{u}$.

In exactly the same way, by considering the solution (3.8) we find that in this case the isotropic correction leads to the appearance of a nonvanishing energy flux $\delta\mathbf{Q} = \delta\mathbf{\Pi}_i = 0$, $\delta\mathbf{P} \sim k^{-d+1}\delta\mu$; the anisotropic term corresponds to the appearance of a momentum flux.

The nonequilibrium deviations from the distributions $N_\omega = \omega^{-1}$ and $N_\omega = 1$ (see Eqs. (3.1), (3.11), and (3.12)) are also related to the appearance of constant fluxes in the spectrum, where the two isotropic terms correspond to an energy flux ($\tilde{t}_2$, $\tilde{t}_3$) and a particle flux (t_2, t_3), and the anisotropic term is associated with a momentum flux.

We note that the laws of energy and momentum conservation in the form (4.1) are also valid for the case of strong turbulence. The Kolmogorov spectrum is associated with the total "charge" of the source. The dipole moment $\mathbf{B}$ of the source should lead to the appearance of momentum flux in the spectrum; a purely dipole source leads to a distribution of the form

$$\mathcal{N}(\mathbf{k}) \sim B^{2/3}k^{-13/3}f(\varkappa),\quad f(\varkappa) \sim 1\tag{4.9}$$

and corresponds to a constant momentum flux (see expression (1.12)), where here $\mathcal{N}(\mathbf{k})$ is normalized to the energy density $\mathcal{E}/\rho L^3 = \int d\mathbf{k}\mathcal{N}(\mathbf{k})$. According to (4.9) the spectral energy density $E(k) = \int d\kappa\mathcal{N}(\mathbf{k})$ is proportional to $E(k) \sim k^{-7/3}$ in contrast to the spectrum for isotropic turbulence,[10,11] where $E(k) \sim k^{-5/3}$.

5. THE LOCALIZABILITY OF THE ANISOTROPIC TURBULENT DISTRIBUTIONS

The distributions obtained above will be local if the region of small and large wave vectors gives a negligible contribution to the collision integral (2.1) and the inertial region, where $k_1, k_2, k_3 \sim k$, gives the major contribution to the integral. This requires convergence of the integrals appearing in the collision term. The localizability of the distributions for gravitational waves on deep water and for plasma oscillations was discussed in[4,6].

First let us consider a system with a non-activational dispersion law, $\omega(k) = k^\beta$ with $\beta < 1$. In this connection, as a consequence of the conservation laws only two wave vectors in the interval (2.1) can be small (large). Let $k_{1,3} \to 0$ ($k_{1,3} \ll k, k_2$). As a consequence of the fact that $N(k) \to \infty$ as $k \to 0$ the most dangerous terms in (2.1) are $N_1N_3(N_2 - N)$. It is convenient to represent the asymptotic behavior of the matrix element in the form

$$U_{\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3} = (k_1k_3)^{m_1/2}(kk_2)^{m_2/2}u\left(\frac{\mathbf{k}+\mathbf{k}_2}{2}\,\middle|\,\mathbf{k}_1,\mathbf{k}_3\right),\tag{5.1}$$

$$k_1,\ k_3 \ll k,\ k_2,\quad m_1 + m_2 = m,$$

where $u \sim 1$ is a function of zero degree of homogeneity, which is symmetric with respect to interchange of the last two arguments (as a consequence of the symmetry of U_k with respect to interchange of pairs of arguments as indicated in (2.4)). The values of the exponents m_1 and m are given in the table for specific systems. Performing the integration over $\mathbf{k}_2$ in (2.1) at the expense of the δ-function in the momentum, we obtain the following result for the dangerous terms in I_{coll}:

$$\begin{aligned}&\int d\mathbf{k}_1\,d\mathbf{k}_3\,\delta[k^\beta + k_1^\beta - k_3^\beta - |\mathbf{k}+\mathbf{q}|^\beta](k_1k_3)^{m_1/2}(|\mathbf{k}+\mathbf{q}|k)^{m_2/2}\\&\times u\left(\mathbf{k}+\frac{\mathbf{q}}{2}\,\middle|\,\mathbf{k}_1,\mathbf{k}_3\right)N(\mathbf{k}_1)N(\mathbf{k}_3)[N(\mathbf{k}+\mathbf{q})-N(\mathbf{k})],\quad q \equiv \mathbf{k}_1-\mathbf{k}_3.\end{aligned}\tag{5.2}$$

Since the principal term of the asymptotic transition probability in (5.2)

$$\delta[k_1^\beta - k_3^\beta - \beta(\mathbf{kq})k^{\beta-2}](k_1k_3)^{m_1/2}k^{m_2}u(\mathbf{k}|\mathbf{k}_1,\mathbf{k}_3)$$

is even under the exchange $\mathbf{k}_1 \rightleftarrows \mathbf{k}_3$, it gives a nonvanishing contribution to the integral upon integration with the quadratic term of the expansion of $N(\mathbf{k}+\mathbf{g}) - N(\mathbf{k})$:

$$\int d\mathbf{k}_1\, d\mathbf{k}_3\, \delta(k_1^\beta - k_3^\beta)(k_2k_3)^{m_1/2}u(\mathbf{k}|\mathbf{k}_1,\mathbf{k}_3)q_iq_jN(\mathbf{k}_1)N(\mathbf{k}_3), \quad (5.3)$$

where we have omitted the factors which do not influence the convergence.

For $\beta < 1$ it is not difficult to verify that the odd (with respect to the interchange $1 \rightleftarrows 3$) part of the transition probability, upon being combined with the linear term of the expansion of $N(\mathbf{k}+\mathbf{q}) - N(\mathbf{k})$, leads to an integral that converges no worse than (5.3). For $N(k)$ of the form (3.1), upon linearizing (5.3) with respect to $\delta\mu$ and $\delta\mathbf{u}$, we obtain the following convergence conditions:

$$\Delta_1 > 0, \quad \Delta_1 + \beta t > 0, \quad \Delta_1 + \beta p > 0,$$
$$\Delta_1 \equiv m_1 + 2d + 2 - \beta + 2\beta s \quad (5.4)$$

for the distribution N_ω (3.1) of the isotropic and anisotropic corrections, respectively.

Without reproducing the analogous calculations, in which the same asymptotic expression (5.1) is used, let us present the convergence condition (2.1) on the distributions (3.1) for large values of k:

$$\Delta_2 < 0, \quad \Delta_2 + \beta t < 0, \quad \Delta_2 + \beta p < 0,$$
$$\Delta_2 \equiv d - 1 + \beta s. \quad (5.5)$$

For an activational dispersion law, β can be either smaller or larger than unity ($\omega_k = k^\beta$ is a small correction to the activation frequency). For $\beta > 1$ the conservation laws permit the vanishing of one of the wave vectors. Therefore it is necessary to consider the convergence of the collision integral in four regions: both $k_{1,3} \ll k, k_2$ and $k_{1,3} \gg k, k_2$ and also for the case of one small argument, for example, $k_1 \ll k, k_{2,3}$ or $k \ll k_{1,2,3}$. Therefore, in order to estimate the resultant integrals, we need in addition to the asymptotic behavior (5.1) also the asymptotic behavior of the matrix element associated with one small argument:

$$U_{\mathbf{kk}_1|\mathbf{k}_2\mathbf{k}_3} = k_1^{m'}v(\mathbf{k}|\mathbf{k}_2,\mathbf{k}_3),$$
$$k_1 \ll k, k_2, k_3, \quad (5.6)$$

where $v(\mathbf{k}|\mathbf{k}_2,\mathbf{k}_3)$ is a homogeneous function of degree $m'' = m - m'$, which is symmetric with respect to the last two arguments. By considering the convergence of the collision term in each of these regions in the same way as has been done above, we obtain the following conditions for localizability of the turbulent distributions (3.1):

$$\Delta' > 0, \quad \Delta' + \beta t > 0, \quad \Delta' + \beta p > 0, \quad \Delta' \equiv m' + d + \beta s,$$
$$\Delta_1 > 0, \quad \Delta_1 + \beta t > 0, \quad \Delta_1 + \beta p > 0, \quad \Delta_1 \equiv m_1 + 2d + 1 + 2\beta s; \quad (5.7)$$
$$\Delta_2 < 0, \quad \Delta_2 + \beta t < 0, \quad \Delta_2 + 1 + \beta p < 0, \quad \Delta_2 \equiv m_2 + d - 3 + \beta s,$$
$$\Delta'' < 0, \quad \Delta'' + \beta t < 0, \quad \Delta'' + \beta p < 0, \quad \Delta'' \equiv m'' + 2d - \beta + 2\beta s, \quad (5.8)$$

where (5.7) guarantee convergence at zero and (5.8) guarantees convergence at infinity.

It is of special interest to note that for the case of Langmuir turbulence considered in[6], when the nonlinearity is due to the interaction with ions,[4)] the matrix element

$$U_{\mathbf{kk}_1|\mathbf{k}_2\mathbf{k}_3} = [(\varkappa\varkappa_3)(\varkappa_1\varkappa_2) + (\varkappa\varkappa_2)(\varkappa_1\varkappa_3)]^2, \quad \varkappa_j = \mathbf{k}_j/k_j, \quad (5.9)$$

is symmetric with respect to a change in the sign of one of the wave vectors, $\mathbf{k}_j \to -\mathbf{k}_j$, which leads to the same property of the functions $u(\mathbf{k}|\mathbf{k}_1,\mathbf{k}_3)$ and $v(\mathbf{k}|\mathbf{k}_2,\mathbf{k}_3)$ in expressions (5.1) and (5.6). In this connection the conditions for convergence at zero of the integrals arising from the anisotropic corrections are improved--in (5.7) it is necessary to make the replacement $\beta p \to \beta p + 1$. The last condition in (5.8) is also weakened—instead of $\Delta'' + \beta p < 0$ we have $\Delta'' - 1 + \beta p < 0$.

The results pertaining to the localizability of the turbulent distributions found for specific systems are given in the accompanying table.

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[1)]In this work we only consider systems of interacting waves under random-phase conditions, and the terms "particle" (quasi-particle) are employed to denote the elementary excitations (wave packets).

[2)]Isotropic distributions for gravitational waves on the surface of a deep liquid are found in [4,5], and the distributions for plasmons are found in [6,7].

[3)]The source may be found in a region where a description with the aid of the kinetic equation is not applicable; however, sources of the conserved quantities D, $\tilde{D}$, and **D**, playing the role of the boundary condition as $k \to 0$, can always be introduced into the conservation laws. Therefore, they are generally independent (i.e., the equalities indicated inside the parentheses may not hold).

[4)]The other case of Langmuir turbulence, when the nonlinearity is purely electronic, is treated in [12]. See [8] for a discussion of the drifting solutions. This case apparently has an extremely narrow region of applicability. [13]

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